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Mathematical model for optimal control in wastewater discharges: the global performance

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Abstract

In this work we show how mathematical models and optimal control techniques can help us to solve some problems of environmental engineering, more precisely, water pollution problems arising from wastewater discharges into coastal areas or rivers. We deal with a complete two-dimensional mathematical model for the evolution of pollutant concentration in a shallow water domain. By integrating this model we obtain a zero-dimensional one and we use it to study the global performance of the system in a realistic situation. Finally, by using the two-dimensional model, we recall two optimal control problems related to the wastewater disposal problem. ***To cite this article: L.J. Alvarez-Vázquez et al., C. R. Biologies 328 (2005).***

Résumé

Un modèle mathématique pour un contrôle optimal des décharges d'eaux usées : la performance globale. On considère des modèles mathématiques et des techniques de contrôle optimisées pour l'étude de quelques problèmes environnementaux et, plus précisément, des problèmes de pollution de la côte et des rivières par les eaux résiduelles. Nous proposons un modèle mathématique bidimensionnel de l'évolution de la concentration en polluants dans un domaine d'eaux peu profondes. Par l'intégration de ce modèle, on obtient un modèle adimensionnel, que nous utilisons pour l'étude de la performance globale du système dans une situation réaliste. Enfin, en utilisant des modèles bidimensionnels, on étudie deux problèmes de contrôle optimal liés à la gestion des eaux d'égout. ***Pour citer cet article : L.J. Alvarez-Vázquez et al., C. R. Biologies 328 (2005).***

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1. Introduction

Usually, wastewater originated from urban areas or industry undergoes a physical/chemical/biological treatment in a purifying plant. From there, wastewater is discharged through a submarine outfall into an aquatic media like a lake, a river or a coastal area, at a suitable distance from protected areas like beaches, fisheries or marine recreation zones. When several purifying plants are going to discharge wastewater into the same domain, the problem of design and management of the whole treatment system arises. To solve this problem, optimal control theory and optimization methods can be very useful: they can help decision makers in formulating rational policies in order to minimize costs while keeping the prescribed levels of water quality. Sections 4 and 5 are devoted to present two typical examples: the optimal management of a wastewater treatment system and the optimal location of the submarine outfalls.

Obviously, the knowledge of mathematical models for the evolution of pollutant concentration is a unavoidable first step if one wants to use optimal control techniques. So, the first part of this work is devoted to study in detail one of them, related with *Biochemical Oxygen Demand* (BOD) and *Dissolved Oxygen* (DO), which is frequently used in the case of domestic discharges. In Section 2 we deal with the complete two-dimensional model consisting of two uncoupled system of partial differential equations which give us the height of water, the depth-averaged horizontal velocity of water and the depth-averaged concentration of BOD and DO. In Section 3 we obtain a new zero-dimensional model, by integrating the previous one, which can be very useful to study the global performance of the system. We apply this model in the *ría* of Arousa (Spain) to obtain the global concentration of BOD and DO along the time.

2. A complete two-dimensional model for *Biochemical Oxygen Demand* and *Dissolved Oxygen*

We consider a domain Ω occupied by shallow water (as can be a *ría* or an estuary), where polluting wastewater are discharged through N_E submarine outfalls located at points $b_j \in \Omega$ and connected to their respective purifying plants located at points $a_j \in \Gamma$ (see Fig. 1). Moreover, we assume the existence of several areas $A_i \subset \Omega$, $i = 1, \ldots, N_\mathrm{Z}$, representing fisheries, beaches or marine recreation zones where it is necessary to guarantee the water quality with pollution levels lower than some allowed threshold levels.

The flow of an effluent from a submarine outfall in a shallow water domain is mainly governed by horizontal transport due to currents (produced by tides, wind ...) and turbulent diffusion. It allows uncouple hydrodynamical equations and transport equations. The former give the height and the velocity field which are used in latter to obtain the pollution levels.

2.1. Hydrodynamic model: the shallow water equations

In this section we recall the shallow water equations which constitute an useful mathematical model for hydrodynamic flows in shallow regions. If we consider Γ, the boundary of Ω, divided into three parts: Γ^- (corresponding to the effluent), Γ^+ (corresponding to open sea) and Γ^0 (corresponding to the cost), then the shallow water equations can be written in the

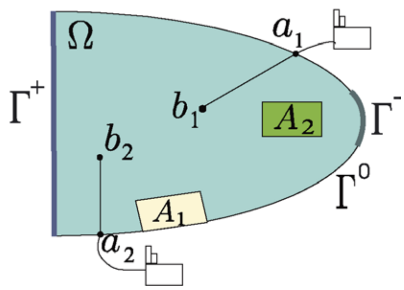

Fig. 1. Domain Ω.

following way:

$$\left.\begin{array}{ll}
\dfrac{\partial h}{\partial t}+\vec{\nabla}\cdot(h\vec{u})=0 & \text{in } \Omega\times(0,T)\\
\dfrac{\partial \vec{u}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{u}-\nu\Delta\vec{u}+g\vec{\nabla}h=\vec{F} & \\
 & \text{in } \Omega\times(0,T)\\
h=\eta & \text{on } \Gamma^{-}\times(0,T)\\
h=\phi & \text{on } \Gamma^{+}\times(0,T)\\
h(0)=h_0 & \text{in } \Omega\\
\vec{u}\cdot\vec{n}=q & \text{on } \Gamma^{-}\times(0,T)\\
\vec{u}\cdot\vec{n}=0 & \text{on } \Gamma^{0}\times(0,T)\\
\vec{\nabla}\cdot\vec{u}=0 & \text{on } \Gamma^{+}\times(0,T)\\
\vec{u}(0)=\vec{u}_0 & \text{in } \Omega
\end{array}\right\} \quad (1)$$

where $h(x,t)$ and $\vec{u}(x,t)$ denote, respectively, the height of water and the depth-averaged horizontal velocity of water, η, ϕ, h_0, q and $\vec{u}_0$ are given functions, ν (kinetic eddy viscosity coefficient) and g (gravity acceleration) are physical parameters experimentally known, $\vec{n}$ denotes the unit outer normal vector to boundary Γ and the second member $\vec{F}$ collects all the effects of atmospheric pressure, wind stress, bottom friction and so on.

There are several works related to the existence, uniqueness and regularity of solution of (1) in particular cases (see, for instance, [1–5]) but in the general case it is still an open problem. A numerical approximation of h and $\vec{u}$ in $\Omega\times(0,T)$ can be obtained by finite difference, finite element or finite volume methods (see, for instance, [6–9]) in order to be employed in the pollutant dispersion model.

2.2. Pollutant dispersion: the BOD-DO model

Firstly, in order to simulate the water quality in Ω, we have to choose some indicators of pollution levels. Two of the most important (especially in the case of domestic discharges) are the *Dissolved Oxygen* (DO) and the organic matter, which can be measured in terms of the need of oxygen to decompose it, the so-called *Biochemical Oxygen Demand* (BOD). If the pollution level is not too high the BOD can be satisfied by the DO. However, if the organic matter increases beyond a maximum value the DO is not enough for its decomposition, leading to important modifications (anaerobic processes) in the ecosystem. To avoid them a threshold value of BOD may not be exceeded and a minimum level of DO must be guaranteed.

The evolution of the BOD and the DO in the domain $\Omega\subset\mathbb{R}^2$ is governed by a system of partial differential equations (cf. [10]). Let us denote by $\rho_1(x,t)$ and $\rho_2(x,t)$ the concentrations of BOD and DO at point $x\in\Omega$ and at time $t\in[0,T]$, respectively. Then these concentrations are obtained as the solution of the following two initial-boundary value problems:

$$\left.\begin{array}{ll}
\dfrac{\partial \rho_1}{\partial t}+\vec{u}\cdot\vec{\nabla}\rho_1-\beta_1\Delta\rho_1 & \\
\quad=-\kappa_1\rho_1+\dfrac{1}{h}\displaystyle\sum_{j=1}^{N_{\mathrm{E}}}m_j\delta(x-b_j) & \text{in } \Omega\times(0,T)\\
\dfrac{\partial \rho_1}{\partial n}=0 & \text{on } \Gamma\times(0,T)\\
\rho_1(x,0)=\rho_{10}(x) & \text{in } \Omega\\
\dfrac{\partial \rho_2}{\partial t}+\vec{u}\cdot\vec{\nabla}\rho_2-\beta_2\Delta\rho_2 & \\
\quad=-\kappa_1\rho_1+\dfrac{1}{h}\kappa_2(d_{\mathrm{s}}-\rho_2) & \text{in } \Omega\times(0,T)\\
\dfrac{\partial \rho_2}{\partial n}=0 & \text{on } \Gamma\times(0,T)\\
\rho_2(x,0)=\rho_{20}(x) & \text{in } \Omega
\end{array}\right\} \quad (2)$$

where $\vec{u}$ and h are given by (1), $m_j(t)$ is the mass flow rate of BOD discharged at point b_j, $\delta(x-b_j)$ denotes the Dirac measure located at b_j and positive parameters β_1 and β_2 (horizontal viscosity coefficients), κ_1, κ_2 (kinetic coefficients related to BOD elimination and oxygen transfer through the surface, respectively) and d_{s} (oxygen saturation density) can be obtained from experimental measurements.

For this system we can prove the existence and uniqueness of solution (cf. [11]). A numerical solution for (2) can be obtained by a method which combines characteristics for time discretization with finite elements for space discretization (cf. [12]).

3. The global performance: a zero-dimensional model

The main aim in this paper is to study the global performance of the BOD-DO model in a *ría*, an estuary or a lake. In order to do it we integrate the system (2) in the region occupied by water. Thus, we obtain a new simple zero-dimensional model.

Let $M_1(t)$ and $M_2(t)$ be, respectively, the total mass of BOD and DO in the region at time t, that is to say,

$$M_1(t) = \int_\Omega h(x,t)\rho_1(x,t)\,\mathrm{d}x$$
$$M_2(t) = \int_\Omega h(x,t)\rho_2(x,t)\,\mathrm{d}x$$

Then, by integrating in Ω Eq. (2) in their conservative form and applying the Green's formula, we obtain

$$\begin{cases} \dfrac{\mathrm{d}M_1}{\mathrm{d}t} = -\kappa_1 M_1 + Q + F_1 \\ \dfrac{\mathrm{d}M_2}{\mathrm{d}t} = -\kappa_1 M_1 + A\kappa_2\left(d_\mathrm{s} - \dfrac{M_2}{V}\right) + F_2 \end{cases} \tag{3}$$

where A is the area of water surface in contact with air, V is the whole water volume, Q denotes the mass flow rate of BOD discharged in the region (that is, $Q = \sum_{j=1}^{N_\mathrm{E}} m_j$), and F_1 and F_2 are, respectively, the mass flow rate of BOD and DO across the boundary ($F_i = \int_\Gamma h\rho_i \vec{u}\vec{n}\,\mathrm{d}\sigma$, for $i = 1, 2$).

If we study a stationary region (as can be a closed lake) A and V are constant along the time. In this case, if we also suppose that Q, F_1 and F_2 are constant, the exact solution of (3) is given by:

$$\begin{aligned} M_1(t) &= M_1(0)\mathrm{e}^{-\kappa_1 t} + \frac{Q+F_1}{\kappa_1}(1-\mathrm{e}^{-\kappa_1 t}) \\ M_2(t) &= M_2(0)\mathrm{e}^{-at} \\ &\quad + \frac{Q+F_1-\kappa_1 M_1(0)}{a-\kappa_1}(\mathrm{e}^{-\kappa_1 t} - \mathrm{e}^{-at}) \\ &\quad + \frac{b-Q-F_1}{a}(1-\mathrm{e}^{-at}) \end{aligned} \tag{4}$$

where $a = \kappa_2 \frac{A}{V}$ and $b = \kappa_2 d_\mathrm{s} A + F_2$.

In order to develop an asymptotic study, for $t \gg \frac{1}{\kappa_1}$ and $t \gg \frac{1}{a}$, the terms with $\mathrm{e}^{-\kappa_1 t}$ and e^{-at} can be negligible in (4) and the asymptotic values of BOD and DO are $M_1(\infty) = (Q + F_1)/\kappa_1$ and $M_2(\infty) = (b - Q - F_1)/a$. However, it is as well to point out that, in a shallow water domain, the typical values of κ_1 and a are, respectively, of the order of 10^{-5} and $10^{-13}\ \mathrm{s}^{-1}$, and then, the previous asymptotic values would be reached 300,000 years later!

For $\frac{1}{\kappa_1} \ll t \ll \frac{1}{a}$ (only about some days later), we can neglect $\mathrm{e}^{-\kappa_1 t}$ and approximate $\mathrm{e}^{-at} \simeq 1 - at$. Then we obtain an asymptotic value for the BOD and a linear approximation for the DO:

$$\begin{aligned} M_1(t) &\simeq \frac{Q+F_1}{\kappa_1} \\ M_2(t) &\simeq \left(M_2(0) + \frac{Q+F_1-\kappa_1 M_1(0)}{\kappa_1 - a}\right) \\ &\quad + \Bigg(b - Q - F_1 \\ &\quad - a\left(M_2(0) + \frac{Q+F_1-\kappa_1 M_1(0)}{\kappa_1 - a}\right)\Bigg)t \end{aligned}$$

When we are working in an open region (as can be a *ría* or an estuary), we must take into account the water renewal. For example, in a *ría*, we can assume that inflow water is more pure than outflow water (in the sense that DO concentration is greater and BOD concentration is lesser) and, moreover, we know that DO concentration in inflow water is always lesser than oxygen saturation density (d_s). In this case, if $\phi(t)$ denotes the height over a fixed level (half tide) at the mouth of the *ría* and $\tilde{V}$ denotes the water volume corresponding to half tide, then, by geometrical reasons, we have

$$V(t) = A\phi(t) + \tilde{V}$$

Moreover, taking into account the purification of the water in the *ría* because of the seawater renewal, we obtain the following expressions for the mass flow rates across the boundary:

$$\begin{aligned} F_1(t) &= \phi'(t)A\frac{\alpha M_1(t)}{V(t)} \\ F_2(t) &= \phi'(t)AC(t) \end{aligned}$$

where $C(t) = \min\{\frac{\beta M_2(t)}{V(t)}, d_\mathrm{s}\}$, and α and β are two experimental parameters verifying $\alpha = \beta = 1$ if $\phi'(t) < 0$, $\alpha < 1 < \beta$ if $\phi'(t) > 0$. According to this, the system (3) is rewritten as follows:

$$\begin{cases} \dfrac{\mathrm{d}M_1(t)}{\mathrm{d}t} = -\kappa_1 M_1(t) + Q + \phi'(t)A\dfrac{\alpha M_1(t)}{V(t)} \\ \dfrac{\mathrm{d}M_2(t)}{\mathrm{d}t} = -\kappa_1 M_1(t) + A\kappa_2\left(d_\mathrm{s} - \dfrac{M_2(t)}{V(t)}\right) \\ \qquad\qquad + \phi'(t)AC(t) \end{cases} \tag{5}$$

This system can be easily solved by finite difference methods. We have used the backward Euler scheme to solve it for data corresponding to the *ría* of

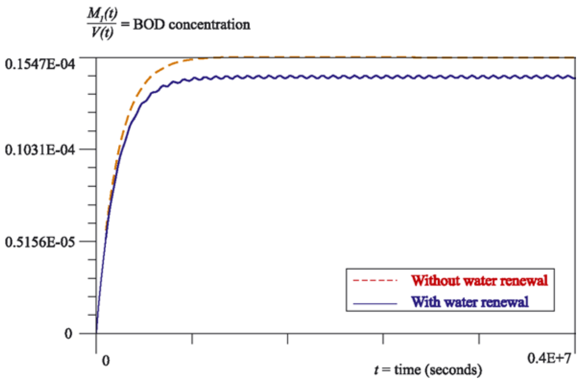

Fig. 2. BOD concentration in the *ría* of Arousa (Spain) along the time.

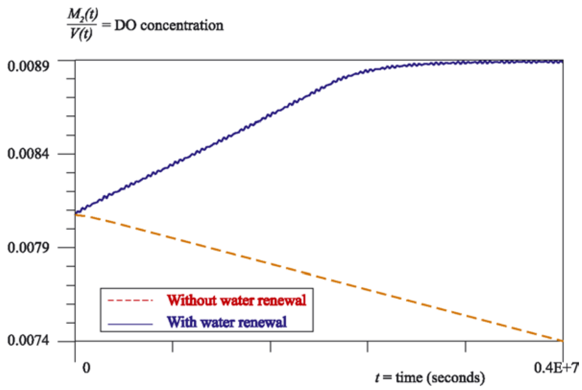

Fig. 3. DO concentration in the *ría* of Arousa (Spain) along the time.

Arousa (Spain). Moreover, in order to study the weight of the water renewal in the *ría*, we have compared the result with the exact solution of (3) with $V(t) = V$ and $F_1 = F_2 = 0$ (in this case the exact solution is given by (4)). As one can see in Figs. 2 and 3, for data given in Table 1, the BOD concentration reaches an asymptotic value in both cases, but the DO concentration observes a linear decrease without water renewal and reaches an asymptotic value (very close to saturation) when the water renewal is considered.

4. Problem 1: Optimal management of a wastewater treatment system

In this section we pose an optimal control problem related to the management of a wastewater treatment system. We recall the situation is Section 2: we consider a domain occupied by shallow water where polluting wastewater is discharged through N_E submarine outfalls (in this section we suppose that these outfalls are located at fixed points $b_j \in \Omega$). Moreover, there exist several *sensitive areas*, $A_i \in \Omega$, in the domain, where it is necessary to guarantee the water quality with pollution levels lower than an allowed threshold. We suppose that wastewater arrives to the purifying plants with a certain BOD concentration. Before discharging it into the sea, its BOD concentration can be reduced in the plants by different biological or biochemical treatments. From the ecological point of view the depuration in each plant must be as high as possible but, from the economical point of view, there is a cost proportional to the developed

Table 1
Data for models (3) and (5) corresponding to the *ría* of Arousa (Spain)

– Total time	$T = 4 \cdot 10^6$ s
– Initial mass of BOD	$M_1(0) = 0$ kg
– Initial mass of DO	$M_2(0) = 1.16311 \cdot 10^7$ kg
– Mass flow rate of BOD discharged in the region	$Q = 0.256$ kg s^{-1}
– Oxygen saturation density	$d_s = 8.98 \cdot 10^{-3}$ kg m^{-3}
– Kinetic coefficients κ_1 and κ_2	$\kappa_1 = 1.15 \cdot 10^{-5}$ s^{-1}
	$\kappa_2 = 9 \cdot 10^{-12}$ m s^{-1}
– Area of water surface in contact with air	$A = 6.77399 \cdot 10^7$ m^2
– Water volume corresponding to half tide	$V = 1.43897 \cdot 10^9$ m^3
Exclusive data for model (5)	
– Height over a fixed level at the mouth of the *ría*	$\phi(t) = 1.4\left(\sin\left(\frac{8\pi t}{178560 - \pi}\right) + 1\right)$
– Parameters α and β if $\phi'(t) < 0$	$\alpha = 1,\ \beta = 1$
– Parameters α and β if $\phi'(t) > 0$	$\alpha = 0.95,\ \beta = 1.02$

depuration. Then, the optimal management problem is determining the depuration at each plant along the time, in such a way that the global depuration cost is minimized and the above constraints on the water quality are satisfied.

4.1. Mathematical formulation

In order to formulate this problem we need to take into account some issues. Firstly, if $\overline{m}_j$ denotes the BOD of wastewater arriving to the jth plant and $\underline{m}_j$ is the BOD corresponding to the maximum depuration at that plant, then determining the depuration at the jth plant is equivalent to finding the mass flow rate of BOD, $m_j(t)$, discharged through the corresponding outfall. We assume that they satisfy the constraints

$$\underline{m} \leqslant m_j(t) \leqslant \overline{m}, \quad j = 1, 2, \ldots, N_{\mathrm{E}} \tag{6}$$

Secondly, if we take BOD and DO as indicators of the water quality, then the environmental constraints on it can be written as follows (see Section 2):

$$\begin{cases} \rho_1|_{A_i} \leqslant \sigma & i = 1, \ldots, N_{\mathrm{Z}} \\ \rho_2|_{A_i} \geqslant \zeta & i = 1, \ldots, N_{\mathrm{Z}} \end{cases} \tag{7}$$

where σ and ζ are, respectively, critical levels for BOD and DO.

Finally, we suppose that the cost of the depuration process at the jth plant is known and it is a strictly convex C^2-function of the BOD discharged through the corresponding outfall. Hence, if f_j denotes the cost function at jth plant, the cost of the whole depuration is given by,

$$J_1(m) = \sum_{j=1}^{N_{\mathrm{E}}} \int_0^T f_j\big(m_j(t)\big)\,\mathrm{d}t \tag{8}$$

According to this, the optimal management problem $(\mathcal{P}_1)$ consists of finding the functions $m_j(t)$, $j = 1, \ldots, N_{\mathrm{E}}$, minimizing the cost function (8) in such a way that the corresponding state of the system given by (2) satisfies the constraints (6) and (7).

This is an optimal control problem with pointwise state constraints and with pointwise control. We can obtain the existence and uniqueness of solution and an optimality system in order to characterize it (see [11] for further details).

4.2. Numerical resolution

The numerical solution of the optimal control problem $(\mathcal{P}_1)$ requires a discretization of the state system (2). Moreover, in practice, because of the particular shape of the functions f_j, the constraints $m_j(t) \leqslant \overline{m}$ can be suppressed. Thus, we are led to consider the discretized constraints:

$$g_1 : m \to g_1(m) = (\rho_1 - \sigma,\ \zeta - \rho_2)$$
$$g_2 : m \to g_2(m) = \underline{m} - m$$

Then, the optimal control problem $(\mathcal{P}_1)$ can be written as follows,

$$(\mathcal{P}_{1\mathrm{D}}) \begin{cases} \min J_1(m) \\ \text{such that } g_i(m) \leqslant 0, \quad i = 1, 2 \end{cases}$$

Now this problem can be solved by different numerical methods (we refer to [13] for a numerical resolution of $(\mathcal{P}_{1\mathrm{D}})$ by using a sequential quadratic programming algorithm and an admissible points method).

4.3. Numerical results

In this work we resolve this problem in a realistic situation posed in the *ría* of A Coruña (Spain). We consider one purifying plant and one protected area (see Fig. 4). We assume pollutant concentration of the wastewater arriving to the purifying plant is 150 $\mathrm{kg\,m^{-3}}$ (then the depuration cost above this value is constant and equal to 100) and we suppose that the complete depuration is not possible. Thus, we take the following cost function

$$f_1(x) = \begin{cases} \dfrac{100 \times 150^3}{x^3 - 3 \times 150x^2 + 3 \times 150^2 x} & \text{if } x \leqslant 150 \\ 100 & \text{if } x \geqslant 150 \end{cases}$$

We fix a maximum value for BOD ($\sigma_1 = 5.2416 \cdot 10^{-4}$ $\mathrm{kg\ m^{-3}}$) and a minimum value for DO ($\zeta_1 = 8.03891 \cdot 10^{-3}$ $\mathrm{kg\,m^{-3}}$) in the protected area and we resolve the problem $(\mathcal{P}_1)$ in order to find the optimal discharge during a complete tidal cycle ($T = 12.4$ h). The result can be seen in Fig. 5. We observe that the worst moment to discharge is low tide ($t = 0$); from this moment, the depuration can be less and less strong until after hide tide when the BOD discharge must be again to decrease (for this discharge, the fulfillment of the BOD constraints in the protected area at the last point of time can be seen in Fig. 4).

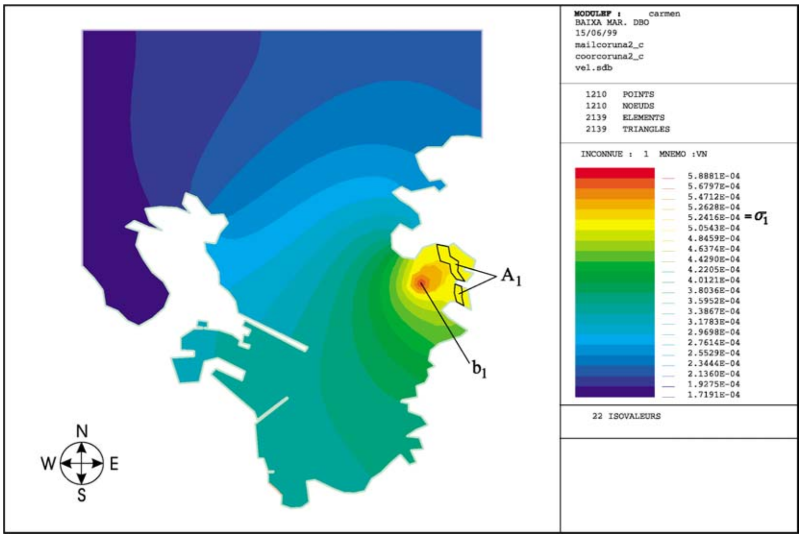

Fig. 4. BOD concentration at the end of the tidal cycle, for the discharge in Fig. 5 (*ría* of A Coruña).

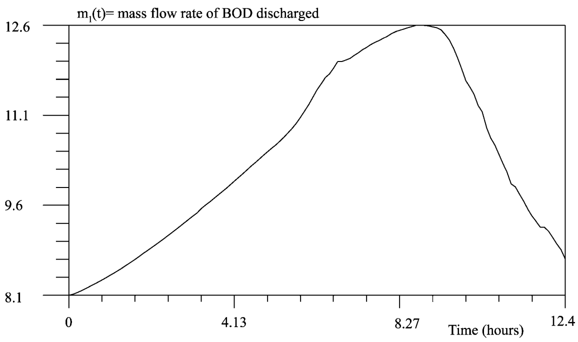

Fig. 5. Optimal discharge during a complete tidal cycle (*ría* of A Coruña).

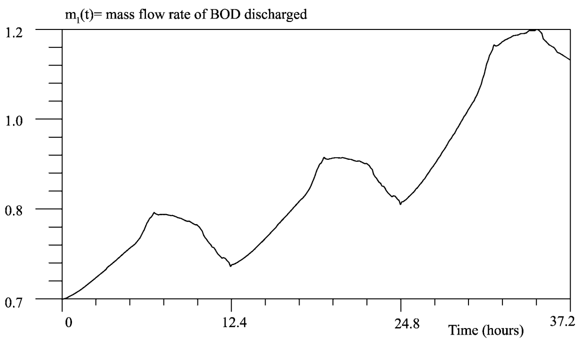

Fig. 6. Optimal discharge during three tidal cycles (*ría* of A Coruña).

In order to make sure that the previous interpretation is correct we have found the optimal discharge during several tidal cycles. In Fig. 6, we show the optimal discharge during three cycles. In effect, we observe some periodicity, however, because of BOD accumulation, the discharge is greater in the next cycle than in the previous one. The DO concentration at the last point of time, when the saturation of the constraints takes place, can be seen in Fig. 7.

5. Problem 2: Optimal location of wastewater outfalls

The second problem is connected with the optimal design of a wastewater treatment system. Particularly, it consists of finding the optimal location of the submarine outfalls.

We consider a similar situation to the previous one: a domain occupied by shallow water where we are going to discharge polluting wastewater through submarine outfalls and where there exist several *sensitive areas* in which we have to guarantee the water quality in terms of BOD and DO. Moreover, we also suppose that there are N_E purifying plants (located at points $a_j \in \Gamma$) but, unlike the previous problem, we now assume that the depuration in every plant is fixed (the functions $m_j(t)$ are known beforehand) and our goal is to determine the points b_j where wastewater will be discharged. These points must be determined in order to minimize the construction cost of the submarine

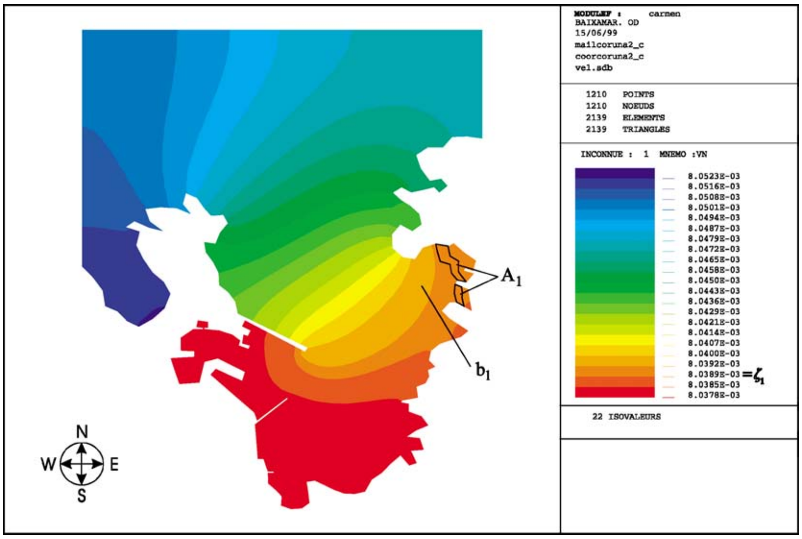

Fig. 7. DO concentration at the end of the three cycles, for the discharge in Fig. 6 (*ría* of A Coruña).

outfalls while guaranteeing the water quality at the protected areas.

5.1. Mathematical formulation

The constraints on the water quality are given in (7). Moreover, taking into account technological limitations, the jth outfall must be placed in a suitable region U_j, where $U_j \subset \Omega \setminus \bigcup_{i=1}^{N_Z} \bar{A}_i$ is a compact convex polyhedral set representing all the admissible points where outfalls can be located. Thus, the optimal locations must verify $b_j \in U_j,\ \forall j = 1, \ldots, N_E$. If we define $U_{ad} = \prod_{j=1}^{N_E} U_j$, this constraint can be written in the simpler way

$$b \in U_{ad} \tag{9}$$

Finally, we suppose that the construction cost of the jth outfall depends on the distance between the purifying plant (located at point a_j) and the point of discharge, $b_j \in \Omega$. Hence we consider that the global cost of the system is given by

$$J_2(b) = \sum_{j=1}^{N_E} \frac{1}{2} \|b_j - a_j\|^2 \tag{10}$$

Then the problem of optimal design, denoted by $(\mathcal{P}_2)$, consists of finding the points $b_j,\ j = 1, \ldots, N_E$ minimizing the cost function (10) under the constraints (7) and (9).

This is a control problem with quadratic cost but with nonconvex pointwise state constraints which makes difficult its analysis and resolution. A complete theoretical analysis of this problem can be seen in [14].

5.2. Numerical resolution

Now, in order to solve the problem $(\mathcal{P}_2)$, we introduce a discretization of the control problem in the same way that the previous problem. Firstly, the function collecting the discretized state constraints is denoted by $\bar{g}_1$,

$$\bar{g}_1 : b \to \bar{g}_1(b) = (\rho_1 - \sigma,\ \zeta - \rho_2)$$

Secondly, we define a function $\bar{g}_2$ collecting all the linear constraints on the control which corresponds to the characterization of U_{ad}, i.e., $\bar{g}_2$ is such that $b \in U_{ad} \Leftrightarrow \bar{g}_2(b) \leqslant 0$.

Then the optimal control problem $(\mathcal{P}_2)$ is approximated by the following *discrete* optimization problem,

$$(\mathcal{P}_{2D}) \begin{cases} \min J_2(b) \\ \text{such that } \bar{g}_i(b) \leqslant 0, \quad i = 1, 2 \end{cases}$$

In [15], three different algorithms are used to obtain the numerical solution of $(\mathcal{P}_{2D})$ namely an admissible points algorithm, the Nelder–Mead simplex method and a duality method. As we can see in that paper, due to the geometric nature of the problem, the three

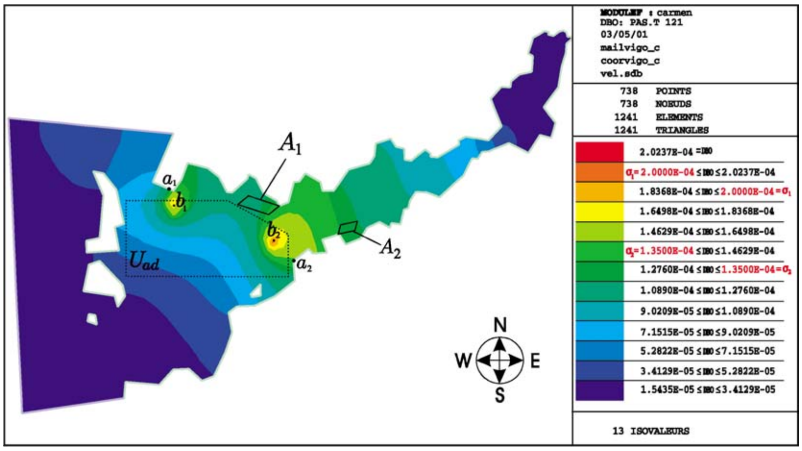

Fig. 8. Optimal BOD concentration (hight tide, *ría* of Vigo).

algorithms present a good performance, specially the Nelder–Mead method.

5.3. Numerical results

In this section we present the numerical results obtained when solving the problem $(\mathcal{P}_2)$ for a realistic situation posed in the *ría* of Vigo (Spain) during a complete tidal cycle. We have taken two purifying plants, located near the coast at points $a_1 = (0, 11000)$ and $a_2 = (6630, 7200)$, and two protected areas (see Fig. 8).

The state constraints for both protected areas corresponds to $\sigma_1 = 0.0002$, $\sigma_2 = 0.000135$, $\zeta_1 = 0.008067$, $\zeta_2 = 0.0000805$. The admissible set U_{ad} and the optimal locations $b_1 = (69, 10224)$ and $b_2 = (5535, 8278)$, given by the Nelder–Mead method can be seen in Fig. 8. Moreover, this figure shows the BOD concentration at high tide, at the end of the tidal cycle that we have simulated.

6. Conclusions

In this work, mathematical modelling and optimal control theory have been successfully applied to an interesting ecological problem: the management and design of a wastewater treatment system for coastal areas. In the last sections of the paper we present numerical results for two different cases related to this ecological problem.

However, the main contribution of this paper is the obtaining of a zero-dimensional model, which can be used in the study of the global performance of the system; we present an application of this simple model in a realistic situation posed in the *ría* of Arousa (Spain).

Obtained results indicate the good performance of our models and show how mathematical tools can be very useful in the study of a wide range of ecological problems.

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