C. R. Biologies 334 (2011) 851–854

Contents lists available at ScienceDirect

Comptes Rendus Biologies

www.sciencedirect.com

Biological modelling / Biomodélisation

Permanence of a stage-structured predator-prey system with a class of functional responses ☆

Zhihui Ma [a,b,1], Shufan Wang [a,1], Wenting Wang [a], Zizhen Li [a,b,*]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, China
[b] Key Laboratory of Arid and Grassland Agroecology of Ministry of Education, Lanzhou, 730000, China

ARTICLE INFO

Article history:
Received 13 August 2009
Accepted after revision 2 August 2011
Available online 15 September 2011

Keywords:
Predator-prey system
Stage structure
Generalized functional response
Permanence
Non-permanence

ABSTRACT

A stage-structured predator-prey system incorporating a class of functional responses is presented in this article. By analyzing the system and using the standard comparison theorem, the sufficient conditions are derived for permanence of the system and non-permanence of predators.

1. Introduction

The past decades have witnessed an enormous interest in predator-prey systems, with most of them focusing on homogeneous populations [1–5]. In fact, no natural population is truly homogeneous, many organisms undergo radical changes in many aspects such as the rates of survival, maturation, reproduction and predation while while they are processing their life history. Therefore, stage-structured predator-prey systems have received much attention recently [6–21].

However, these works [7–21] are largely using constant coefficient systems and have stage structures on only one of the interactive species. The research objectives mainly include the stability and permanence of the system, often with one particular type of functional response. In this article, we propose the following variable coefficient predator-prey system with time delays and stage structures for both interactive species, as well as different functional responses, Specifically, we will consider the condition for the permanence of the two species and the non-permanence condition for predators to become extinct:

$$\begin{aligned}
\dot{x}_1(t) &= r_1(t)x_2(t) - d_{11}x_1(t) - r_1(t)e^{-d_{11}\tau_1}x_2(t-\tau_1) \\
\dot{x}_2(t) &= r_1(t)e^{-d_{11}\tau_1}x_2(t-\tau_1) - d_{12}x_2(t) - b_1(t)x_2^2 \\
&\quad - c_1(t)\varphi(x_2(t))x_2(t)y_2(t) \\
\dot{y}_1(t) &= r_2(t)y_2(t) - d_{22}y_1(t) - r_2(t)e^{-d_{22}\tau_2}y_2(t-\tau_2) \\
\dot{y}_2(t) &= r_2(t)e^{-d_{22}\tau_2}y_2(t-\tau_2) - d_{21}y_2(t) - b_2(t)y_2^2 \\
&\quad + c_2(t)\varphi(x_2(t))x_2(t)y_2(t)
\end{aligned} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ denote the immature and mature densities of the prey at time t, respectively. $y_1(t)$ and $y_2(t)$ denote the immature and mature densities of the predator at time t, respectively. $r_i(t)$, $b_i(t)$, $c_i(t)$ $(i = 1, 2)$ are positive and continuous functions for all $t \geq 0$.

The above system assumes that the mature predators only feed on the mature prey population, the immature are produced by the mature populations. The birth rate of prey and predators $(r_i(t) > 0;\ i = 1, 2)$ is proportional to the existing mature population sizes.

☆ This work was supported by the Fundamental Research Funds for the Central Universities (No. lzujbky-2011-48).

* Corresponding author.
E-mail addresses: mazhh@lzu.edu.cn (Z. Ma), zizhenlee@lzu.edu.cn (Z. Li).
[1] Zhihui Ma and Shufan Wang contributed equally to the manuscript.

1631-0691/$ – see front matter © 2011 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crvi.2011.08.002

The other parameters have the following biological meanings: $c_1(t)$ denotes the capturing rate of mature predators at time t, and $c_2(t)/c_1(t)$ is the rate of conversion of nutrients from the mature prey into the reproduction of the mature predator; $b_i(t)>0$ $(i=1,\ 2)$ represents the intraspecific competition rate of mature prey and predators at time t, respectively; d_{11} and d_{12} are the death rate of immature and mature population of prey respectively, d_{22} and d_{21} are the death rate of immature and mature population of predator respectively; $\tau_i>0(i=1,\ 2)$ is the length of time from the birth to maturity of the species i. The term $r_1(t)e^{-d_{11}\tau}{}_1x_2(t-\tau_1)$ represents the survived prey population born at time $t-\tau_1$, that is, the transition from the immature to mature prey. The term $r_2(t)e^{-d_{22}\tau}{}_2y_2(t-\tau_2)$ denotes the survived predator population born at time $t-\tau_2$, that is, the transition from the immature to mature predator.

The term $\varphi(x_2)x_2$, the number of prey captured per predator per unit time, is called the predator functional response and satisfies the following assumptions

$$0<\varphi(x_2)<L<+\infty,\quad \frac{\mathrm{d}}{\mathrm{d}x_2}(\varphi(x_2)x_2)\ge 0(x_2>0). \tag{2}$$

The initial conditions of system (1) are given by

$$\begin{aligned}&x_i(\theta)=\varphi_i(\theta)>0,\quad y_i(\theta)=\psi_i(\theta)>0,\quad \varphi_i(0)>0,\\&\psi_i(0)>0\quad (i=1,2),\quad \theta\in[-\tau,0],\\&\tau=\max\{\tau_1,\tau_2\}.\end{aligned} \tag{3}$$

For the continuity of initial conditions, we require further that

$$\begin{aligned}x_1(0)&=\int_{-\tau_1}^{0} r_1(s)\varphi_2(s)\,\mathrm{d}s,\\ y_1(0)&=\int_{-\tau_2}^{0} r_2(s)\psi_2(s)\,\mathrm{d}s.\end{aligned} \tag{4}$$

Again, we define that

$$\begin{aligned}&r_k^s=\sup_{t\ge 0} r_k(t)>0, c_k^s=\sup_{t\ge 0} c_k(t)>0, b_k^s=\sup_{t\ge 0} b_k(t)>0,\\&r_k^i=\inf_{t\ge 0} r_k(t)>0, c_k^i=\inf_{t\ge 0} c_k(t)>0, b_k^i=\inf_{t\ge 0} b_k(t)>0\\&\quad (k=1,2).\end{aligned} \tag{5}$$

2. Positivity and boundedness

Theorem 2.1. *Solutions of system (1) with initial conditions (3) and (4)are positive and bounded for all $t\ge 0$.*

Proof. First we show $x_2(t)>0$ for all $t\ge 0$. Otherwise, noticing that $x_2(t)=\varphi_2(t)>0$ for all $-\tau_1<t<0$.

Then there exists a $t^*>0$, such that $x_2(t^*)=0$.

Now denoting $t_0=\inf\{t>0 \mid x_2(t)=0\}$.

Then $t_0>0$ and from system (1), we have

$$0\le t_0\le \tau_1 \Rightarrow \dot{x}_2(t_0)=r_1(t_0)e^{-d_{11}\tau_1}\varphi_2(t_0-\tau_1)>0.$$

$$t_0>\tau_1 \Rightarrow \dot{x}_2(t_0)=r_1(t_0)e^{-d_{11}\tau_1}x_2(t_0-\tau_1)>0.$$

Hence $\dot{x}_2(t_0)>0$. By the definition of t_0, we have $\dot{x}_2(t_0)\le 0$. A contradiction.

Thus $x_2(t)>0$ for all $t>0$.

Again, considering the following equation

$$\begin{aligned}&\dot{u}(t)=-d_{11}u(t)-r_1(t)e^{-d_{11}\tau_1}u(t-\tau_1)\\&u(0)=x_2(0)\end{aligned} \tag{6}$$

We can easily obtain that

$$u(t)=e^{-d_{11}t}\left[\varphi_2(0)-\int_0^t r_1(s)\varphi_2(s)\,\mathrm{d}s\right].$$

and

$$x_1(t)>u(t)(0\le t<\tau_1).$$

From the initial conditions (3), we have $u(\tau_1)=0$, then $x_1(t)>u(t)>0$ for $0\le t<\tau_1$.

By induction, we can show $x_1(t)>0$ for all $t\ge 0$.

Similarly, we can prove that $y_1(t)>0$ and $y_2(t)>0$ for all $t\ge 0$.

Next, we will prove the boundedness of the solutions of system (1).

Defining the function

$$V(t)=c_2^s x_1(t)+c_2^s x_2(t)+c_1^i y_1(t)+c_1^i y_2(t).$$

The derivative of $V(t)$ along the positive solutions of system (1) is

$$\begin{aligned}\dot{V}(t)&=c_2^s\dot{x}_1(t)+c_2^s\dot{x}_2(t)+c_1^i\dot{y}_1(t)+c_1^i\dot{y}_2(t)\\&\le (c_2^s r_1^s-c_2^s d_{12})x_2(t)-c_2^s b_1^i x_2^2(t)-c_2^s d_{11}x_1(t)\\&\quad +(c_1^i r_2^s-c_1^i d_{21})y_2(t)-c_1^i b_2^i y_2^2(t)-c_1^i d_{22}y_1(t).\end{aligned}$$

For a positive constant $d\le\min\{d_{11},\ d_{22}\}$, then

$$\begin{aligned}\dot{V}(t)+dV(t)&\le c_2^s(r_1^s-d_{12}+d)x_2(t)-c_2^s b_1^i x_2^2(t)\\&\quad +c_1^i(r_2^s-c_1^i d_{21}+d)y_2(t)-c_1^i b_2^i y_2^2(t).\end{aligned}$$

Hence there exists a positive number M, such that $\dot{V}(t)+dV(t)\le M$. Then we get

$$V(t)\le Md^{-1}+(V(0)-Md^{-1})e^{-dt}.$$

Therefore, the positive solutions of system (1) are bounded. This completes the Proof. □

3. Permanence and non-permanence

Definition 3.1. If there exist positive constants m_x, M_x, m_y and M_y, such that each solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ of system (1) satisfies

$$0<m_x\le \liminf_{t\to+\infty} x_i(t)\le \limsup_{t\to+\infty} x_i(t)\le M_x\quad (i=1,2),$$

$$0<m_y\le \liminf_{t\to+\infty} y_i(t)\le \limsup_{t\to+\infty} y_i(t)\le M_y\quad (i=1,2).$$

Then system (1) is permanent. Otherwise, it is non-permanent.

Lemma 3.2 ([22]).
Consider the following equation:

$$\dot{u}(t) = au(t-\tau) - bu(t) - cu^2(t)$$

where $a, b, c > 0; u(t) > 0$ *for* $-\tau \le t \le 0$, *we have*

(1). *If* $a > b$, *then* $\lim_{t\to+\infty} u(t) = \frac{a-b}{c}$,

(2). *If* $a < b$, *then* $\lim_{t\to+\infty} u(t) = 0$.

Theorem 3.3. *If the following assumptions hold*

$$\begin{aligned} &(H_1). \quad r_1^i x_2^i > r_1^s x_2^s e^{-d_{11}\tau_1} > 0 \\ &(H_2). \quad r_2^i y_2^i > r_2^s y_2^s e^{-d_{22}\tau_2} > 0 \end{aligned} \tag{7}$$

where

$$x_2^i = \frac{r_1^i e^{-d_{11}\tau_1} - d_{12} - c_1^s L y_2^s}{b_1^s}, \quad x_2^s = \frac{r_1^s e^{-d_{11}\tau_1} - d_{12}}{b_1^i},$$

$$y_2^i = \frac{r_2^i e^{-d_{22}} + c_2^i \varphi(x_2^i) x_2^i - d_{21}}{b_2^s},$$

$$y_2^s = \frac{r_2^s e^{-d_{22}\tau_2} + c_2^s \varphi(x_2^s) x_2^s - d_{21}}{b_2^i}.$$

Then system (1) *is permanent.*

Proof. Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1) for $t \ge 0$. Firstly, from the second equation of system (1), we have

$$\begin{aligned} \dot{x}_2(t) &\le r_1(t) e^{-d_{11}\tau_1} x_2(t-\tau_1) - d_{12} x_2(t) - b_1(t) x_2^2 \\ &\le r_1^s e^{-d_{11}\tau_1} x_2(t-\tau_1) - d_{12} x_2(t) - b_1^i x_2^2(t). \end{aligned}$$

By Lemma 3.2 and standard comparison theorem, we get

$$limsup_{t\to+\infty} x_2(t) \le \frac{r_1^s e^{-d_{11}\tau_1} - d_{12}}{b_1^i} \doteq x_2^s > 0.$$

That is, for any $\varepsilon > 0$, there exists a $T_1 > 0$, for any $t > T_1 > 0$, such that $x_2(t) < x_2^s + \varepsilon$.

Again, from the fourth equation of system (1), there exists a $T_2 > T_1 > 0$, for any $t > T_2$, we have

$$\begin{aligned} \dot{y}_2(t) &\le r_2(t) e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21} y_2(t) - b_2(t) y_2^2 \\ &\quad + c_2(t)\varphi(x_2^s+\varepsilon)(x_2^s+\varepsilon) y_2(t) \\ &\le r_2^s e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21} y_2(t) - b_2^i y_2^2 \\ &\quad + c_2^s \varphi(x_2^s+\varepsilon)(x_2^s+\varepsilon) y_2(t). \end{aligned}$$

We obtain that using Lemma 3.2 and standard comparison theorem

$$limsup_{t\to+\infty} y_2(t) \le \frac{r_2^s e^{-d_{22}\tau_2} + c_2^s \varphi(x_2^s+\varepsilon)(x_2^s+\varepsilon) - d_{21}}{b_2^i}.$$

Since ε is sufficiently small, then

$$limsup_{t\to+\infty} y_2(t) \le \frac{r_2^s e^{-d_{22}\tau_2} - d_{21} + c_2^s \varphi(x_2^s) x_2^s}{b_2^i} \doteq y_2^s > 0.$$

Hence, for this $\varepsilon > 0$, there exists a $T_3 > T_2 > 0$, for any $t > T_3 > 0$, such that $y_2(t) < y_2^s + \varepsilon$.

Now, substituting it into the second equation of system (1), we have

$$\begin{aligned} \dot{x}_2(t) &\ge r_1(t) e^{-d_{11}\tau_1} x_2(t-\tau_1) - d_{12} x_2(t) - b_1(t) x_2^2(t) \\ &\quad - c_1(t)(y_2^s+\varepsilon)\varphi(x_2(t)) x_2(t) \\ &\ge r_1^i e^{-d_{11}\tau_1} x_2(t-\tau_1) - d_{12} x_2(t) - b_1^s x_2^2(t) \\ &\quad - c_1^s L(y_2^s+\varepsilon) x_2(t). \end{aligned}$$

By the Lemma 3.2 and the standard comparison theory, noticing that ε is sufficiently small, we obtain that

$$liminf_{t\to+\infty} x_2(t) \ge \frac{r_1^i e^{-d_{11}\tau_1} - d_{12} - c_1^s L y_2^s}{b_1^s} \doteq x_2^i > 0.$$

That is, for this $\varepsilon > 0$, there exists a $T_4 > T_3 > 0$, for any $t > T_4 > 0$, such that $x_2(t) \ge x_2^i - \varepsilon$.

From the fourth equation of system (1), there exists a $T_5 > T_4 > 0$, for any $t > T_5 > 0$, we have

$$\begin{aligned} \dot{y}_2(t) &\ge r_2(t) e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21} y_2(t) - b_2(t) y_2^2(t) \\ &\quad + c_2(t)\varphi(x_2^i-\varepsilon)(x_2^i-\varepsilon) y_2(t) \\ &\ge r_2^i e^{-d_{22}\tau_2} y_2(t-\tau_2) - d_{21} y_2(t) - b_2^s y_2^2(t) \\ &\quad + c_2^i \varphi(x_2^i-\varepsilon)(x_2^i-\varepsilon) y_2(t). \end{aligned}$$

Similarly, we obtain that

$$liminf_{t\to+\infty} y_2(t) \ge \frac{r_2^i e^{-d_{22}} - d_{21} + c_2^i \varphi(x_2^i) x_2^i}{b_2^s} \doteq y_2^i > 0.$$

That is, for this $\varepsilon > 0$, there exists a $T_6 > T_5 > 0$, for any $t > T_6 > 0$, such that $y_2(t) \ge y_2^i - \varepsilon$.

Again, from the first equation of system (1), we have

$$\begin{aligned} \dot{x}_1(t) &\le r_1(t)(x_2^s+\varepsilon) - d_{11} x_1(t) - r_1(t) e^{-d_{11}\tau_1}(x_2^i-\varepsilon) \\ &\le r_1^s(x_2^s+\varepsilon) - d_{11} x_1(t) - r_1^i e^{-d_{11}\tau_1}(x_2^i-\varepsilon). \end{aligned}$$

Noticing that ε is sufficiently small, we get

$$limsup_{t\to+\infty} x_1(t) \le \frac{r_1^s x_2^s - r_1^i e^{-d_{11}\tau_1} x_2^i}{d_{11}} \doteq x_1^s > 0.$$

According to the third equation of system (1) for any $t > T_6$, and by Lemma 3.2, we have

$$limsup_{t\to+\infty} y_1(t) \le \frac{r_2^s y_2^s - r_2^i e^{-d_{22}\tau_2} y_2^i}{d_{22}} \doteq y_1^s > 0.$$

By the analogous way, we obtain that

$$liminf_{t\to+\infty} x_1(t) \ge \frac{r_1^i x_2^i - r_1^s e^{-d_{11}\tau_1} x_2^s}{d_{11}} \doteq x_1^i > 0.$$

$$liminf_{t\to+\infty} y_1(t) \ge \frac{r_2^i y_2^i - r_2^s e^{-d_{22}\tau_2} y_2^s}{d_{22}} \doteq y_1^i > 0.$$

Therefore, by Definition 3.1, system (1) is permanent. This completes the proof. □

Theorem 3.4. *If* (H_1) *and the following assumption hold*

$$(H_3). \quad r_2^s e^{-d_{22}\tau_2} + c_2^s \varphi(x_2^s) x_2^s < d_{21} \tag{8}$$

where

$$x_2^s = \frac{r_1^s e^{-d_{11}\tau_1} - d_{12}}{b_1^i}.$$

Then the prey population is permanent while the predators are non-permanent.

Proof. Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1) for $t \geq 0$. In order to prove the non-permanence of predators and permanence of prey population, we only show that

$$\lim_{t\to+\infty} y_i(t) = 0 \quad (i = 1, 2).$$

If the assumption (H_3) holds, then we obtain that $y_2^s < 0$ and $y_2^i < 0$.

From the Proof of Theorem 3.3, there exists a $T_7 > T_6 > 0$, for any $t > T_7$, we have

$$\begin{aligned} &limsup_{\, t\to+\infty} y_2(t) \leq 0, \\ &liminf_{\, t\to+\infty} y_2(t) \geq 0. \end{aligned} \tag{9}$$

Hence, we get

$$\lim_{t\to+\infty} y_2(t) = 0.$$

Again, substituting it into third equation of the system (1), we have

$$-d_{22} y_1(t) \leq y_1'(t) \leq -d_{22} y_1(t).$$

Integrate it from 0 to t, we get

$$0 < y_1(o) e^{-d_{22}t} \leq y_1(t) \leq y_1(o) e^{-d_{22}t}.$$

Thus, we obtain that

$$\lim_{t\to+\infty} y_1(t) = 0.$$

Therefore, the prey population is permanent and predators are non-permanent. This completes the proof. □

4. Discussion

In this article, we have considered a stage-structured (for both interactive populations) predator-prey system with a class of functional response incorporating discrete time delay, and obtain the sufficient conditions for the permanence of system (1) and the non-permanence of predators. Our work has provided some valuable suggestions for regulating populations and saving endangered species. We see that the predators only forage on mature prey as food resource. Consequently, when the prey population size is lower than a certain level predators will not sustain. This is especially true for mature predators which are responsible for recruitment. In other words, as long as the population sizes of prey and predators maintain at a certain positive level, the predator-prey interaction will continue.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (No. 30970478,31100306).

References

[1] A. Worz-Busekros, Global stability in ecological systems with continous time delay, SIAM J. Appl. Math. 35 (1978) 123–134.
[2] L. Liou, K. Cheng, Global stability of a predator-prey system, J. Math. Biol. 26 (1988) 65–71.
[3] J. Hainzl, Stability and Hopf Bifurcation in a Predator-Prey System with Several Parameters, SIAM J. Appl. Math. 48 (1988) 170–190.
[4] S. Hsu, T. Huang, Global Stability for a Class of Predator-Prey Systems, SIAM J. Appl. Math. 55 (1995) 763–783.
[5] A. Nindjin, M. Aziz-Alaoui, M. Cadivel, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay, Nonlinear Analysis: RWA 7 (2006) 1104–1118.
[6] Z. Ma, Z. Li, S. Wang, T. Li, F. Zhang, Permanence of a predator-prey system with stage structure and time delay, Appl. Math. Comput. 201 (2008) 65–71.
[7] W. Aiello, H. Freedman, A time-delay of single-species growth, Math. Biol. 101 (1990) 139–153.
[8] H. Freedman, J. Wu, Persistence and global asymptotic stability of single species dispersal models with stage structure, Quart. Appl. Math. 49 (1991) 351–371.
[9] W. Aiello, H. Freedman, J. Wu, Analysis of a model representing stage-structure population growth with state-dependent time delay, SIAM J. Appl. Math. 52 (1992) 855–869.
[10] W. Wang, L. Chen, A predator-prey system with stage structure for predator, Comput. Math. Appl. 33 (8) (1997) 83–91.
[11] X. Zhang, L. Chen, The stage-structured predator-prey model and optimal harvesting policy, Math. Biosci. 168 (2000) 201–210.
[12] X. Song, L. Chen, Optimal harvesting and stability for a two sepecise competetive system with stage structure, Math. Biol. 170 (2001) 173–186.
[13] X. Zhang, L. Chen, U.A. Numan, The stage-structured predator-prey model and optimal havesting policy, Math. Biol. 168 (2000) 201–210.
[14] X. Song, L. Chen, A predator-prey system with stage-structured and havesting for prey, Acta. Math. Appl. Son. 18 (3) (2002) 423–430.
[15] Z. Teng, L. Chen, Permanence and extinction of periodic predator. prey systems in a patchy environment with delay, Nonlinear Analysis: RWA. 4 (2003) 335–364.
[16] Y. Muroya, Permanence and Global Stability in a Lotka-Volterra Predator-Prey System with Delays, Appl. Math. Letter. 16 (2003) 1245–1250.
[17] R. Xu, M. Chaplain, F. Davidson, Global stability of a Lotka-Volterra type predator-prey model with stage structure and time delay, Appl. Math. Comput. 159 (2004) 863–880.
[18] J. Cui, X. Song, Permanence of a predator-prey system with stage structure, Discret. Contin. Dyn. Syst. Ser. B 4 (3) (2004) 547–554.
[19] S. Liu, L. Chen, Z. Liu, Extinction and permanence in nonautonomous competitive system with stage structure, J. Math. Anal. Appl. 274 (2002) 667–684.
[20] J. Cui, Y. Takeuchib, A predator-prey system with a stage structure for the prey, Math. Comput. Model. 44 (2006) 1126–1132.
[21] H. Zhang, L. Chen, R. Zhu, Permanence and extinction of a periodic predator-prey delay system with functional response and stage structure for prey, Appl. Math. Comput. 184 (2007) 931–944.
[22] X. Song, L. Chen, Optimal harvesting and stability for a two-species competitive system with stage structure, Math. Biosci. 170 (2001) 173–186.