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On conservativity and shattering for an equation of phytoplankton dynamics

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Presented by Pierre Auger

This article is dedicated to the memory of Ovide Arino

Abstract

A model of phytoplankton dynamics introduced by Arino describes the evolution of aggregates of phytoplankton by a kinetic-type equation composed of terms describing the growth of the aggregates and their splitting, where the latter is modelled by a singular integral operator of the same form as in the classical fragmentation theory. In this paper we shall show that despite the presence of the growth term, the model displays the typical properties of the fragmentation models; in particular, if the fragmentation rate is unbounded as the size of aggregates tends to zero, then there occurs an unaccounted for loss of the phytoplankton though formally nothing is taken out of the system. ***To cite this article: J. Banasiak, C. R. Biologies 327 (2004).***

Résumé

Sur la conservativité et l'éclatement d'une équation de dynamique du phytoplancton. Un modèle de dynamique du phytoplancton introduit par Arino et Rudnicki décrit l'évolution d'agrégats de phytoplancton au moyen d'une équation de type cinétique composée de termes décrivant la croissance d'agrégats et leur éclatement, où ce dernier phénomène est modélisé par un opérateur intégral singulier de la même forme que dans la théorie classique de la fragmentation. Nous montrerons dans cet article que, malgré la présence du terme de croissance, le modèle présente les propriétés typiques des modèles de fragmentation ; en particulier, si le taux de fragmentation est illimité alors que la taille des agrégats tend vers zéro, alors il y a un terme non pris en compte traduisant une disparition de phytoplancton, quoique rien n'ait formellement quitté le système. ***Pour citer cet article : J. Banasiak, C. R. Biologies 327 (2004).***

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1. Introduction

In their recent paper [1], O. Arino and R. Rudnicki considered a model of phytoplankton at the level of aggregates of cells. The aggregates are structured by their size and the phytoplankton system consists of aggregates of all possible sizes. The aggregate size can change due to the usual birth and death of individual cells, but also there are two other mechanisms acting at the level of aggregate: splitting of an aggregate into several parts and combining of two or more aggregates into a bigger one. The latter two are known in physics and chemical engineering as fragmentation–coagulation processes and describe a variety of phenomena ranging from polymerization/polymer degradation, droplets break-up and build-up, through rock crushing and grinding, solid drugs break-up in organisms, to blood cell aggregation and fragmentation. In phytoplankton, the major role in fragmentation and coagulation processes is played by the substance called TEP (Transparent Exopolymer Particles) that is a by-product of the growth of phytoplankton, and its stickiness causes the cells to remain together [2–5]. On the contrary, a low level of concentration of TEP results in fragmentation of the aggregate due to external causes, like currents or turbulence on one hand, and internal unspecified forces of biotic nature on the other.

In [1], the authors considered a relatively simple model of binary fragmentation and coagulation with bounded fragmentation and coagulation rates, as their aim was to investigate the long-time behaviour of the solution, and they succeeded in proving the existence of a time-invariant distribution to which the population of aggregates converges as time tends to infinity, whatever the initial population might be.

Our aim in this paper is to analyze more closely the inter-relation between the growth and fragmentation of aggregates so that we shall disregard the coagulation part. By the very nature of the model, the fragmentation process itself should be conservative, that is, the total amount (mass, the number of particles or cells) of the described quantity, say Q, contained in all the aggregates before and after a fragmentation event should be the same. Thus, if in some system the fragmentation occurs alongside another process of growth or decay determined by a certain law, then the evolution of the total amount of Q should follow this law due to the conservativity of the fragmentation process. If this is the case, then such a process is said to be *honest*. However, for pure fragmentation models and models combining fragmentation of clusters with their dissolution in the surrounding solute, it has been known for some time [6–9], that if the fragmentation rate of small clusters is large enough, then there appears an unexpected leakage of Q from the system, that is, the amount of Q in the system is strictly smaller than predicted by the laws of nature used to build the model.

In the existing physical literature, op. cit., this unaccounted for loss of Q (in this case, mass-loss), termed *shattering fragmentation*, is attributed to a phase transition and formation of a 'dust' of particles with zero size and non-zero mass (a similar but in some sense opposite process of forming an 'infinitely large' particle is known in coagulation as a *gelation*). For some relatively simple models, shattering fragmentation was analyzed in [4,7] by probabilistic methods. In a series of recent papers [10–14], the shattering and non-shattering fragmentation was fully characterized by the properties of the generator of the semigroup describing the evolution and the theory was applied to a wide range of processes providing a comprehensive classification of fragmentation models.

In particular, in [10], a model where fragmentation occurs together with a continuous mass loss due to dissolving of the substance has been analyzed and conditions ensuring conservativity and shattering have been provided. A crucial rôle in the analysis is played by the theory of substochastic, that is, positivity preserving and contractive semigroups. In this paper, we shall show that the model introduced by Arino and Rudnicki, though obviously not substochastic due to the appearance of the growth term, can be nevertheless transformed into one, and treated by a generalization of the theory developed in [10] yielding similar results, that is, the process is honest for rates of fragmentation bounded at 0, otherwise shattering fragmentation occurs irrespective of the growth rate (within the limits of the model).

It is, however, fair to admit that shattering fragmentation, as related to the creation of infinitesimally small aggregates, is not really a biological (or physical) phenomenon as in the real world there is always a lowest size of objects beyond which we cannot reach without encountering quantum effects. If one adopts such a point of view, then our results can be restated as saying that the models with fragmentation rates that are unbounded at 0 are non-biological.

2. The model

Following [1], we consider the following fragmentation model with mass loss:

$$\begin{aligned}\partial_t u(x,t) &= -\partial_x\big[b(x)u(x,t)\big] - d(x)u(x,t) - p(x)u(x,t)\\ &\quad + \int_x^\infty p(y)k(x,y)u(y,t)\,\mathrm{d}y \end{aligned} \tag{2.1}$$

where u is the distribution function of all the aggregates according to their size x, which, depending on the model, can be the number of cells, the total mass of the cells, or the total length of cells forming the aggregate. By the total size of the system, we understand the sum of sizes of all the aggregates the system consists of, that is, $\int_0^\infty u(x)x\,\mathrm{d}x$. Thus a natural requirement is that the total size of the system is finite at all finite times, which leads to the natural setting for (2.1), which is:

$$\begin{aligned}X &= L_1(\mathbb{R}_+, x\,\mathrm{d}x)\\ &= \left\{u;\ \|u\| := \int_0^\infty \big|u(x)\big|x\,\mathrm{d}x < \infty\right\}\end{aligned}$$

Further, the growth rate b is a sufficiently smooth function on $[0,\infty)$, satisfying:

$$0 < b(x) \leqslant \tilde{b}x, \quad x > 0 \tag{2.2}$$

for some constant $\tilde{b} > 0$. From (2.2) we have $b(0) = 0$ and we assume also that:

$$b'(0) > 0 \tag{2.3}$$

The function d is the death rate, which we assume to be continuous and bounded.

The fragmentation is characterized by two functions: p and k. The function p is the fragmentation rate, that is, the number of fragmentation events of aggregates of size x per unit time. We assume that $p \in L_{\infty,\mathrm{loc}}(\mathbb{R}_+)$ and $p \geqslant 0$ *a.e.* Further, k is a non-negative measurable function that describes the distribution of particle masses x spawned by the fragmentation of a particle of mass y. Formal balance of mass in fragmentation requires:

$$\int_0^y xk(x,y)\,\mathrm{d}x = y \tag{2.4}$$

that expresses the fact that the sizes of all daughter aggregates after fragmentation must add up to the size of the parent. The integral:

$$\int_0^y k(x,y)\,\mathrm{d}x = M_y \tag{2.5}$$

gives the expected number of daughter aggregates resulting from the fragmentation of a parent of size y; in general, M_y may be infinite. Note that in [1] the authors considered only binary fragmentation, that is, $M_y = 2$, and the normalized function $K(x,y) = k(x,y)/2$.

The typical choices for k used in the literature are: the power law $(\nu+2)x^\nu/y^{\nu+1}$ with $-2 < \nu \leqslant 0$, and its generalization

$$k(x,y) = \frac{1}{y}h\left(\frac{x}{y}\right) \tag{2.6}$$

which describes the situation when the fragmentation depends on the daughter size/parent size ratio and not on their sizes separately.

Integrating (2.1) multiplied by x, we obtain the formal equation governing the evolution of the total size of the system:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty u(x,t)x\,\mathrm{d}x &= -\int_0^\infty d(x)u(x,t)x\,\mathrm{d}x\\ &\quad + \int_0^\infty b(x)u(x,t)\,\mathrm{d}x\end{aligned} \tag{2.7}$$

where we used (2.4) and integration by parts. It is to be stressed that (2.7) is far from obvious – apart from the validity of the integration by parts, each term of (2.1) should be an integrable function so that we can use Fubini's theorem and (2.4). In fact, in this paper we determine classes of coefficients for which (2.7) is valid and for which it is not.

3. Transport semigroup

In this section we consider the differential part of Eq. (2.1), that is, the Cauchy problem:

$$\begin{aligned}\partial_t u(x,t) &= -\partial_x\big[b(x)u(x,t)\big] - d(x)u(x,t)\\ &\quad - p(x)u(x,t), \quad x>0,\ t>0 \qquad (3.1)\\ u(x,0) &= g(x)\end{aligned}$$

The solution of this problem can be written down explicitly. However, for the purpose of this paper, we shall need a precise characterization of the domain of the generator of the semigroup solving (3.1) and this is not obvious due to possible singularities of the fragmentation rate p and degeneracy of b at $x=0$. It turns out that direct estimates of the resolvent of the right-hand side of the equation in (3.1) are not easy, thus we shall simplify the problem even further and as the first step we shall deal with the Cauchy problem:

$$\begin{aligned}\partial_t u(x,t) &= -\partial_x\big[b(x)u(x,t)\big], \quad x>0,\ t>0\\ u(x,0) &= g(x)\end{aligned} \qquad (3.2)$$

Define the operator

$$[T_B u](x) = -\big(b(x)u(x)\big)_x$$

on the domain

$$D(T_B) = \big\{u\in X;\ bu \text{ is a.a.c. and } (bu)_x \in X\big\}$$

where the abbreviation 'a.a.c.' stands for almost absolutely continuous, that is, absolutely continuous on each compact interval of $(0,\infty)$.

Denoting by B a fixed antiderivative of $1/b$, say, $B(x) = \int_1^x \frac{\mathrm{d}s}{b(s)}$, we see, due to $0<b(x)<\tilde{b}x$ for $x>0$, that:

$$\lim_{x\to\infty} B(x) = +\infty, \qquad \lim_{x\to 0} B(x) = -\infty \qquad (3.3)$$

thus B is globally invertible on $\mathbb{R}$. Hence, defining $Y(t,x) := B^{-1}(B(x)-t)$, $x>0$, $0\leqslant t<\infty$, we can prove as in [1] that:

$$\big[S_{T_B}(t)g(\cdot)\big](x) = \frac{b(Y(t,x))g(Y(t,x))}{b(x)}$$

is a C_0-semigroup generated by $(T_B, D(T_B))$, that satisfies:

$$\big\|S_{T_B}(t)u\big\| \leqslant \mathrm{e}^{\tilde{b}t}\|u\| \qquad (3.4)$$

In particular, by the Hille–Yosida theorem, we obtain for $g\in X$ and $\lambda>\tilde{b}$:

$$\big\|R(\lambda, T_B)g\big\| \leqslant \frac{1}{\lambda-\tilde{b}}\|g\| \qquad (3.5)$$

Using the above we can prove the following result for the the semigroup solving (3.1).

Proposition 3.1. *The operator T defined by the formal expression:*

$$[Tu](x) = -\big(b(x)u(x)\big)_x - a(x)u(x)$$

on the domain:

$$D(T) = \big\{u\in X, au\in X, bu \text{ is a.a.c. and } (bu)_x\in X\big\}$$

where $a(x)=p(x)+d(x)$, generates a positive semigroup, say $(S_T(t))_{t\geqslant 0}$, satisfying for any $u\in X$:

$$\big\|S_T(t)u\big\| \leqslant \mathrm{e}^{\tilde{b}t}\|u\| \qquad (3.6)$$

where $\tilde{b}$ is defined in (2.2).

Proof. Let us consider the resolvent equation of (3.1):

$$\big(b(x)u(x)\big)_x + a(x)u(x) + \lambda u(x) = f(x)$$

Solving the above equation, we see that a good candidate for the resolvent is:

$$\big[R(\lambda)g\big](x) = \frac{\mathrm{e}^{-\lambda B(x)-A(x)}}{b(x)}\int_0^x \mathrm{e}^{\lambda B(y)+A(y)}g(y)\,\mathrm{d}y$$

where $A(x)$ is a fixed antiderivative of $a(x)/b(x)$. Direct integration gives:

$$\begin{aligned}&\big\|R(\lambda)g\big\|\\ &\quad\leqslant \int_0^\infty\left(\frac{\mathrm{e}^{-\lambda B(x)-A(x)}}{b(x)}\int_0^x \mathrm{e}^{\lambda B(y)+A(y)}\big|g(y)\big|\,\mathrm{d}y\right)x\,\mathrm{d}x\\ &\quad\leqslant \frac{1}{\lambda-\tilde{b}}\|g\|\end{aligned}$$

where we used the fact that $\mathrm{e}^{-A(x)}$ is non-increasing, and (3.5). Further, we have:

$$\begin{aligned}&\frac{a(x)}{b(x)}\mathrm{e}^{-\lambda B(x)-A(x)}\\ &\quad= -\frac{\lambda}{b(x)}\mathrm{e}^{-\lambda B(x)-A(x)} - \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-\lambda B(x)-A(x)} \qquad (3.7)\end{aligned}$$

so that

$$
\begin{aligned}
&\|aR(\lambda)g\| \\
&\quad\leqslant \int_0^\infty \left(\frac{e^{\lambda B(y)+A(y)}}{y} \int_y^\infty \frac{x a(x) e^{-\lambda B(x)-A(x)}}{b(x)} \, dx \right) \\
&\qquad \times |g(y)| y \, dy \\
&\quad\leqslant \int_0^\infty \left(1 + \frac{e^{\lambda B(y)+A(y)}}{y} \int_y^\infty e^{-\lambda B(x)-A(x)} \, dx \right) \\
&\qquad \times y |g(y)| \, dy \\
&\quad\leqslant \left(1 + (\lambda - \tilde{b})^{-1}\right) \|g\|
\end{aligned}
$$

where we again used monotonicity of $e^{-A(x)}$ and (3.5).

Next we observe that for $f \in X$,

$$
b(x)u(x) = e^{-\lambda B(x)-A(x)} \int_0^x e^{\lambda B(y)+A(y)} f(y) \, dy
$$

and both $e^{-\lambda B(x)-A(x)}$ and the integral (as a function of its upper limit) are almost absolutely continuous and bounded over any fixed interval $[\alpha, \beta] \subset]0, \infty[$. Therefore, it follows that the product is absolutely continuous on $[\alpha, \beta]$ and therefore bu is almost absolutely continuous. Moreover,

$$
\begin{aligned}
-\left(b(x)u(x)\right)_x &= \left(\lambda + a(x)\right) \frac{e^{-\lambda B(x)-A(x)}}{b(x)} \\
&\quad \times \int_0^x e^{\lambda B(y)+A(y)} f(y) \, dy - f(x) \\
&= \left(\lambda + a(x)\right) u(x) - f(x) \in X
\end{aligned}
$$

so that $R(\lambda)X \subset D(T)$. Since clearly $(\lambda - T)D(T) \subset X$, we have $(\lambda I - T)R(\lambda)f = f$ for any $f \in X$. To show that $R(\lambda)$ is the resolvent for T, it is enough to show that $\lambda I - T$ is injective on $D(T)$. We see that the only solution (up to a multiplicative constant) to

$$
\left(b(x)u(x)\right)_x + a(x)u(x) + \lambda u(x) = 0
$$

is $u_\lambda(x) = e^{-\lambda B(x)-A(x)}/b(x)$. Firstly, we observe that since $e^{-A(x)}$ is positive and decreasing, $e^{-A(x)} \geqslant c > 0$ in some interval $[0, \alpha]$. Moreover, since $b(x) \leqslant \tilde{b}x$, we have for $x \leqslant 1$:

$$
\begin{aligned}
e^{-\lambda B(x)} &= e^{-\lambda \int_1^x \frac{ds}{b(s)}} = e^{\lambda \int_x^1 \frac{ds}{b(s)}} \geqslant e^{-\frac{\lambda}{\tilde{b}} \ln x} \\
&= x^{-\frac{\lambda}{\tilde{b}}}
\end{aligned}
$$

Therefore, for $\alpha < 1$:

$$
\begin{aligned}
\|u_\lambda\| &= \int_0^\infty \frac{e^{-\lambda B(x)-A(x)}}{b(x)} x \, dx \geqslant c \int_0^\alpha \frac{e^{-\lambda B(x)}}{b(x)} x \, dx \\
&\geqslant \frac{c}{\tilde{b}} \int_0^\alpha x^{-\frac{\lambda}{\tilde{b}}} \, dx = \infty
\end{aligned}
\tag{3.8}
$$

as $\lambda > \tilde{b}$. Hence, $\lambda I - T$ is injective for $\lambda > \tilde{b}$ (even on its maximal domain) and $R(\lambda) = R(\lambda, T)$. The resolvent is clearly a positive operator so that, by the Hille–Yosida theorem, $(T, D(T))$ generates a positive semigroup satisfying (3.6).

From this proposition it follows that the operator

$$
\left(\tilde{T}, D(T)\right) = \left(T - \tilde{b}I, D(T)\right) \tag{3.9}
$$

generates a positive semigroup of contractions given by

$$
S_{\tilde{T}}(t)u = e^{-\tilde{b}t} S_T(t)u \tag{3.10}
$$

This shows that to prove the existence of a semigroup solving (a realization of) (2.1), characterize its generator and thus analyze the dynamics of the process, we can use the substochastic semigroup theory developed recently in a series of papers [10,11,13,15–17]. Below we shall recall the basic results of this theory.

4. Substochastic semigroups

In this section we shall summarize relevant facts from substochastic semigroup theory as developed in [10]. To avoid confusion, we shall use the same notation for the abstract operators as for the particular application discussed in this paper, however the theory is fairly general and requires only that the assumptions (A1)–(A3) be satisfied.

Let (Ω, μ) be a measure space and let $X = L_1(\Omega, \mu)$. If $Z \subset X$ is a subspace, then Z_+ denotes the cone of nonnegative elements of Z and for $f \in X$ the symbols $f_\pm$ denote the positive and negative part of f, that is, $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$. Let $(S(t))_{t \geqslant 0}$ be a strongly continuous semigroup on X. We say that $(S(t))_{t \geqslant 0}$ is a *substochastic semigroup* if for any $t \geqslant 0$, $S(t) \geqslant 0$ and $\|S(t)\| \leqslant 1$, and a *stochastic semigroup* if additionally $\|S(t)f\| = \|f\|$ for $f \in X_+$.

Accordingly, we consider linear operators in X: $\tilde{T} \subset T_B + \tilde{A}$ with $D(\tilde{T}) \subset D(T_B) \cap D(\tilde{A})$, and K, that have the following properties:

(A1) $(\tilde{T}, D(\tilde{T}))$ generates a substochastic semigroup $(S_{\tilde{T}}(t))_{t \geqslant 0}$;
(A2) $D(K) \supset D(\tilde{T})$ and $Ku \geqslant 0$ for $u \in D(\tilde{T})_+$;
(A3) for all $u \in D(\tilde{T})_+$

$$\int_\Omega (\tilde{T}u + Kf)\,\mathrm{d}\mu \leqslant 0 \tag{4.1}$$

Theorem 4.1 [11,17]. *Under the above assumptions, there exists a smallest substochastic semigroup $(S_{\tilde{G}}(t))_{t \geqslant 0}$ generated by an extension $\tilde{G}$ of the operator $\tilde{T} + K$. This semigroup, for arbitrary $u \in D(\tilde{G})$ and $t > 0$, satisfies:*

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{\tilde{G}}(t)u = \tilde{G} S_{\tilde{G}}(t)u \tag{4.2}$$

$(S_{\tilde{G}}(t))_{t \geqslant 0}$ can be obtained as a strong limit in X of semigroups $(S_r(t))_{t \geqslant 0}$ generated by $(\tilde{T} + rK, D(\tilde{T}))$ as $r \nearrow 1^-$; if $f \in X_+$, then the limit is monotonic.

The generator $\tilde{G}$ of $(S_{\tilde{G}}(t))_{t \geqslant 0}$ is characterized by:

$$(\lambda I - \tilde{G})^{-1} f = \sum_{n=0}^{\infty} (\lambda I - \tilde{T})^{-1} \big[K(\lambda I - \tilde{T})^{-1}\big]^n f \qquad f \in X \tag{4.3}$$

Formula (4.3) does not provide any explicit information as to how large an extension of $\tilde{T} + K$ the generator $\tilde{G}$ is and this problem is closely related to the behaviour of $(S_{\tilde{G}}(t))_{t \geqslant 0}$. To make this remark precise, we adapt the concept of honesty and dishonesty from the theory of Markov processes [18].

Firstly, note that (4.1) can be written as:

$$\int_\Omega (\tilde{T} + K)u\,\mathrm{d}\mu = -c(u), \quad u \in D(\tilde{T})_+ \tag{4.4}$$

where c is a nonnegative (possibly zero) functional defined on $D(\tilde{T})$. In this paper, we shall consider only the situation when c can be written as an integral functional, that is:

$$c(u) = \int_\Omega \varsigma(x) u(x)\,\mathrm{d}\mu_x \tag{4.5}$$

for some positive measurable function ς. We do not assume that c is bounded or closed.

Definition 4.1. We say that a substochastic semigroup $(S_{\tilde{G}}(t))_{t \geqslant 0}$ (generated by an extension $\tilde{G}$ of the operator $\tilde{T} + K$) is honest if c is finite on $D(\tilde{G})$, and, for any $0 \leqslant \mathring{u} \in D(\tilde{G})$, the solution $u(t) = S_{\tilde{G}}(t)\,\mathring{u}$ of (4.2) satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(t)\,\mathrm{d}\mu = \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\| = -c\big(u(t)\big) \tag{4.6}$$

Remark 4.1. The definition of honesty is not restricted to contractive semigroups and is valid even if c in (4.4) is of undetermined sign. In fact, for the original model (2.1) we shall be using this definition with a positive right-hand side in (4.6). However, for a general c, the existence part of the theory is usually not a trivial matter and this is why we prefer to present a complete theory for substochastic semigroups, and then apply it to a wider class of models that can be transformed to a substochastic case.

It can be proved that the honesty of $(S_{\tilde{G}}(t))_{t \geqslant 0}$, (4.6) is equivalent to its integral version: $(S_{\tilde{G}}(t))_{t \geqslant 0}$ is honest if and only if for any $f \in X_+$ and $t \geqslant 0$:

$$\|S_{\tilde{G}}(t)f\| = \|f\| - c\left(\int_0^t S_{\tilde{G}}(s) f\,\mathrm{d}s\right) \tag{4.7}$$

Dishonesty of a semigroup is manifested both in the time domain for semigroups, and at the level of resolvents. In the time domain, we introduce the defect function: for $f \in X_+$ and $t \geqslant 0$ we define it by:

$$\eta_f(t) = \|S_{\tilde{G}}(t)f\| - \|f\| + \int_0^t c\big(S_{\tilde{G}}(s)f\big)\,\mathrm{d}s \tag{4.8}$$

For resolvents, we have the following important result:

Theorem 4.2. *For any fixed $\lambda > 0$, there is $0 \leqslant \beta_\lambda \in X^*$ with $\|\beta_\lambda\| \leqslant 1$ such that:*

$$\lambda \|R(\lambda, \tilde{G})f\| = \|f\| - \langle \beta_\lambda, f\rangle - c\big(R(\lambda, \tilde{G})f\big) \tag{4.9}$$

Moreover, c extends to a nonnegative continuous linear functional on $D(\tilde{G})$, given again by (4.5).

The properties of η_f and its relation to β_λ are summarized in the proposition below.

Proposition 4.1. *The following holds:*

(*i*) *for any $f \in X_+$, η_f is a non-positive and a non-increasing function for $t \geqslant 0$;*

(*ii*)

$$\int_0^\infty e^{-\lambda t}\eta_f(t)\,dt = -\frac{1}{\lambda}\langle \beta_\lambda, f\rangle$$

hence $(S_{\tilde G}(t))_{t\geqslant 0}$ is honest if and only if $\beta_\lambda \equiv 0$ for any (some) $\lambda > 0$;

(*iii*) *if $(S_{\tilde G}(t))_{t\geqslant 0}$ is dishonest, then for some $f \in X_+$ and any $t > 0$:*

$$\left\| S_{\tilde G}(t) f\right\| < \|f\| + \int_0^t c\big(S_{\tilde G}(s)f\big)\,ds$$

An important characterization of honesty is given in the following theorem.

Theorem 4.3. *The following are equivalent:*

(a) *The semigroup $(S_{\tilde G}(t))_{t\geqslant 0}$ is honest;*

(b) $\tilde G = \overline{\tilde T + K}$*;*

(c) *For any $u \in R(\lambda, \tilde G)X_+$, where $\lambda > 0$ is arbitrary, we have:*

$$\int_\Omega \tilde G u\,d\mu \geqslant -c(u) \tag{4.10}$$

The problem with the characterization results given above is that they require the knowledge of the generator itself and therefore they are not immediately useful. To circumvent this problem, we shall be using certain extensions of the involved operators, that are defined below.

Define by E the set of measurable functions that are defined on Ω and take values in the extended set of real numbers and by E_f the subspace of E consisting of functions that are finite almost everywhere. E is a vector lattice with respect to the usual relation: $\leqslant$ almost everywhere, $X \subset \mathsf{E}_f \subset \mathsf{E}$ with X and E_f being sublattices of E.

In what follows, we shall denote by $\tilde{\mathcal{T}}$, $\mathcal{K}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{L}}_\lambda$ extensions of the operators $\tilde T$, K, $\tilde G$ and $R(\lambda, \tilde T)$, respectively. By $\mathcal{L}$ we abbreviate $\mathcal{L}_1$. At this moment, we shall require only that all the extensions have domains and ranges in E_f, that $\mathcal{K}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}_\lambda$ are positive operators on their domains and that $\tilde{\mathcal{G}} \subset \tilde{\mathcal{T}} + \mathcal{K}$.

We shall present here a theorem giving a sufficient condition for dishonesty in terms of these extensions.

Theorem 4.4. *Assume that there exists $u \in D(\tilde{\mathcal{G}})_+$ such that*

(*i*) *$[\tilde{\mathcal{L}}_\lambda(\lambda I - \tilde{\mathcal{T}})u](x) = u(x)$, a.e., for some $\lambda > 0$,*

(*ii*) *for some $\lambda > 0$, $\lambda u(x) - [\tilde{\mathcal{G}}u](x) = g(x) \in X_+$,*

(*iii*) *$c(u)$ is finite and*

$$\int_\Omega \tilde{\mathcal{G}} u\,d\mu < -c(u) \tag{4.11}$$

Then the semigroup $(S_{\tilde G}(t))_{t\geqslant 0}$ is dishonest.

5. Back to the growth–fragmentation equation

Let us look at the problem (2.1) from the point of view of the developed theory. Let us recall that we consider the operator K defined by the expression:

$$[Ku](x) = \int_x^\infty p(y)k(x,y)u(y,t)\,dy$$

on the domain $D(\tilde T)$. Firstly, by standard calculations, see [13 (Lemma 4.1)] we obtain that for any $u \in D(\tilde T)_+$:

$$\int_0^\infty (\tilde T u + Ku)x\,dx = -\int_0^\infty \big(\tilde b x - b(x)\big)u(x)\,dx - \int_0^\infty d(x)u(x)x\,dx \tag{5.1}$$

which, due to (2.2), shows that the assumptions (A1)–(A3) of Section 4 are satisfied. Hence, there is an extension $\tilde G$ of the operator $\tilde T + K$ that generates a substochastic semigroup $(S_{\tilde G}(t))_{t\geqslant 0}$. The relation of $(S_{\tilde G}(t))_{t\geqslant 0}$ and the solution to (2.1) is given in the next proposition.

Proposition 5.1. *There is an extension G of $T + K$ given by $(G, D(G)) = (\tilde G + \tilde b I, D(\tilde G))$ that generates* *a positive semigroup* $(S_G(t))_{t\geqslant 0} = (\mathrm{e}^{\tilde{b}t}S_{\tilde{G}}(t))_{t\geqslant 0}$. *Moreover, the generator* G *is characterized by:*

$$(\lambda I - G)^{-1}f = \sum_{n=0}^{\infty}(\lambda I - T)^{-1}\big[K(\lambda I - T)^{-1}\big]^n f \tag{5.2}$$

for $f \in X$ *and* $\lambda > \tilde{b}$.

Proof. The operator $\tilde{T}$ was constructed from T by subtracting the bounded operator $\tilde{b}I$. Let us consider the approximating semigroups $(S_r(t))_{t\geqslant 0}$, mentioned in Theorem 4.1. They are generated by $(T - \tilde{b}I + rK, D(T))$, $0 < r < 1$ and

$$\lim_{r\to 1^-} S_r(t)f = S_{\tilde{G}}(t)f \tag{5.3}$$

in X, uniformly in t on bounded intervals. Define semigroups $(S'_r(t))_{t\geqslant 0} = (\mathrm{e}^{\tilde{b}t}S_r(t))_{t\geqslant 0}$ generated by $T + rK$. As multiplication by $\mathrm{e}^{\tilde{b}t}$ does not affect convergence, we see in (5.3) that $(S'_r(t))_{t\geqslant 0}$ converges strongly to the semigroup $(S_G(t))_{t\geqslant 0} = (\mathrm{e}^{\tilde{b}t}S_{\tilde{G}}(t))_{t\geqslant 0}$ which is generated by $G = \tilde{G} + \tilde{b}I$ and thus is an extension of $T + K$, defined on the same domain as $\tilde{G}$, $D(G) = D(\tilde{G})$.

Formula (5.2) follows immediately from (4.3) by noting that since $\lambda I - G = (\lambda - \tilde{b})I - \tilde{G}$, we have $(\lambda I - G)^{-1} = (\lambda' I - \tilde{G})^{-1}$ for $\lambda > \tilde{b}$ and the same holds for the resolvent of T. □

Formula (5.1) for $T + K$ takes the form:

$$\int_0^\infty (Tu + Ku)x\,\mathrm{d}x = \int_0^\infty b(x)u(x)\,\mathrm{d}x - \int_0^\infty d(x)u(x)x\,\mathrm{d}x =: C(u) \tag{5.4}$$

and, as mentioned in Remark 4.1, we shall say that $(S_G(t))_{t\geqslant 0}$ is honest if and only if:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty u(t)x\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\| = C\big(u(t)\big) \tag{5.5}$$

Thus, all the results characterizing honesty and dishonesty can be applied to $(S_G(t))_{t\geqslant 0}$ with $-c(u)$ replaced by $C(u)$. In particular, $(S_G(t))_{t\geqslant 0}$ is honest if and only if $G = \overline{T + K}$, which in turn is equivalent to:

$$\int_0^\infty Gux\,\mathrm{d}x \geqslant C(u) \tag{5.6}$$

for any $u \in R(\lambda, G)X_+$, with $\lambda > \tilde{b}$.

To proceed, we have to specify the extensions of the operators which we will be working with. Possibly the most general choice is as follows. For $u \in D(\mathcal{T}) := \{u \in L_1([0,\infty), x\,\mathrm{d}x);\ bu \in \text{a.a.c.}\}$ we denote:

$$[\mathcal{T}u](x) = -\big(b(x)u(x)\big)_x - a(x)u(x) \tag{5.7}$$

where, as before $a(x) = d(x) + p(x)$; thus $\mathcal{T}: D(\mathcal{T}) \to \mathsf{E}_f$. By $\mathcal{K}$ we denote the operator defined by the expression:

$$[\mathcal{K}u](x) = \int_x^\infty p(y)k(x,y)u(y)\,\mathrm{d}y \tag{5.8}$$

defined on $D(\mathcal{K}) = \{u \in L_1([0,\infty), x\,\mathrm{d}x);\ x \to [\mathcal{K}u](x)$ is finite $a.e.\}$. Using these two concepts, we can define an operator that can be thought of as the maximal extension of $T + K$ in X:

$$[\mathcal{G}u](x) := [\mathcal{T}u](x) + [\mathcal{K}u](x) \tag{5.9}$$

defined on the domain $D(\mathcal{G}) = \{u \in D(\mathcal{T}) \cap D(\mathcal{K});\ x \to [\mathcal{G}u](x) \in L_1([0,\infty), x\,\mathrm{d}x)\}$. In a similar way, we consider the operator $\mathcal{L}_\lambda$ extending $R(\lambda, T)$, $\lambda > \tilde{b}$ defined by the expression:

$$[\mathcal{L}_\lambda f](x) := \frac{\mathrm{e}^{-\lambda B(x) - A(x)}}{b(x)}\int_0^x \mathrm{e}^{\lambda B(y) + A(y)} f(y)\,\mathrm{d}y \tag{5.10}$$

that is considered on $D(\mathcal{L}_\lambda) = \{f \in \mathsf{E};\ x \to [\mathcal{L}_\lambda f](x)$ is finite $a.e.\}$. Since the kernels of both $\mathcal{K}$ and $\mathcal{L}$ are nonnegative, the existence of the respective integrals is equivalent to the existence of the integrals of both the positive and negative parts of the integrands. It can be proved as in [10 (Lemma 4.1)] that $G \subset \mathcal{G}$, so that the extensions are defined correctly.

We illustrate the usefulness of the concept of extensions in the following observation.

Proposition 5.2. *Any function* $u \in D(G)$ *is continuous on* $(0, \infty)$.

Proof. Let first $f \in X_+$ and $\lambda > \tilde{b}$. Since $\lambda I - T$ extends to a positive integral operator $\mathcal{L}_\lambda$ on E, by (5.2) the element $\bar{u}_+ = (\lambda I - G)^{-1} f = \mathcal{L}_\lambda g$, where $g = \sum_{n=0}^{\infty} [K(\lambda I - T)^{-1}]^n f$, is a well-defined element of E as the series is increasing. However, as $\bar{u}_+ \in D(G) \subset X$, it must be finite almost everywhere. From (5.10) we have:

$$u(x) = \frac{e^{-\lambda B(x) - A(x)}}{b(x)} \int_0^x e^{\lambda B(y) + A(y)} g(y)\, dy$$

and as the functions B, A and b can have zeroes or singularities only at 0 and infinity, we see that $e^{\lambda B(y) + A(y)} g(y)$ is integrable over $[0, N]$ for any $N < +\infty$ and therefore u is continuous with a possible exception at $x = 0$. If we take now arbitrary u, we see that $u = \bar{u}_+ - \bar{u}_- = (\lambda I - G)^{-1} f_+ - (\lambda I - G)^{-1} f_-$, where f_+, f_- are the positive and negative parts of f. For $f_\pm$, the corresponding $g_\pm$ are also positive and hence $\bar{u}_\pm$ are continuous on $(0, N)$, which yields continuity of u. □

The following technical result can be proved as in [10 (Lemma 4.2)]

Lemma 5.1. *Let $\mathcal{K}$ and $\mathcal{L}_\lambda$ be the extensions introduced above. If for some $g \in D(\mathcal{L})_+$, both g and $\mathcal{K}\mathcal{L}_\lambda g$ belong to $L_1([\alpha, N], x\, dx)$, where $0 \leqslant \alpha < N \leqslant \infty$, then:*

$$\begin{aligned}
&\int_\alpha^N \big(-g(x) + [\mathcal{K}\mathcal{L}_\lambda g](x) + \lambda[\mathcal{L}_\lambda g](x)\big) x\, dx \\
&\quad = \alpha b(\alpha)[\mathcal{L}_\lambda g](\alpha) \\
&\qquad - \int_\alpha^N p(y)[\mathcal{L}_\lambda g](y) \left(\int_0^\alpha k(x, y) x\, dx \right) dy \\
&\qquad - N b(N)[\mathcal{L}_\lambda g](N) \\
&\qquad + \int_N^\infty p(y)[\mathcal{L}_\lambda g](y) \left(\int_\alpha^N k(x, y) x\, dx \right) dy \\
&\qquad + \int_\alpha^N b(x)[\mathcal{L}_\lambda g](x)\, dx - \int_\alpha^N d(x)[\mathcal{L}_\lambda g](x) x\, dx
\end{aligned} \tag{5.11}$$

A crucial rôle in the following considerations is played by the next theorem.

Theorem 5.1. *If $u \in D(G)$, then there are sequences $\alpha_k \to 0^+$ and $N_k \to \infty$ as $k \to \infty$ such that:*

$$\begin{aligned}
&\int_0^\infty [Gu](x) x\, dx \\
&\quad = \lim_{k\to\infty} \left(- \int_{\alpha_k}^{N_k} p(y) u(y) \left(\int_0^{\alpha_k} k(x, y) x\, dx \right) dy \right. \\
&\qquad \left. + \int_{N_k}^\infty p(y) u(y) \left(\int_{\alpha_k}^{N_k} k(x, y) x\, dx \right) dy \right) \\
&\qquad + \int_0^\infty b(x) u(x)\, dx - \int_0^\infty d(x) u(x) x\, dx
\end{aligned} \tag{5.12}$$

Proof. Using a similar argument to Proposition 5.2 we see that if $g \in \mathsf{E}_+$ is such that $\mathcal{L}_\lambda g \in X$, then $g \in L_1([\alpha, N], x\, dx)$ for any $0 < \alpha < N < \infty$. Following [10 (Corollary 4.1)], we observe that if $u = R(\lambda, G) f$, $f \in X_+$ there is $g \in \mathsf{E}_{f,+}$, constructed as in the proof of Proposition 5.2, such that $u = \mathcal{L}_\lambda g$ and:

$$Gu = \lambda \mathcal{L}_\lambda g - g + \mathcal{K}\mathcal{L}_\lambda g$$

and, as $g \in L_1([\alpha, N], x\, dx)$, we have $\mathcal{K}\mathcal{L}_\lambda g \in L_1([\alpha, N], x\, dx)$ and, by Lemma 5.1,

$$\begin{aligned}
&\int_0^\infty [Gu](x) x\, dx \\
&\quad = \lim_{k\to\infty} \left(\alpha_k b(\alpha_k) u(\alpha_k) - \int_{\alpha_k}^{N_k} p(y) u(y) \right. \\
&\qquad \times \left(\int_0^{\alpha_k} k(x, y) x\, dx \right) dy - N_k b(N_k) u(N_k) \\
&\qquad \left. + \int_{N_k}^\infty p(y) u(y) \left(\int_{\alpha_k}^{N_k} k(x, y) x\, dx \right) dy \right) \\
&\qquad + \int_0^\infty b(x) u(x)\, dx - \int_0^\infty x\, d(x) u(x)\, dx
\end{aligned}$$

for any sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ converging to 0 and ∞, respectively. This can be extended to arbitrary u using the decomposition of Proposition 5.2.

Since we know that $u \in L_1([0,\infty), x\,\mathrm{d}x) \cap C(0,\infty)$, we have $\liminf_{x\to\infty} x^2|u(x)| = 0$. Thus, there is a sequence $(N_k)_{k\in\mathbb{N}}$ converging to ∞ such that $\lim_{k\to\infty} N_k^2|u(N_k)| = 0$. Similarly, we obtain a sequence $(\alpha_k)_{k\in\mathbb{N}}$ that converges to 0 as $k\to\infty$, such that $\lim_{k\to\infty}\alpha_k^2|u(\alpha_k)| = 0$. Since $b(x) \leqslant \tilde{b}x$ for $x > 0$, we obtain the thesis. □

Theorem 5.2. *If*

$$\lim_{x\to 0^+} p(x) + d(x) < +\infty \tag{5.13}$$

then $G = \overline{T+K}$, *thus* $(S_G(t))_{t\geqslant 0}$ *is honest.*

Proof. As in the previous proof, it is enough to consider $u = R(\lambda, G)f$, $f \in X_+, \lambda > \tilde{b}$; for such f we have also $u = \mathcal{L}_\lambda g$ for some $g \in \mathsf{E}_+$. Since $u \in X$, by (5.10) and Tonelli's theorem, we obtain:

$$\int_0^\infty (\mathcal{L}_\lambda g)(x)x\,\mathrm{d}x = \int_0^\infty y g(y)\left(\frac{\mathrm{e}^{\lambda B(y)+A(y)}}{y}\int_y^\infty \frac{x\mathrm{e}^{-\lambda B(x)-A(x)}}{b(x)}\,\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty y g(y)\psi(y)\,\mathrm{d}y$$

The function $\psi(y)$ is continuous and non-negative, and the only points where it may be zero are at $y=0$ or as $y\to\infty$. As $y\to 0$, the integral term tends to infinity, see (3.8). Since a is bounded at 0, the other term tends to 0 by (3.3) and the l'Hospital rule gives:

$$\lim_{y\to 0^+}\psi(y) = \lim_{y\to 0^+}\frac{1}{-\frac{b(y)}{y}+\lambda+p(y)+d(y)} > 0$$

as $\lambda > \tilde{b}$ and $\lim_{y\to 0} b(y)/y = b'(0) \leqslant \tilde{b}$. Thus $g \in L([0,N], x\,\mathrm{d}x)$ for any $N < +\infty$, and we can put $\alpha = 0$ in (5.11), and thus in (5.12), getting:

$$\int_0^\infty [Gu](x)x\,\mathrm{d}x = \lim_{N\to+\infty}\left(\int_N^\infty p(y)u(y)\left(\int_0^N k(x,y)x\,\mathrm{d}x\right)\mathrm{d}y\right) + \int_0^\infty b(x)u(x)\,\mathrm{d}x - \int_0^\infty x d(x)u(x)\,\mathrm{d}x \geqslant C(u)$$

so that (5.6) is obviously satisfied. □

The theorem on dishonesty below is intended primarily as an example so that the regularity assumptions on the coefficients are not optimal. We shall also put $d\equiv 0$ as adding or subtracting a bounded operator does not change the domain of the generator; hence $a \equiv p$. Moreover, we restrict our attention to k given by (2.6): $k(x,y) = y^{-1}h(x/y)$ and satisfying:

$$-\int_0^1 z h(z)\ln z\,\mathrm{d}z < +\infty \tag{5.14}$$

Theorem 5.3. *Assume that* $b \in C^1([0,\infty))$ *with* $\inf_{0\leqslant x<\infty} b'(x) > -\infty$,

$$\frac{1}{xp(x)} \in L_1\big([0,\eta]\big), \qquad \frac{1}{x^k p(x)} \in L_1\big([N,\infty)\big) \tag{5.15}$$

for some $\eta, N, k > 0$, $p \in C^1((0,\infty)), p > 0$ *on* $(0,\infty)$ *and:*

$$\sup_{x\in[0,\infty)}\left|\frac{xp'(x)}{p(x)}\right| = L < +\infty \tag{5.16}$$

Then $(S_G(t))_{t\geqslant 0}$ *is dishonest.*

Proof. To simplify notation, we put $\eta = 1$. We use Theorem 4.4 so that we work with the operator extensions introduced at the beginning of this section and construct $u \in \mathcal{D}(\mathcal{G})_+$ satisfying the assumptions of this theorem. Let us define:

$$u(x) = \begin{cases} \frac{1}{x^2 p(x)} & \text{for } 0 < x < 1 \\ \frac{1}{x^{2+m}p(x)} & \text{for } x \geqslant 1 \end{cases} \tag{5.17}$$

where $m > 0$ and $m+1 \geqslant k$, see (5.15). Clearly $u \in X$ and it is continuous on $(0,\infty)$. Moreover, $pu \in L_1([N,\infty), x\,\mathrm{d}x)$ for any $N > 0$, and therefore we can pass to the limit with $N\to\infty$ in the integral terms on the right-hand side of (5.11) (taking into account that $\int_\alpha^N k(x,y)x\,\mathrm{d}x = \int_\alpha^y k(x,y)x\,\mathrm{d}x \leqslant y$). Thanks to the continuity, we can repeat the argument of Theorem 5.1

getting:

$$\int_0^\infty [\mathcal{G}u](x)x\,\mathrm{d}x = -\lim_{k\to\infty}\int_{\alpha_k}^\infty p(y)u(y)\left(\int_0^{\alpha_k} k(x,y)x\,\mathrm{d}x\right)\mathrm{d}y + \int_0^\infty b(x)u(x)\,\mathrm{d}x \tag{5.18}$$

for some $(\alpha_k)_{k\in\mathbb{N}}$ converging to zero, where we used the estimate (2.2) to pass to the limit in the last term.

Consider first the interval $(0,1]$ where we have $u(x) = 1/x^2 a(x)$. Using $k(x,y) = h(x/y)/y$, we have:

$$\int_\alpha^1 \left(\int_0^\alpha k(x,y)x\,\mathrm{d}x\right)\frac{1}{y^2}\,\mathrm{d}y = \int_\alpha^1\left(\int_0^r zh(z)\,\mathrm{d}z\right)\frac{1}{r}\,\mathrm{d}r$$

and since $\int_0^1(\int_0^r zh(z)\,\mathrm{d}z)\frac{1}{r}\,\mathrm{d}r = -\int_0^1 zh(z)\ln z\,\mathrm{d}z$, the Tonelli's theorem gives:

$$-\lim_{\alpha\to 0^+}\int_\alpha^1\left(\int_0^\alpha k(x,y)x\,\mathrm{d}x\right)a(y)u(y)\,\mathrm{d}y = \int_0^1 zh(z)\ln z\,\mathrm{d}z < 0$$

Furthermore, the integral:

$$\int_1^\infty p(y)u(y)\left(\int_0^\alpha k(x,y)x\,\mathrm{d}x\right)\mathrm{d}y$$

converges to zero as $pu \in L_1([1,\infty))$ and $\int_0^\alpha k(x,y)\cdot x\,\mathrm{d}x \leqslant y$, by Lebesgue's dominated convergence theorem. Thus, (5.18) shows that assumption (*iii*) of Theorem 4.4 is satisfied. Let us turn our attention to assumption (*ii*). Let us write:

$$\big(\lambda u(x) + \big(b(x)u(x)\big)'\big) + \left(p(x)u(x) - \int_x^\infty p(y)h\left(\frac{x}{y}\right)y^{-1}u(y)\,\mathrm{d}y\right) = I_1 + I_2$$

Consider first the interval $(0,1]$. We have:

$$I_1 = \frac{1}{x^2 p(x)}\left(\lambda + b'(x) - \frac{2b(x)}{x} - \frac{b(x)}{x}\frac{xp'(x)}{p(x)}\right) \tag{5.19}$$

and by assumption all the terms within the brackets are bounded on $[0,1]$ so that $I_1 > 0$ for sufficiently large λ. Moreover, $I_1 \in L_1([0,1], x\,\mathrm{d}x)$ by (5.15). Furthermore, (2.4) in our case reduces to $\int_0^1 zh(z)\,\mathrm{d}z = 1$ so that:

$$\frac{1}{x^2} = \frac{1}{x^2}\int_0^1 zh(z)\,\mathrm{d}z = \int_x^\infty \frac{1}{y^3}h\left(\frac{x}{y}\right)\mathrm{d}y \geqslant \int_x^1 \frac{1}{y^3}h\left(\frac{x}{y}\right)\mathrm{d}y + \int_1^\infty \frac{1}{y^{3+m}}h\left(\frac{x}{y}\right)\mathrm{d}y$$

for $m \geqslant 0$. Hence

$$0 \leqslant \frac{1}{x^2} - \int_x^1 \frac{1}{y^3}h\left(\frac{x}{y}\right)\mathrm{d}y - \int_x^\infty \frac{1}{y^{3+m}}h\left(\frac{x}{y}\right)\mathrm{d}y = I_2 \leqslant \frac{1}{x^2} - \int_x^1 \frac{1}{y^3}h\left(\frac{x}{y}\right)\mathrm{d}y = \frac{1}{x^2}\int_0^x zh(z)\,\mathrm{d}z$$

which is integrable on $[0,1]$ with respect to $x\,\mathrm{d}x$ by (5.14).

For $x \in [1,\infty)$ we have similarly to (5.19)

$$I_1 = \frac{1}{x^{2+m}p(x)} \times \left(\lambda + b'(x) - \frac{(2+m)b(x)}{x} - \frac{b(x)}{x}\frac{xp'(x)}{p(x)}\right)$$

which is positive and integrable on $[1,\infty)$ with respect to $x\,\mathrm{d}x$, possibly with larger λ. For I_2 we have:

$$I_2 = \frac{1}{x^{2+m}} - \int_x^\infty \frac{1}{y^{3+m}}h\left(\frac{x}{y}\right)\mathrm{d}y = \frac{1}{x^{2+m}} - \frac{1}{x^{2+m}}\int_0^1 z^{1+m}h(z)\,\mathrm{d}z \geqslant \frac{1}{x^{2+m}}\left(1 - \int_0^1 zh(z)\,\mathrm{d}z\right) = 0$$

and clearly, as $m > 0$,

$$0 \leqslant I_2 \leqslant \frac{1}{x^{2+m}}\left(1 - \int_0^1 z^{1+m} h(z)\,dz\right) \in L_1\big([1,\infty), x\,dx\big)$$

It remains to prove (*i*). Integrating by parts, we get:

$$\begin{aligned}
&\big[\mathcal{L}_\lambda((bu)')\big](x) \\
&\quad = \frac{e^{-\lambda B(x)-A(x)}}{b(x)} \int_0^x e^{\lambda B(y)+A(y)}\big(b(y)u(y)\big)'\,dy \\
&\quad = u(x) - \frac{e^{-\lambda B(x)-A(x)}}{b(x)} \lim_{y\to 0^+} b(y)e^{\lambda B(y)+A(y)}u(y) \\
&\qquad - \frac{e^{-\lambda B(x)-A(x)}}{b(x)} \int_0^x e^{\lambda B(y)+A(y)} \\
&\qquad \times \big(\lambda + p(y)\big)u(y)\,dy
\end{aligned}$$

Since close to zero $e^{\lambda B(x)+A(x)} \leqslant x^{\lambda/\tilde{b}}$, with $\lambda > \tilde{b}$, and both $(bu)'$ and $(\lambda + p)u$ behave as $1/x^2 p(x)$ and $1/x^2$, respectively, we see that both integrals, and hence the limit, exist. Since $1/xp(x)$ is integrable and differentiable except at 0, we can prove as in Theorem 5.1 that there is a sequence $(x_n)_{n\in\mathbb{N}}$ converging to zero such that $1/p(x_n) \to 0$. Hence, using this sequence we have:

$$b(x_n)e^{\lambda B(x_n)+A(x_n)}u(x_n) \leqslant \frac{\tilde{b}x_n x_n^{\lambda/\tilde{b}}}{x_n^2 p(x_n)} = \tilde{b}x_n^{\lambda/\tilde{b}-1}\frac{1}{p(x_n)} \to 0$$

and thus u satisfies assumption (*i*). □

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