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Approximation for limit cycles and their isochrons

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Abstract

Local analysis of trajectories of dynamical systems near an attractive periodic orbit displays the notion of asymptotic phase and isochrons. These notions are quite useful in applications to biosciences. In this note, we give an expression for the first approximation of equations of isochrons in the setting of perturbations of polynomial Hamiltonian systems. This method can be generalized to perturbations of systems that have a polynomial integral factor (like the Lotka–Volterra equation). ***To cite this article: J. Demongeot, J.-P. Françoise, C. R. Biologies 329 (2006).***

Résumé

Approximation pour les cycles limites et leurs isochrones. Pour analyser le comportement des trajectoires au voisinage des orbites périodiques attractives, les notions de phase asymptotique et d'isochrone apparaissent naturellement. Mais c'est leur importance dans les applications, en particulier dans les sciences de la vie (biologie, physiologie, pharmacocinétique, ...), qui a le plus fortement motivé leur étude. L'objet de cette note est de donner, dans le cadre des pertubations de systèmes hamiltoniens polynomiaux, une approximation au premier ordre pour les équations des isochrones. Cette méthode peut ensuite s'étendre à des perturbations de systèmes qui ont un facteur intégrant polynomial (comme l'équation de Lotka–Volterra). ***Pour citer cet article : J. Demongeot, J.-P. Françoise, C. R. Biologies 329 (2006).***

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Mots-clés : Rythmes biologiques ; Synchronisation ; Phase asymptotique

1. Introduction

The interest for isochrons of limit cycles appeared first in relation with mathematics of biosciences. Their use was particularly emphasized by Winfree [1] and latter appeared repeatedly in many classical references related with biological rhythms [2–5]. Under the influence of Winfree's articles, Guckenheimer [6] identified the isochrons with the stable manifold of a point on an attractive hyperbolic limit cycle. Another proof of their existence and a discussion of the case of non-hyperbolic limit cycles appeared in [7]. M. Sabatini [8,9] proved

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that the existence of isochron sections for a limit cycle of a vector field X is equivalent to the existence of a vector field Y such that the bracket $[X, Y]$ is proportional to Y. More recently, the need for finding a closed formula for isochrons was frequently formulated in view of applications. In this article, we focus on a perturbative situation where the system is a perturbation of a polynomial Hamiltonian system:

$$\dot{x} = \frac{\partial H}{\partial y} + \epsilon f(x, y, \epsilon)$$
$$\dot{y} = -\frac{\partial H}{\partial x} + \epsilon g(x, y, \epsilon).$$

We provide a closed form to the first-order approximation in ϵ for an equation of the limit cycles and for their isochrons. The essential idea is to use a one-parameter family of integrating factors. This method has been first used to compute the successive derivatives of return maps [10] and periods [11]. A more general and abstract setting has been recently proposed in [12]. This general mathematical setting uses iterated integrals and Leray residue. In this note, we stick to the first order of approximation and use only ramified integrals that are not iterated.

2. Definition of isochrons and of asymptotic phase

Existence of isochrons can be deduced from [13]. Let $\gamma : [0, T] \to R^n$, $\gamma : t \mapsto \gamma(t)$ be an attractive hyperbolic periodic orbit and let Σ be a section transverse to the flow. First-return mapping writes:

$$P : x \mapsto x_1 = A \cdot x + D(x)$$

with $a = \|A\| < 1$. Denote x_n, the successive intersection points of the orbit with the transverse section. Let τ be the time of first return to the transverse section along the orbit passing by x. Denote $\tau_1 = \tau(x), \ldots, \tau_n = \tau_{n-1} + \tau(x_n)$. Function $\tau(x)$ is at least of class C^1, hence from x small enough, there is a constant L such that:

$$\left|\tau(x_{n-1}) - T\right| \leqslant L\|x\| \leqslant La^{n-1}\|x\|$$

The series of general term $[(\tau_n - nT) - (\tau_{n-1} - (n-1)T)]$ is normally convergent and thus the sequence $\tau_n - nT$ is convergent. The limit

$$t_0(x) = \lim_{n\to\infty} (\tau_n - nT)$$

is called the asymptotic phase of the point x. Isochrons are defined by the equations:

$$\left\{x \mid t_0(x) = c.\right\}$$

3. One-parameter family of integrating factor and approximation to limit cycle

Given a one-form ω, a function H and a transverse section Σ, consider the continuous family of regular ovals $H = c$, which fills up a domain homeomorphic to an annulus denoted A.

Define the ramified primitive $f_1(P) : A - \Sigma \to R$ of ω:

$$f_1(P) = \oint_{P_0(t)}^{P} \omega$$

where the integral is taken along the solution $H(x, y) = H(P) = t$ of the Hamiltonian flow between the first intersection point $P_0(t)$ of the solution with the transverse section and the point P.

Define the polynomial D by

$$\mathrm{d}\omega = D(x, y)\,\mathrm{d}x \wedge \mathrm{d}y$$

Given a polynomial H, there always exists a polynomial $m(H)$ that belongs to the Jacobian ideal of H (cf. [12]). There is a 1-form, unique up to a multiple of $\mathrm{d}H$, $\tilde{\omega}$, such that:

$$m(H)\,\mathrm{d}x \wedge \mathrm{d}y = \tilde{\omega} \wedge \mathrm{d}H$$

and take:

$$\omega_1 = D(x, y)\frac{\tilde{\omega}}{m(H)}$$

Define finally the function g_1 as:

$$g_1 = \oint \omega_1$$

We have thus obtained that any polynomial one-form ω can be written

$$\omega = g_1\,\mathrm{d}H + \mathrm{d}f_1$$

with ramified functions f_1 and g_1. The function f_1 is kind of primitive of the form ω along the level lines of H. The function $-g_1$ is a primitive of the Leray derivative of ω, and this can be conveniently denoted $\mathrm{d}g_1 = -\frac{\mathrm{d}\omega}{\mathrm{d}H}$.

This construction yields (recursively) a construction of a 1-parameter family of integrating factor (cf. [12]). We stick here to the first-order approximation, which displays

$$(1 - \epsilon g_1)(\mathrm{d}H + \epsilon\omega) = \mathrm{d}(H + \epsilon f_1) + O\left(\epsilon^2\right)$$

This approximation provides an approximation to an equation of an eventual limit cycle as:

$$H + \epsilon f_1 = c$$

Note that f_1 is ramified, but (up to $O(\epsilon^2)$) the ramified part of f_1 vanishes identically on the limit cycle. Thus this formal construction usually yields a tractable expression for this approximation, as it will appear more clearly in the foregoing example.

4. Van der Pol oscillator

Consider the van der Pol oscillator:

$$\dot{x} = y$$

$$\dot{y} = -x + \epsilon F(x) y$$

where $F(x) = x^2 - 1$. The case of Liénard equations where F can be any polynomial can be treated completely similarly.

The associated foliation of the plane can be defined by the 1-form $\mathrm{d}H + \epsilon\omega$, $\omega = -F(x)\, y\, \mathrm{d}x$. We obtain in that case:

$$f_1 = \left[\frac{(x^2+y^2)^2}{8} - \frac{(x^2+y^2)}{2}\right] \operatorname{Arctan}\left(\frac{y}{x}\right) + \frac{xy^3}{8} - \frac{x^3 y}{8} + \frac{xy}{2}$$

This is a ramified function but the limit cycle sits (up to terms of order $O(\epsilon)$) near the circle

$$\left[\frac{(x^2+y^2)^2}{8} - \frac{(x^2+y^2)}{2}\right] = 0$$

Hence the first approximation to the equation for the limit cycle that we obtain is of the form:

$$\frac{x^2+y^2}{2} + \epsilon\left[\frac{xy^3}{8} - \frac{x^3 y}{8} + \frac{xy}{2}\right] = c$$

5. Isochronous forms to a vector field

Assume that X_0 is a Hamiltonian planar vector field,

$$i_{X_0}\, \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}H$$

Assume furthermore that H is polynomial and that for $c_0 \leqslant c \leqslant c_1$, the level line $H^{-1}(c)$ contains an oval γ_c (closed smooth compact curve). The ovals γ_c are periodic orbits of X_0 of period T_c.

Definition 1. A 1-form ω_0 is said to be isochronous to X_0 if $\omega_0(X_0) = 1$, or equivalently $\mathrm{d}H \wedge \omega_0 = \mathrm{d}x \wedge \mathrm{d}y$.

For instance, if $H = \frac{1}{2}(x^2+y^2)$, then we can choose $\omega_0 = \mathrm{d}\theta = \mathrm{d}\operatorname{Arctan}\frac{y}{x}$. In general, for any polynomial H, there is a polynomial $m(H)$ that belongs to the Jacobian ideal of H. This yields two polynomials p and q such that we can choose:

$$\omega_0 = [p\,\mathrm{d}x + q\,\mathrm{d}y]/m(H)$$

A polynomial H is said to be quasi-homogeneous if there are three integers d, m and n such that:

$$H = \frac{d}{m}\frac{\partial H}{\partial x} + \frac{d}{n}\frac{\partial H}{\partial y}$$

In that case, a possible choice is:

$$\omega_0 = \left[\frac{d}{m}\,\mathrm{d}y - \frac{d}{n}\,\mathrm{d}x\right]/H$$

Consider now a 1-parameter perturbation:

$$\omega_\epsilon = \mathrm{d}H + \epsilon\omega$$

and the associated vector field X_ϵ defined by

$$i_{X_\epsilon}\, \mathrm{d}x \wedge \mathrm{d}y = \omega_\epsilon$$

The following notion was first discussed in [11].

Definition 2. A 1-form $\omega_0 + \epsilon\omega_1$ is 1-isochronous to ω_ϵ if:

$$(\mathrm{d}H + \epsilon\omega) \wedge (\omega_0 + \epsilon\omega_1) = \left[1 + O\left(\epsilon^2\right)\right] \mathrm{d}x \wedge \mathrm{d}y.$$

Construction of a 1-isochronous form proceeds in the same way as above. The unknown form ω_1 must satisfy:

$$\omega \wedge \omega_0 + \mathrm{d}H \wedge \omega_1 = 0$$

and this 1-form can be formally defined as the Leray residue:

$$\omega_1 = -\frac{\omega \wedge \omega_0}{\mathrm{d}H}$$

A transversal section to the flow of a vector field near a limit cycle is a section such that the first return time is constant. Guckenheimer pointed out the fact that the isochrons are permuted by the flow. Isochronous sections and isochrons are not exactly the same notions. For instance, in the case of the Hamiltonian flow, there is no limit cycle (and hence no isochrons), despite the fact that there may be isochronous sections if the Hamiltonian flow is isochronous. But if there is an hyperbolic attractive limit cycle and if there is an isochronous section, then it is of course an isochron. Our method displays an approximation for an isochronous section. If we denote:

$$f_\epsilon = \oint (\omega_0 + \epsilon\omega_1)$$

the ramified primitive of the 1-isochronous form, then an approximated equation for an isochronous section is:

$$f_\epsilon = c$$

6. Isochrons for the van der Pol system

In the case of the van der Pol system, we obtain that:

$$\omega_0 = \mathrm{d}\theta$$

and

$$\omega_1 = (r^2 \cos^2\theta - 1)\sin\theta\cos\theta\,\mathrm{d}\theta$$

An approximated equation for isochrons is thus obtained as:

$$\theta + \epsilon\left[r^2\left(\frac{\cos^4}{4} - 1\right) + \frac{1}{2}\sin^2\theta\right] = c$$

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