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Oscillations for a delayed predator–prey model with Hassell–Varley-type functional response☆

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ABSTRACT

In this paper, a delayed predator–prey model with Hassell–Varley-type functional response is investigated. By choosing the delay as a bifurcation parameter and analyzing the locations on the complex plane of the roots of the associated characteristic equation, the existence of a bifurcation parameter point is determined. It is found that a Hopf bifurcation occurs when the parameter τ passes through a series of critical values. The direction and the stability of Hopf bifurcation periodic solutions are determined by using the normal form theory and the center manifold theorem due to Faria and Maglhalaes (1995). In addition, using a global Hopf bifurcation result of Wu (1998) for functional differential equations, we show the global existence of periodic solutions. Some numerical simulations are presented to substantiate the analytical results. Finally, some biological explanations and the main conclusions are included.

1. Introduction

Since the pioneering work of Volterra [1] and Lotka [2] in the mid-1920s, there has been increasing interest in investigating the dynamical behaviors of predator–prey models in both ecology and mathematical ecology [3–11]. In particular, one of the important dynamical predator–prey behaviors, such as periodic phenomena and bifurcation has become even more interesting [6–10,12–24]. In 1980, Freeman [25] proposed a most popular predator–prey model with Michaelis–Menten-type functional response:

$$\begin{cases} \dfrac{dx_1}{dt} = rx_1\left(1-\dfrac{x_1}{K}\right) - \dfrac{cx_1x_2}{m+x_1}, \\ \dfrac{dx_2}{dt} = x_2\left(-d+\dfrac{fx_1}{m+x_1}\right), \\ x(0)>0, y(0)>0, \end{cases} \tag{1}$$

where x_1, x_2 denote the population of preys and predators at time t, respectively. r, K, c, m, d, and f are positive constants that denote the prey's intrinsic growth rate, carrying capacity, capturing rate, half-saturation constant, predator death rate, maximal predator growth rate, respectively. For more details about the model, the reader is referred to [25].

Considering that in many situations, predators must search and share or compete for food, Arditi and Ginzburg

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[3] introduced and studied the following ratio-dependent-type functional response model:

$$\begin{cases} \frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1}{K}\right) - \frac{cx_1x_2}{mx_2 + x_1}, \\ \frac{dx_2}{dt} = x_2\left(-d + \frac{fx_1}{mx_2 + x_1}\right), \\ \qquad x(0) > 0, y(0) > 0 \end{cases} \tag{1.2}$$

Since the functional response depends on the predator density in a different way, Hassel and Varley [26] reconstructed the predator–prey model with Hassell–Varly-type functional response, which takes the following form:

$$\begin{cases} \frac{dx_1}{dt} = rx_1\left[1 - \frac{x_1}{K}\right] - \frac{cx_1x_2}{mx_2^{\gamma} + x_1}, \\ \frac{dx_2}{dt} = x_2\left[-d + \frac{fx_1}{mx_2^{\gamma} + x_1}\right], \\ \qquad x(0) > 0, y(0) > 0, \end{cases} \tag{1.3}$$

where $\gamma \in (0,1)$ is called the HV constant. Generally, the consumptions of prey by the predator throughout its past history governs the present birth rate of the predator. Motivated by this point of view, Wang [27] introduced and investigated the periodic solutions to the following delayed predator–prey model:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\left[a(t) - b(t)x_1(t-\tau(t)) - \frac{c(t)x_2(t)}{mx_2^{\gamma}(t) + x_1(t)}\right], \\ \qquad \frac{dx_2(t)}{dt} = x_2(t)\left[-d(t) + \frac{r(t)x_1(t)}{mx_2^{\gamma} + x_1}\right], \\ \qquad\qquad x(0)0, y(0) > 0 \end{cases} \tag{1.4}$$

with the following initial condition:

$$\begin{cases} x_1(t) = \varphi(\theta), \theta \in [-\delta, 0], \varphi(0) = \varphi_0 > 0, \\ x_2(t) = \psi(\theta), \theta \in [-\delta, 0], \psi(0) = \psi_0 > 0, \end{cases} \tag{1.5}$$

where $\delta = \sup_{t \in [0,\omega]}\{\tau(t)\}, \varphi, \psi \in C([-\delta, 0])$ with the norm $\|x\| = \sup_{t \in [-\delta,0]}|x(t)|$. It is worth pointing out that during the course of the predator–prey interaction when predators do not form groups, one can assume that the HV constant is equal to 1, that is, $\gamma = 1$. Moreover, it is more reasonable to incorporate the delay into Hassell–Varly-type functional response. From the point of view of biology, we will consider the following model with delayed Hassell–Varly-type functional response:

$$\begin{cases} \frac{dx_1}{dt} = x_1\left[a - bx_1(t-\tau) - \frac{cx_2(t-\tau)}{mx_2(t-\tau) + x_1(t-\tau)}\right], \\ \qquad \frac{dx_2}{dt} = x_2\left[-d + \frac{rx_1(t-\tau)}{mx_2(t-\tau) + x_1(t-\tau)}\right] \end{cases} \tag{1.6}$$

In this paper, we will devote our attention to investigating the properties of a Hopf bifurcation of system (1.6), that is to say, we shall take the delay τ as the bifurcation parameter and show that when τ passes through a certain critical value, the positive equilibrium loses its stability and a Hopf bifurcation will take place. Furthermore, when the delay τ takes a sequence of critical values containing the above critical value, the positive equilibrium of system (1.6) will undergo a Hopf bifurcation. In particular, by using the normal form theory and the center manifold reduction due to Faria and Maglhalaes [28], the formulae for determining the direction of Hopf bifurcations and the stability of bifurcating periodic solutions are obtained. In addition, the existence of periodic solutions for τ far away from the Hopf bifurcation values is also established by means of the global Hopf bifurcation result of Wu [29].

In order to obtain the main results of our paper, throughout this paper, we assume that the coefficients of system (1.6) satisfy the following condition:

H_1. $am^2 + cd - cr > 0$, $r > d$

This paper is organized as follows. In Section 2, the stability of the positive equilibrium and the existence of a Hopf bifurcation at the positive equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. In Section 5, some conditions that guarantee the global existence of the bifurcating periodic solutions to the model are given. Biological explanations and some main conclusions are drawn in Section 6.

2. Stability of the equilibrium and existence of the local Hopf bifurcation

In the section, by analyzing the characteristic equation of the linearized system of system (1.6) at the positive equilibrium, we investigate the stability of the positive equilibrium and the existence of the local Hopf bifurcations occurring at the positive equilibrium.

Considering the biological meaning, we only study the property of a unique positive equilibrium (i.e., coexistence equilibrium). It is easy to see that under the hypothesis (H_1), system (1.6) has a unique positive equilibrium $E_*(x_1^*, x_2^*)$, where

$$x_1^* = \frac{am^2 + cd - cr}{abm}, x_2^* = \frac{(r-d)(am^2 + cd - cr)}{abdm^2}$$

Let $u_1(t) = x_1(t) - x_1^*, u_2(t) = x_2(t) - x_2^*$, then, system (1.6) takes the following form:

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = m_1u_1(t-\tau)+m_2u_2(t-\tau)+m_3u_1^2(t-\tau)+m_4u_2^2(t-\tau) \\ \qquad +m_5u_1(t-\tau)u_2(t-\tau)+m_6u_1(t)u_1(t-\tau)+m_7u_1(t)u_2(t-\tau) \\ \qquad +m_8u_1(t)u_1^2(t-\tau)+m_9u_1(t)u_2^2(t-\tau)+h.o.t., \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = n_1u_1(t-\tau)+n_2u_2(t-\tau)+n_3u_1^2(t-\tau)+n_4u_2^2(t-\tau) \\ \qquad +n_5u_1(t-\tau)u_2(t-\tau)+n_6u_1(t-\tau)u_2(t)+n_7u_2(t)u_2(t-\tau) \\ \qquad +n_8u_1(t-\tau)u_2^2(t-\tau)+n_9u_1^2(t-\tau)u_2(t-\tau)+n_{10}u_2(t)u_2^2(t-\tau) \\ \qquad +n_{11}u_1(t-\tau)u_2(t)u_2(t-\tau)+n_{12}u_1^2(t-\tau)u_2(t)+h.o.t., \end{cases} \tag{2.1}$$

where

$$m_1=\left(\frac{cx_2^*}{mx_2^*+x_1^*}-b\right)x_1^*, m_2=\left(\frac{mcx_2^*}{(mx_2^*+x_1^*)^2}-\frac{c}{mx_2^*+x_1^*}\right)x_1^*,$$

$$m_3=-\frac{cx_1^*x_2^*}{(mx_2^*+x_1^*)^3}, m_4=\frac{mcx_1^*}{(mx_2^*+x_1^*)^2}-\frac{m^2cx_2^*}{(mx_2^*+x_1^*)^3},$$

$$m_5=\frac{c}{(mx_2^*+x_1^*)^2}-\frac{2mcx_2^*}{(mx_2^*+x_1^*)^2}, m_6=\frac{cx_2^*}{mx_2^*+x_1^*}-b,$$

$$m_7=\frac{mcx_2^*}{(mx_2^*+x_1^*)^2}-\frac{c}{mx_2^*+x_1^*}, m_8=-\frac{cx_2^*}{(mx_2^*+x_1^*)^3},$$

$$m_9=\frac{mc}{(mx_2^*+x_1^*)^2}-\frac{m^2cx_2^*}{(mx_2^*+x_1^*)^3}, n_1=\frac{rx_2^*}{mx_2^*+x_1^*}-\frac{rx_1^*x_2^*}{(mx_2^*+x_1^*)^2},$$

$$n_2=-\frac{mrx_1^*x_2^*}{(mx_2^*+x_1^*)^2}, n_3=-\frac{rx_2^*}{(mx_2^*+x_1^*)^2}, n_4=-\frac{m^2rx_1^*x_2^*}{(mx_2^*+x_1^*)^3},$$

$$n_5=\frac{2mrx_1^*x_2^*}{(mx_2^*+x_1^*)^3}-\frac{mrx_2^*}{(mx_2^*+x_1^*)^2}, n_6=\frac{r}{mx_2^*+x_1^*}-\frac{rx_1^*}{(mx_2^*+x_1^*)^2},$$

$$n_7=-\frac{mrx_1^*}{(mx_2^*+x_1^*)^2}, n_8=-\frac{m^2rx_2^*}{(mx_2^*+x_1^*)^3}, n_9=\frac{2mrx_2^*}{(mx_2^*+x_1^*)^3},$$

$$n_{10}=-\frac{m^2rx_1^*}{(mx_2^*+x_1^*)^3}, n_{11}=\frac{2mrx_1^*}{(mx_2^*+x_1^*)^3}-\frac{mr}{(mx_2^*+x_1^*)^2}, n_{12}=-\frac{r}{(mx_2^*+x_1^*)^2}$$

Then, we obtain the linearized system of (2.1)

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = m_1u_1(t-\tau)+m_2u_2(t-\tau), \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = n_1u_1(t-\tau)+n_2u_2(t-\tau) \end{cases} \tag{2.2}$$

with characteristic equation:

$$\lambda^2-(m_1+n_2)\lambda\ \mathrm{e}^{-\lambda\tau}+(m_1n_2-m_2n_1)\ \mathrm{e}^{-2\lambda\tau}=0 \tag{2.3}$$

In order to investigate the distribution of roots of the transcendental equation (2.3), the following Lemma stated in [30] is helpful.

Lemma 2.1. [30] *For the transcendental equation*

$$P\left(\lambda,\mathrm{e}^{-\lambda\tau_1},\cdots,\mathrm{e}^{-\lambda\tau_m}\right)=\lambda^n+p_1^{(0)}\lambda^{n-1}+\cdots+p_{n-1}^{(0)}\lambda+p_n^{(0)} + \left[p_1^{(1)}\lambda^{n-1}+\cdots+p_{n-1}^{(1)}\lambda+p_n^{(1)}\right]\ \mathrm{e}^{-\lambda\tau_1}+\cdots + \left[p_1^{(m)}\lambda^{n-1}+\cdots+p_{n-1}^{(m)}\lambda+p_n^{(m)}\right]\ \mathrm{e}^{-\lambda\tau_m}=0,$$

as $(\tau_1,\tau_2,\tau_3,\cdots,\tau_m)$ vary, the sum of orders of the zeros of $P(\lambda,\mathrm{e}^{-\lambda\tau_1},\cdots,\mathrm{e}^{-\lambda\tau_m})$ *in the open right half plane can change, and only a zero appears on or crosses the imaginary axis.*

Regarding τ as the parameter, we can apply Lemma 2.1 to (2.3), which is a special case of $P\left(\lambda,\mathrm{e}^{-\lambda\tau_1},\cdots,\mathrm{e}^{-\lambda\tau_m}\right)$

Obviously, if $m_1n_2 \neq m_2n_1$, then, $\lambda = 0$ is not a root of (2.3). For $\tau = 0$ the characteristic equation (2.3) becomes:

$$\lambda^2 - (m_1 + n_2)\lambda + m_1n_2 - m_2n_1 = 0 \tag{2.4}$$

It is easy to see that Eq. (2.4) have two negative real roots if the following condition:

H₂. $m_1 + n_2 < 0,\ m_1n_2 - m_2n_1 > 0$

holds.

Multiplying $e^{\lambda\tau}$ on both sides of (2.3), it is obvious to obtain:

$$\lambda^2 e^{\lambda\tau} - (m_1 + n_2)\lambda + (m_1n_2 - m_2n_1)\ e^{-\lambda\tau} = 0 \tag{2.5}$$

For $\omega_0 > 0$, $i\omega_0$ is a root of (2.5) if and only if

$$-\omega_0^2\ e^{i\omega_0\tau} - (m_1 + n_2)i\omega + (m_1n_2 - m_2n_1)\ e^{-i\omega_0\tau} = 0 \tag{2.6}$$

Separating the real and imaginary parts of (2.6), we get:

$$\begin{cases} (m_1n_2 - m_2n_1 - \omega_0^2)\cos\omega_0\tau = 0, \\ (\omega_0^2 + m_1n_2 - m_2n_1)\sin\omega_0\tau = (m_1 + n_2)\omega_0 \end{cases} \tag{2.7}$$

If the condition (H_2) holds, we can easily check that $\cos\omega_0\tau \neq 0$. Then, it follows from (2.7) that:

$$m_1n_2 - m_2n_1 = \omega_0^2$$

which leads to:

$$\omega_0 = \sqrt{m_1n_2 - m_2n_1} \tag{2.8}$$

From the second equation of (2.7), we can easily obtain:

$$\tau_k = \frac{1}{\omega_0}\left[\arcsin\frac{(m_1 + n_2)\sqrt{m_1n_2 - m_2n_1}}{2(m_1n_2 - m_2n_1)} + 2k\pi\right], k = 0, 1, 2, \cdots. \tag{2.9}$$

From (2.7), we know that (2.3) has a simple pair of purely imaginary roots $\pm i\omega_0$ at $\tau_k (k = 0, 1, 2, \cdots)$

Let $\lambda_k(\tau) = \alpha_k(\tau) + i\omega_k(\tau)$ be the root of Eq. (2.3) satisfying $\alpha_k(\tau_k) = 0, \omega_k(\tau_k) = \omega_0$

Assume that

H₃. $4(m_1n_2 - m_2n_1)^2$

$$+(m_1 + n_2)\sqrt{4(m_1n_2 - m_2n_1)^2 - (m_1 + n_2)^2} > 2(m_1 + n_2)^2$$

Then, the following transversality condition holds.

Lemma 2.2. *If* (H_1), (H_2) *and* (H_3) *are satisfied, then* $\frac{d\alpha_k(\tau)}{d\tau}|\tau = \tau k > 0$

Proof. Differentiating the equation (2.3) with respect to τ leads to:

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2\lambda\ e^{\lambda\tau} + (m_1 + n_2)\tau - 2(m_1n_2 - m_2n_1)\tau\ e^{-\lambda\tau}}{2(m_1n_2 - m_2n_1)\tau\ e^{-\lambda\tau}\lambda - (m_1 + n_2)\lambda}$$

When $\tau = \tau_k$, $i\omega_0$ is a purely imaginary root of (2.3). We then easily get:

$$\left[\frac{d\alpha_k(\tau)}{d\tau}\right]^{-1}|_{\tau=\tau_k} = \frac{4\omega_0^4\cos 2\omega_0\tau_k - 2\omega_0^3(m_1 + n_2)\cos\omega_0\tau_k}{[2\omega_0^3\sin\omega_0\tau_k]^2 + [2\omega_0^3\cos\omega_0\tau_k - (m_1 + n_2)\omega_k]^2} \tag{2.10}$$

From (2.7), we have:

$$\sin\omega_0\tau_k = \frac{m_1 + n_2}{2\omega_0}, \cos\omega_0\tau_k = \pm\frac{\sqrt{4\omega_0^2 - (m_1 + n_2)^2}}{2\omega_0}$$

Hence,

$$\left[\frac{d\alpha_k(\tau)}{d\tau}\right]^{-1}|_{\tau=\tau_k} = \frac{\omega_0^2\left[4(m_1n_2 - m_2n_1) - 2(m_1 + n_2)^2\right]^2}{[2\omega_0^3\sin\omega_0\tau_k]^2 + [2\omega_0^3\cos\omega_0\tau_k - (m_1 + n_2)\omega_k]^2}$$
$$\pm\frac{\omega_0^2\sqrt{4(m_1n_2 - m_2n_1) - (m_1 + n_2)^2}}{[2\omega_0^3\sin\omega_0\tau_k]^2 + [2\omega_0^3\cos\omega_0\tau_k - (m_1 + n_2)\omega_k]^2}$$

Under the condition (H_3), we know that

$$\frac{d\alpha_k(\tau)}{d\tau}|\tau = \tau_k > 0,$$

completing the proof. Lemma 2.2 implies that the roots $\lambda_k(\tau)$ of characteristic equation (2.3) near τ_k crosses the imaginary axis from the left to the right as τ continuously varies from a number less than τ_k to one greater than τ_k by Rouché's theorem [31].

Applying Lemma 2.1, we obtain the following results:

Lemma 2.3. *If* (H_1), (H_2) *and* (H_3) *are satisfied, then*

(i) if $\tau \in [0, \tau_0)$, *all roots Eq.* (2.3) *have negative real parts;*

(ii) if $\tau = \tau_0$, *all roots of Eq.* (2.3) *except* $\pm i\omega_0$ *have negative real parts;*

(iii) if $\tau \in [\tau_k, \tau_{k+1})$ *for* $k = 0, 1, 2, \ldots$, *Eq.* (2.3) *have 2 (k + 1) roots with positive real parts.*

Spectral properties of Eq. (2.3) immediately lead to the properties of the zero solutions to system (2.2), and equivalently, of the positive equilibrium E_* for system (1.6).

Theorem 2.4. *Suppose that* (H_1), (H_2) *and* (H_3) *are satisfied. Then, the positive equilibrium* E_* *of system* (1.6) *is asymptotically stable when* $\tau \in [0, \tau_0)$, *and unstable when* $\tau > \tau_0$. *Moreover, at* $\tau = \tau_k$, $k = 0, 1, 2, \ldots, \pm i\omega_0$ *are a simple pair imaginary roots of* (2.3), *and* (1.6) *undergoes a Hopf bifurcation near* E_*.

3. Direction and stability of the Hopf bifurcation

In the previous section, we have obtained conditions for Hopf bifurcations to occur when $\tau = \tau_k$. In this section, we will employ the algorithm of Faria and Maglhalaes [28] to compute explicitly the normal forms of system (1.6) on the center manifold. After that, we will investigate the direction of the Hopf bifurcation and stability of the bifurcating periodic orbits from the positive equilibrium E_* of system (1.6) at these critical values of τ_k. We know that Eq. (2.3) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_k$ and system (1.6) undergoes a Hopf bifurcation from

E_*. Let $\mu = \tau - \tau_k$, then $\mu = 0$ is the Hopf bifurcation value of system (1.6).

Throughout this section, we refer to Faria and Maglhalaes [28] for the meaning of the notations involved.

Normalizing the delay τ by the time-scaling $t \to t/\tau$, then the system (2.1) can be rewritten as a functional differential equation in $C\left([-1,0],R^2\right)$:

Let $u = (u_1(t), u_2(t))^T$, then, system (3.1) can be rewritten in the following vector form:

$$\begin{cases} \dfrac{du_1}{dt} = \tau[m_1u_1(t-1) + m_2u_2(t-1) + m_3u_1^2(t-1) + m_4u_2^2(t-1) \\ \quad + m_5u_1(t-1)u_2(t-1) + m_6u_1(t)u_1(t-1) + m_7u_1(t)u_2(t-1) \\ \quad + m_8u_1(t)u_1^2(t-1) + m_9u_1(t)u_2^2(t-1) + h.o.t.], \\ \dfrac{du_2}{dt} = \tau[n_1u_1(t-1) + n_2u_2(t-1) + n_3u_1^2(t-1) + n_4u_2^2(t-1) \\ \quad + n_5u_1(t-1)u_2(t-1) + n_6u_1(t-1)u_2(t) + n_7u_2(t)u_2(t-1) \\ \quad + n_8u_1(t-1)u_2^2(t-1) + n_9u_1^2(t-1)u_2(t-1) + n_{10}u_2(t)u_2^2(t-1) \\ \quad + n_{11}u_1(t-1)u_2(t)u_2(t-1) + n_{12}u_1^2(t-1)u_2(t) + h.o.t.] \end{cases} \tag{3.1}$$

$$\frac{du}{dt} = L(\tau)(u_t) + F(u_t, \tau), \tag{3.2}$$

where

$$L(\tau)(\phi) = \tau\begin{pmatrix} m_1\phi_1(-1) + m_2\phi_2(-1) \\ n_1\phi_1(-1) + n_2\phi_2(-1) \end{pmatrix},$$

$$F(\phi,\tau) = \tau\begin{pmatrix} F_1(\phi,\tau) \\ F_2(\phi,\tau) \end{pmatrix}$$

where $\phi = (\phi_1, \phi_2)^T \in C([-1,0]), R^2)$, and

$$\begin{aligned} F_1(\phi,\tau) &= m_3\phi_1^2(-1) + m_4\phi_2^2(-1) + m_5\phi_1(-1)u_2(-1) + m_6\phi_1(0)\phi_1(-1) \\ &\quad + m_7\phi_1(0)\phi_2(-1) + m_8\phi_1(0)\phi_1^2(-1) + m_9\phi_1(0)\phi_2^2(-1) + h.o.t., \\ F_2(\phi,\tau) &= n_3\phi_1^2(-1) + n_4\phi_2^2(-1) + n_5\phi_1(-1)\phi_2(-1) + n_6\phi_1(-1)\phi_2(0) \\ &\quad + n_7\phi_2(0)\phi_2(-1) + n_8\phi_1(-1)\phi_2^2(-1) + n_9\phi_1^2(-1)\phi_2(-1) \\ &\quad + n_{10}\phi_2(0)\phi_2^2(-1) + n_{11}\phi_1(-1)\phi_2(0)\phi_2(-1) + n_{12}\phi_1^2(-1)\phi_2(0) + h.o.t.. \end{aligned}$$

Obviously, $L(\tau)$ is a continuous linear function mapping $C([-1,0]), R^2)$ into R^2. By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta,\tau), -1 \le \theta \le 0$, whose elements are of bounded variation such that

$$\phi \mapsto \Phi(\Psi, \phi) \tag{3.3}$$

In fact, we can choose

$$\eta(\theta,\tau) = -\tau\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}\delta(\theta+1), \tag{3.4}$$

where δ denotes Dirac delta function. Then, (3.3) is satisfied. If ϕ is any a given function in $C([-1,0]), R^2)$ and $u(\phi)$ is the unique solution to the linearized equation $\frac{du}{dt} = L(\tau)u_t$ of Eq. (3.2) with the initial function ϕ at zero, then the solution operator $T(t) : C\left([-1,0], R^2\right) \to C([-1,0]), R^2)$ is defined by

$$T(t)\phi = u_t(\phi), t \ge 0 \tag{3.5}$$

From Lemma 7.1.1 in Hale [32], we know that $T(t), t \ge 0$, is a strongly continuous semigroup of linear transformation on $[0,\infty)$ and the infinitesimal generator $A(\tau)$ of $T(t)$, $t \ge 0$ is given by

$$A(\tau)\phi(\theta) = \dot{\phi}(\theta) + X_0(\theta)\left[L(\tau)(\phi) - \dot{\phi}(0)\right] \quad \text{for } \phi \in C([-1,0]), R^2), \tag{3.6}$$

where

$$X_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ I, & \theta = 0 \end{cases} \tag{3.7}$$

For $\psi \in C([-1,0]), \left(R^2\right)^*)$, define

$$A^*\psi(s) = -\dot{\psi}(s) + X_0(-s)\left[\int_{-1}^{0} \psi(-t)d\eta(t,\tau_k) + \dot{\psi}(0)\right] \tag{3.8}$$

and a linear inner product

$$<\psi,\phi> = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.9}$$

where $\eta(\theta) = \eta(\theta,0)$ the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$, respectively. Let $\Lambda = \{-i\omega_0\tau_k, i\omega_0\tau_k\}$ and denote by P the invariant space of $A(\tau_k)$ associated with Λ, where the dimension of P equals to 2. Now, we can decompose the space $C := C([-1,0]), R^2\big)$ as $C = P \oplus Q$ by applying the formal adjoint theory for functional differential equations in [32].

Consider the complex coordinates and still denote $C([-1,0]), R^2\big)$ as C. Suppose that $\Phi = (\Phi_1, \Phi_2)$ is a basis of P and

$$\Phi_1(\theta) = e^{i\omega_0\tau_k\theta}(1,\xi,)^T,\ \Phi_2(\theta) = \overline{\Phi_1(\theta)},\ -1 \le \theta \le 0,$$

where

$$\xi = \frac{i\omega_0\ e^{i\omega_0\tau_k} - m_1}{m_2}.$$

Also, the two eigenvectors Ψ_1, Ψ_2 of A^* corresponding to the eigenvalues $i\omega_0\tau_k$, $-i\omega_0\tau_k$, respectively, construct a basis $\Psi = (\Psi_1, \Psi_2)^T$ of the adjoint space P^* of P and

$$\Psi_1(\theta) = Ee^{-i\omega_0\tau_k s}(1,\eta)^T,\ \Psi_2(\theta) = \overline{\Psi_1(\theta)},\ 0 \le s \le 1,$$

where

$$\eta = -\frac{i\omega_0\ e^{i\omega_0\tau_k} + m_1}{n_1},$$
$$E = \frac{1}{1 + \bar{\xi}\eta + m_1\ e^{i\omega_0\tau_k} + m_2\eta\ e^{i\omega_0\tau_k} + m_2\ e^{i\omega_0\tau_k} + n_2\bar{\xi}\eta\ e^{i\omega_0\tau_k}}.$$

Thus, $(\Psi, \Phi) = \big((\Psi_j, \Phi_i), i, j = 1, 2\big) = I_2$, where I_2 is a second-order identical matrix. It is known that $\dot{\Phi} = \Phi B$, where

$$B = \begin{pmatrix} i\omega_0\tau_k & 0 \\ 0 & -i\omega_0\tau_k \end{pmatrix}.$$

Take the enlarged phase space C by considering the space $BC := \big\{\phi : [-1,0] \to C^2 | \phi$ is continuous on $[-1,0]$ and $lim_{\theta \to 0^-}\phi(\theta)$ exists$\}$. The projection $\phi \mapsto \Phi(\Psi, \phi)$ of C upon P, associated with the decomposition $C = P \oplus Q$, is now replaced by $\pi : BC \to P$ such that $\pi(\phi + X_0\alpha) = \Phi[(\Psi, \phi) + \Psi(0)\alpha]$.

Thus, we have the decomposition $BC = P \oplus Ker\pi$. Using the decomposition $u_t = \phi x(t) + y_t$, $x(t) \in C^2$, $y_t \in Ker\pi \cap C^1 = Q^1$, and by Theorem 7.6.1 in [32], we can decompose (3.2) as

$$\begin{cases} \dfrac{dx}{dt} = Bx + \psi(0)F_0(\Phi x + y, \mu), \\ \dfrac{dy}{dt} = A(\tilde{\tau})\Big|Q^1 y + (I - \pi)X_0F_0(\Phi x + y, \mu), \end{cases} \tag{3.10}$$

where $F_0(\phi, \mu) = L(\mu)(\phi) + F(\phi, \tau_k + \mu)$. In view of the Taylor expansion, we denote, respectively, $\Psi(0)F_0(\Phi x + y, \mu)$ and $(I - \pi)X_0F_0(\Phi x + y, \mu)$ as:

$$\begin{cases} \Psi(0)F_0(\Phi x + y, \mu) = \dfrac{1}{2}f_2^1(x, y, \mu) + \dfrac{1}{3}f_3^1(x, y, \mu) + \cdots, \\ (I - \pi)X_0F_0(\Phi x + y, \mu) = \dfrac{1}{2}f_2^2(x, y, \mu) + \dfrac{1}{3}f_3^2(x, y, \mu) + \cdots, \end{cases} \tag{3.11}$$

where $f_j^1(x, y, \mu)$ and $f_j^2(x, y, \mu)$ are homogeneous polynomials in (x, y, μ) of degree j, $j = 2, 3, \ldots$, with coefficients in C^2 and $Ker\pi$, respectively. The normal form method gives a normal form on the center manifold of the origin for (3.10) as

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x, 0, \mu) + \frac{1}{3}g_3^1(x, 0, \mu) + \cdots, \tag{3.12}$$

where $g_j^1(x, 0, \mu)$ are homogeneous polynomials in (x, μ) of degree j, $j = 2, 3, \ldots$

In what follows, we first define the operators M_j^1 as

$$M_j^1(p)(x, \mu) = D_x p(x, \mu)Bx - Bp(x, \mu),\ j \ge 2 \tag{3.13}$$

In particular, $M_j^1\big(\mu^l x^q e_k\big) = i\omega_0\tau_k\Big(q_1 - q_2 + (-1)^k\Big)\mu^l x^q e_k$, $l + q_1 + q_2 = j, k = 1, 2$ for $j = 2, 3$, $q = (q_1, q_2) \in N_0^2$, $l \in N_0$ and $\{e_1, e_2\}$ is the canonical basis for C^2. Therefore, we have

$$\mathrm{Ker}(M_2^1) = \mathrm{span}\left\{\begin{pmatrix} x_1\mu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu \end{pmatrix}\right\},$$

$$\mathrm{Ker}(M_3^1) = \mathrm{span}\left\{\begin{pmatrix} x_1^2x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\mu^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu^2 \end{pmatrix}\right\}$$

By (3.10), we have

$$f_2^1(x, y, \mu) = 2\Psi(0)[L(\mu)(\Phi x + y) + F_2(\Phi x + y, \tau_k)] \tag{3.14}$$

Noting that $L(\mu) = \mu / \tau_k L(\tau_k)$, we have

$$f_2^1(x, 0, u) = \begin{pmatrix} 2A_1x_1\mu + 2A_1\mu x_2 + a_{20}x_1^2 + 2a_{11}x_1x_2 + a_{02}x_2^2 \\ 2\bar{A}_1x_1\mu + 2\bar{A}_2x_2\mu x_2 + \bar{a}_{20}x_1^2 + 2\bar{a}_{11}x_1x_2 + \bar{a}_{02}x_2^2 \end{pmatrix} \tag{3.15}$$

where

$$\begin{aligned} A_1 &= E\Big[m_1\ e^{-i\omega_0\tau_k} + m_2\ e^{-i\omega_0\tau_k}\xi + \eta\Big(n_1\ e^{-i\omega_0\tau_k} + n_2\ e^{-i\omega_0\tau_k}\xi\Big)\Big], \\ A_2 &= E\Big[m_1\ e^{i\omega_0\tau_k} + m_2\ e^{i\omega_0\tau_k}\bar{\xi} + \eta\Big(n_1\ e^{i\omega_0\tau_k} + n_2\ e^{i\omega_0\tau_k}\bar{\xi}\Big)\Big], \\ a_{20} &= 2E\tau_k\Big[m_3\ e^{-2i\omega_0\tau_k} + m_4\ e^{-2i\omega_0\tau_k}\xi^2 + m_5\ e^{-2i\omega_0\tau_k}\xi + m_6\ e^{-i\omega_0\tau_k} + m_7\ e^{-i\omega_0\tau_k}\xi\Big], \\ a_{11} &= 2E\tau_k\Big[m_3 + m_4|\xi|^2 + m_5Re\{\xi\} + m_6Re\Big\{e^{i\omega_0\tau_k}\Big\} + m_7\ e^{i\omega_0\tau_k}\bar{\xi}\Big], \\ a_{02} &= 2E\tau_k\Big[m_3\ e^{2i\omega_0\tau_k} + m_4\ e^{2i\omega_0\tau_k} + m_5\ e^{2i\omega_0\tau_k}\bar{\xi} + m_6\ e^{i\omega_0\tau_k} + m_7\ e^{i\omega_0\tau_k}\bar{\xi}\Big] \end{aligned} \tag{3.16}$$

Since the second-order terms in (μ, x) on the center manifold are given by

$$\frac{1}{2}g_2^1(x, 0, \mu) = \frac{1}{2}\mathrm{Proj}_{\mathrm{Ker}(M_2^1)} f_2^1(x, 0, \mu), \tag{3.17}$$

it follows that

$$\frac{1}{2}g_2^1(x, 0, \mu) = \begin{pmatrix} A_1x_1\mu \\ \bar{A}_1x_2\mu \end{pmatrix}, \tag{3.18}$$

where A_1 defined by (3.16).

In the following, we shall compute the cubic term $g_3^1(x,0,\mu)$. First, we note that

$$g_3^1(x,0,\mu)\in Ker(M_3^1) = \mathrm{Span}\left\{\begin{pmatrix} x_1^2x_2 \\ 0\end{pmatrix},\begin{pmatrix} x_1\mu^2 \\ 0\end{pmatrix},\begin{pmatrix} 0 \\ x_1x_2^2\end{pmatrix},\begin{pmatrix} 0 \\ x_2\mu^2\end{pmatrix}\right\}$$

However, the terms $O(0|x|\mu^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, it is only needed to compute the coefficients of $\begin{pmatrix} x_1^2x_2 \\ 0\end{pmatrix}$ and $\begin{pmatrix} 0 \\ x_1x_2^2\end{pmatrix}$. Let

$$s := \mathrm{span}\left\{\begin{pmatrix} x_1\mu \\ 0\end{pmatrix},\begin{pmatrix} 0 \\ x_2\mu\end{pmatrix}\right\},$$

then we have

$$\frac{1}{3}g_3^1(x,0,\mu) = \frac{1}{3!}\mathrm{Proj}_{Ker(M_3^1)}\overline{f}_3^1(x,0,\mu) = \frac{1}{3!}\mathrm{Proj}_s\overline{f}_3^1(x,0,0)+O(0|x|\mu^2),$$

where

$$\overline{f}_3^1(x,0,0) = f_3^1(x,0,0)+\frac{3}{2}\left[(D_xf_2^1)U_2^1-(D_xU_2^1g_2^1)\right]_{(x,0,0)} +\frac{3}{2}\left[(D_yf_2^1)h\right]_{(x,0,0)}$$

is the third-order term of the equation, which is obtained after computing the second-order terms of the normal form, $U_2^1(x,0)$ is the solution to equation $M_2^1U_2^1(x,0)=f_2^1(x,0,0)$, and $h=(h_1,\ h_2)^{\mathrm{T}}$ is a second homogeneous polynomial in (x_1,x_1,μ) with coefficients in Q^1.

From (3.1) and (3.2) and from (3.18) we can easily see that $g_2^1(x,0,0)=0$ and

$$\mathrm{Proj}_s f_3^1(x,0,0) = \begin{pmatrix} 3a_{21}x_1^2x_2 \\ 3\bar{a}_{21}x_1x_2^2\end{pmatrix},$$

where

$$a_{21} = E\left(2m_8+2m_9|\xi|^2\right)+E\eta\left(2n_8|\xi|^2\ \mathrm{e}^{-\mathrm{i}\omega_0\tau_k}+2n_9\xi\ \mathrm{e}^{-\mathrm{i}\omega_0\tau_k} +2n_{10}|\xi|^2\xi+n_{11}\left(|\xi|^2\ \mathrm{e}^{-\mathrm{i}\omega_0\tau_k}+\xi^2+\bar{\xi}\right)+2n_{12}\xi\right.$$

Therefore, we have $[(D_xU_2^1)g_2^1]_{(x,0,0)}=0$. In the sequel, we only need to compute $U_2^1(x,0)$ and $h(x)(\theta)$.

From (3.15), we have:

$$f_2^1(x,0,0) = \begin{pmatrix} a_{20}x_1^2+2a_{11}x_1x_2+a_{02}x_2^2 \\ \bar{a}_{02}x_1^2+2\bar{a}_{11}x_1x_2+\bar{a}_{20}x_2^2\end{pmatrix}$$

In view of the definition of M_2^1, the equation $M_2^1U_2^1(x,0)=f_2^1(x,0,0)$ can be written as the following partial differential equations:

$$\begin{cases} x_1\dfrac{\partial u_1}{\partial x_1}-x_2\dfrac{\partial u_1}{\partial x_2}-u_1 = \dfrac{1}{i\omega_0\tau_k}(a_{20}x_1^2+2a_{11}x_1x_2+a_{02}x_2^2), \\ x_1\dfrac{\partial u_2}{\partial x_1}-x_2\dfrac{\partial u_2}{\partial x_2}+u_2 = \dfrac{1}{i\omega_0\tau_k}(\bar{a}_{02}x_1^2+2\bar{a}_{11}x_1x_2+\bar{a}_{20}x_2^2)\end{cases} \tag{3.19}$$

From (3.19), we can easily obtain:

$$U_2^1(x,0) = \begin{pmatrix} \dfrac{1}{\mathrm{i}\omega_0\tau_k}\left(a_{20}x_1^2-2a_{11}x_1x_2-\dfrac{1}{3}a_{02}x_2^2\right) \\ \dfrac{1}{\mathrm{i}\omega_0\tau_k}\left(\dfrac{1}{3}\bar{a}_{02}x_1^2+2\bar{a}_{11}x_1x_2-\bar{a}_{20}x_2^2\right)\end{pmatrix}$$

Thus, we obtain:

$$\mathrm{Proj}_s\left[(D_xf_2^1)U_2^1\right]_{(x,0,0)} = \begin{pmatrix} \dfrac{2\,\mathrm{i}}{\omega_0\tau_k}\left(a_{20}a_{11}-2|a_{11}|^2-\dfrac{1}{3}|a_{02}|^2\right)x_1^2x_2) \\ \dfrac{-2\,\mathrm{i}}{\omega_0\tau_k}\left(\bar{a}_{20}\bar{a}_{11}-2|a_{11}|^2-\dfrac{1}{3}|a_{02}|^2\right)x_1^2x_2)\end{pmatrix}$$

In what follows, we shall compute $\mathrm{Proj}_s[D_xf_2^1)h]_{(x,0,0)}$. From (3.14), we know

$$f_2^1(x,y,0) = 2\Psi(0)F(\Phi x+y,\tau_k) = 2\tau_k\begin{pmatrix} E(m_3e_1^2+m_4e_2^2+m_5e_1e_2+m_6e_1e_3+m_7e_2e_3) \\ +E\eta(n_3e_1^2+n_5e_2^2+n_6e_1e_4+n_7e_2e_4) \\ \bar{E}(m_3e_1^2+m_4e_2^2+m_5e_1e_2+m_6e_1e_3+m_7e_2e_3) \\ +\bar{E}\bar{\eta}(n_3e_1^2+n_5e_2^2+n_6e_1e_4+n_7e_2e_4)\end{pmatrix}, \tag{3.20}$$

where

$$e_1 = \mathrm{e}^{-\mathrm{i}\omega_0\tau_k}x_1+\mathrm{e}^{\mathrm{i}\omega_0\tau_k}x_2+y_1(-1),$$
$$e_2 = \mathrm{e}^{-\mathrm{i}\omega_0\tau_k}\xi x_1+\mathrm{e}^{\mathrm{i}\omega_0\tau_k}\bar{\xi}x_2+y_2(-1),$$
$$e_3 = x_1+x_2+y_1(0),$$
$$e_4 = x_1+x_2+y_2(0)$$

since $h=(h^{(1)},\ h^{(2)})^{\mathrm{T}}$ is a second-order homogeneous polynomial in (x_1,x_2,μ) with coefficients in Q^1. Hence, we can let:

$$h = h_{110}x_1x_2+h_{101}x_1\mu+h_{011}x_2\mu+h_{200}x_1^2+h_{020}x_2^2 +h_{002}\mu^2 \tag{3.21}$$

Thus, from (3.20), we obtain

$$[D_yf_2^1)h]_{(x,0,0,)} = 2\tau_k\begin{pmatrix} K_1 \\ K_2\end{pmatrix},$$

where

$$K_1 = E\Big[2m_3e_1^0h^{(1)}(-1) + m_5e_2^0h^{(1)}(-1) + e_1^0h^{(1)}(0)\Big] + m_7e_2^0h^{(1)}(0)] + E\eta\Big[2n_3e_1^0h^{(1)}(-1) + n_6e_4^0h^{(1)}(-1)\Big] + E\Big[2m_4e_2^0h^{(2)}(-1), + m_5e_1^0h^{(2)}(-1) + m_7e_3^0h^{(2)}(-1)] + E\eta\Big[2n_5e_2^0h^{(2)}(-1) + n_6e_1^0h^{(2)}(0) + n_7\Big(e_4^0h^{(2)}(-1) + e_2^0h^{(2)}(0)\Big)\Big],$$

$$K_2 = \bar{E}\Big[2m_3e_1^0h^{(1)}(-1) + m_5\Big(e_2^0h^{(1)}(-1) + e_1^0h^{(1)}(-1)\Big) + m_6e_3^0h^{(1)}(-1) + m_7e_2^0h^{(1)}(0)\Big] + \bar{E}\bar{\eta}\Big[2n_3e_1^0h^{(1)}(-1) + n_6e_4^0h^{(1)}(-1)\Big] + \bar{E}\Big[2m_4e_2^0h^{(2)}(-1) + m_5e_1^0h^{(2)}(-1) + m_7e_3^0h^{(2)}(-1)\Big] + \bar{E}\bar{\eta}\Big[2n_5e_2^0h^{(2)}(-1) + n_6e_1^0h^{(2)}(0) + n_7(e_4^0h^{(2)}(-1) + e_2^0h^{(2)}(0)\Big],$$

where

$$e_1^0 = e^{-i\omega_0\tau_k}x_1 + e^{i\omega_0\tau_k}x_2, e_2^0 = e^{-i\omega_0\tau_k}\xi x_1 + e^{i\omega_0\tau_k}\bar{\xi}x_2,$$
$$e_3^0 = x_1 + x_2, e_4^0 = x_1 + x_2$$

Therefore,

$$[D_y f_2^1)h]_{(x,0,0,)} = \begin{pmatrix} 2B_3x_1^2x_2 \\ 2\bar{B}_3x_1x_2^2 \end{pmatrix},$$

where

$$B_3 = -mE\tau_k\Big(\xi_3h_{110}^{(1)}(0) + \bar{\xi}_3h_{200}^{(1)}(0) + h_{110}^{(3)}(0) + h_{200}^{(3)}(0)\Big)$$

Since $h_{110}(\theta)$ and $h_{200}(\theta)$ for $\theta \in [-1, 0]$ appear in B_3, we still need to compute them.

Following [28], we know that $h = h(\theta; x_1, x_2, \mu)$ is the unique solution in the linear space of homogeneous polynomials of degree 2 in 3 real variable $(x, \mu) = (x_1, x_2, \mu)$ of the equation

$$(M_2^2(h)(x,\mu) = 2(I-\pi)X_0[L(\mu)(\Phi x) + F(\Phi x, \tau_k)], \tag{3.22}$$

since

$$\Big(M_2^2(h)(x,\mu) = D_xh(x,\mu)Bx - A(\tau_k)\Big|Q^1(h(x,\mu)) \tag{3.23}$$

Combining the definition (3.6) of the operator $A(\tau)$, we can obtain:

$$D_xh(x,\mu)Bx - \dot{h}(x,\mu) - X_0(\theta)\Big[L(\tau)(h(x,\mu)) - \dot{h}(x,\mu)(0)\Big] = 2(I-\pi)X_0[L(\mu)(\Phi x) + F(\Phi x, \tau_k)]$$

For the sake of simplicity, let:

$$h = h_{110}(\theta)x_1x_2 + h_{200}(\theta)x_1^2 + h_{020}(\theta)x_2^2 \tag{3.24}$$

Then, h_0 can be evaluated by the system

$$\dot{h}_0(x) - D_xh_0(x,\mu)Bx = 2\Phi\Psi(0)[L(0)(\Phi x) + F(\Phi x, \tau_k)], \tag{3.25}$$

$$\dot{h}_0(x) - L(\tau_k)(h_0(x)) = 2[L(0)(\Phi x) + F(\Phi x, \tau_k)], \tag{3.26}$$

where $\dot{h}_0$ denote the derivative of h_0 with respect to θ.

In view of (3.14), (3.15), (3.25) and (3.26), we know that $h_{110} = (h_{110}^1, h_{110}^2)^T$ and $h_{200} = (h_{200}^1, h_{200}^2)^T$ are the solution to the following two equations:

$$\begin{cases} \dot{h}_{110} = 2(a_{11}\Phi_1 + \bar{a}_{11}\Phi_2), \\ \dot{h}_{110}(0) - L(\tau_k)(h_0(x)) = 4\tau_k\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \end{cases} \tag{3.27}$$

$$\begin{cases} \dot{h}_{200} - 2i\omega_0\tau_kh_{200} = a_{20}\Phi_1 + \bar{a}_{02}\Phi_2), \\ \dot{h}_{200}(0) - L(\tau_k)(h_{200}) = \tau_k\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \end{cases} \tag{3.28}$$

respectively, where

$$p_1 = m_3 + m_4|\xi|^2 + m_5\,\mathrm{Re}\{\xi\} + m_6\,\mathrm{Re}\Big\{e^{i\omega_0\tau_k}\Big\} + m_7\,\mathrm{Re}\Big\{\xi\ e^{-i\omega_0\tau_k}\Big\}$$
$$p_2 = n_3 + n_4|\xi|^2 + n_5\,\mathrm{Re}\{\xi\} + n_6\,\mathrm{Re}\Big\{\xi\ e^{-i\omega_0\tau_k}\Big\} + n_7\,\mathrm{Re}\Big\{\xi\ e^{-i\omega_0\tau_k}\Big\},$$
$$q_1 = m_3\ e^{-2i\omega_0\tau_k} + m_4\ e^{-2i\omega_0\tau_k}\xi^2 + m_5\xi + m_6e^{-i\omega_0\tau_k} + m_7\xi\ e^{-i\omega_0\tau_k},$$
$$q_2 = n_3\ e^{-2i\omega_0\tau_k} + n_4\ e^{-2i\omega_0\tau_k}\xi^2 + n_5\xi + n_6\xi\ e^{-i\omega_0\tau_k} + n_7\xi\ e^{-i\omega_0\tau_k}$$

Solving the Eq. (3.27) and (3.28), we have

$$\begin{cases} h_{110}(\theta) = \dfrac{2}{i\omega_0\tau_k}(a_{11}\Phi_1 - \bar{a}_{11}\Phi_2) + C, \\ h_{200}(\theta) = -\dfrac{a_{20}}{i\omega_0\tau_k}\Phi_1 - \dfrac{\bar{a}_{20}}{3i\omega_0\tau_k}\Phi_2 + D\ e^{2i\omega_0\tau_k}, \end{cases} \tag{3.29}$$

where $C = (c_1, c_2)^T$, $D = (d_1, d_2)^T$ are the solution to the following linear algebra equations:

$$\begin{pmatrix} m_1\ e^{-i\omega_0\tau_k} & m_2\ e^{-i\omega_0\tau_k} \\ n_1\ e^{-i\omega_0\tau_k} & n_2\ e^{-i\omega_0\tau_k} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -4\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{3.30}$$

$$\begin{pmatrix} 2i\omega_0 - m_1\ e^{2i\omega_0\tau_k} & -m_2\ e^{2i\omega_0\tau_k} \\ -n_1\ e^{2i\omega_0\tau_k} & 2i\omega_0 - m_2\ e^{2i\omega_0\tau_k} \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{3.31}$$

respectively. We know that

$$\frac{1}{3}g_3^1(x,0,0)=\begin{pmatrix}A_3x_1^2x_2\\ \bar{A}_3x_1x_2^2\end{pmatrix}, \tag{3.32}$$

where

$$A_3=\frac{\mathrm{i}}{2\omega_0\tau_k}\left(a_{20}a_{11}-2|a_{11}|^2-\frac{1}{3}|a_{02}|^2\right)-\frac{1}{2}B_3 \tag{3.33}$$

Accordingly, the normal form (3.12) of (3.10) has the following form

$$=Bx+\begin{pmatrix}A_1x_1\mu\\ \bar{A}_1x_2\mu\end{pmatrix}+\begin{pmatrix}A_3x_1^2x_2\\ \bar{A}_3x_1x_2^2\end{pmatrix}+O\left(|x|\mu^2+|x|^4\right)$$

The normal form (3.12) relative to P can be written in real coordinates (ω_1, ω_2) through the change of variables $x_1=\omega_1-\mathrm{i}\omega_2$, $x_2=\omega_1+\mathrm{i}\omega_2$. Setting $\omega_1=\rho\cos\nu$, $\omega_2=\rho\sin\nu$, then this form becomes

$$\begin{cases}\dot{\rho}=k_1\mu\rho+k_2\rho^3+O\left(\mu^2\rho+|(\rho,\mu)|^4\right),\\ \dot{\nu}=-\sigma_k+O(|(\rho,\mu)|),\end{cases} \tag{3.34}$$

where $k_1=\mathrm{Re}\{A_1\}, k_2=\mathrm{Re}\{A_3\}$. Following [33], we know that the sign of k_1k_2 determines the direction of the Hopf bifurcation and the sign of k_2 determines the stability of the nontrivial periodic solution bifurcating from the Hopf bifurcation. The Hopf bifurcation is supercritical when $k_1k_2<0$ and subcritical if $k_1k_2>0$. In addition, the bifurcating periodic solution on the center manifold is stable provided that $k_2<0$ and unstable if $k_2>0$.

Summarizing the above analysis, we have the following result.

Theorem 3.1. *The flow of Eq. (3.2) with $\mu=0$ on the center manifold of the origin is given by (3.34). Hopf bifurcation is supercritical if $k_1k_2<0$ and subcritical if $k_1k_2>0$. Moreover, the nontrivial periodic solution is stable if $k_2<0$ and unstable if $k_2>0$.*

4. Numerical examples

In this section, we present some numerical results for some particular values of the parameters associated with the model system (1.6). We consider the system (1.6) with $a=1$, $b=2$, $c=0.3$, $m=0.5$, $d=0.8$, $r=2$. That is,

$$\begin{cases}\dfrac{dx_1}{dt}=x_1\left[1-2x_1(t-\tau)-\dfrac{0.3x_2(t-\tau)}{0.5x_2(t-\tau)+x_1(t-\tau)}\right],\\ \dfrac{dx_2}{dt}=x_2\left[-0.8+\dfrac{2x_1(t-\tau)}{0.5x_2(t-\tau)+x_1(t-\tau)}\right],\end{cases} \tag{4.35}$$

which has a positive equilibrium $E_*=(0.32, 0.96)$. By means of software Matlab, we obtain:

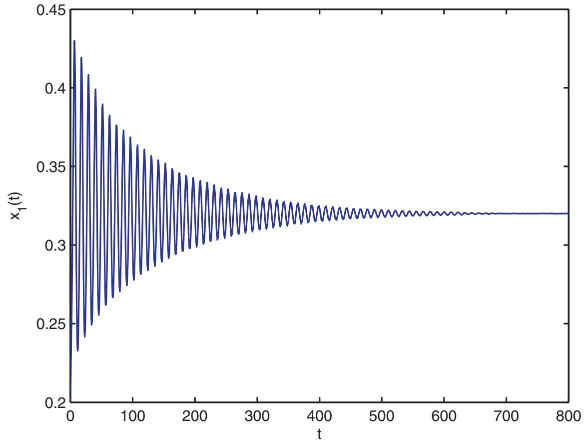

Fig. 1. Trajectory portrait and phase portrait of system (4.35) with $\tau=1.9<\tau_0\approx 1.94$. The positive equilibrium $E_*(0.32, 0.96)$ is asymptotically stable. The initial value is (0.2, 1.5).

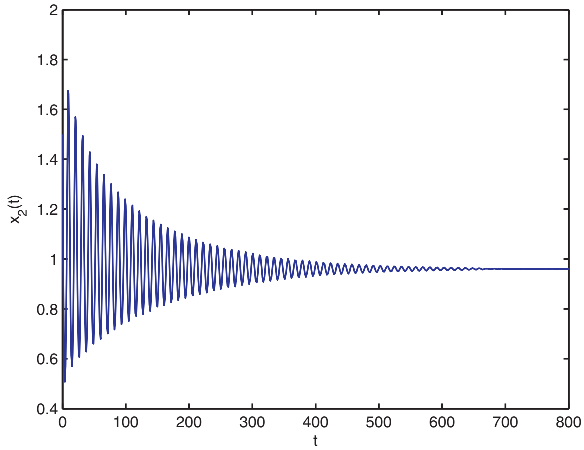

Fig. 2. Trajectory portrait and phase portrait of system (4.35) with $\tau=1.9<\tau_0\approx 1.94$. The positive equilibrium $E_*(0.32, 0.96)$ is asymptotically stable. The initial value is (0.2, 1.5).

Then, the stability determining quantities for Hopf bifurcating periodic solutions are given by $k_1=0.1912$, $k_2=-0.1377$. From Theorem 2.4, we know that the positive equilibrium $E_*(0.32, 0.96)$ is asymptotically stable when $\tau\in[0, 1.94)$ and is unstable when $\tau>1.94$. The numerical simulations for $\tau=1.9$ and $\tau=2$ are shown on Figs. 1–8, respectively. From Theorem 3.1, we know that system (4.35) with $\tau=1.94$ has a supercritical Hopf bifurcation and the nontrivial periodic solution bifurcating from the Hopf

$$\begin{aligned}
&\omega_0=0.9254, \tau_0=1.940, \xi=10.2032-5.3126\,\mathrm{i}, \eta=0.1073+0.2435\,\mathrm{i},\\
&E=0.0107-0.04131\,\mathrm{i}, A_1=0.1912+0.3103\,\mathrm{i}, A_2=-0.2956-0.2391\,\mathrm{i},\\
&a_{20}=0.7443-0.3476\,\mathrm{i}, a_{11}=0.0252-0.0395\,\mathrm{i}, a_{02}=-0.7005-0.4669\,\mathrm{i},\\
&c_1=1.0411+0.0012\,\mathrm{i}, c_2=-0.3589-0.8993\,\mathrm{i}, d_1=2.1251+0.9696\,\mathrm{i},\\
&d_2=0.4998+0.5933\,\mathrm{i}, h_{110}^{(1)}(0)=0.2013-10.0882\,\mathrm{i}, h_{200}^{(1)}(0)=2.3266+7.0852\,\mathrm{i},\\
&B_3=0.3092-0.5112\,\mathrm{i}, A_3=-0.1377-1.3126\,\mathrm{i}
\end{aligned}$$

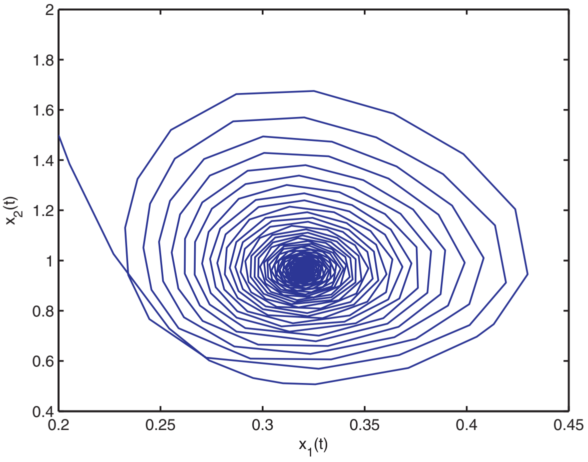

Fig. 3. Trajectory portrait and phase portrait of system (4.35) with $\tau = 1.9 < \tau_0 \approx 1.94$. The positive equilibrium $E_*(0.32, 0.96)$ is asymptotically stable. The initial value is (0.2, 1.5).

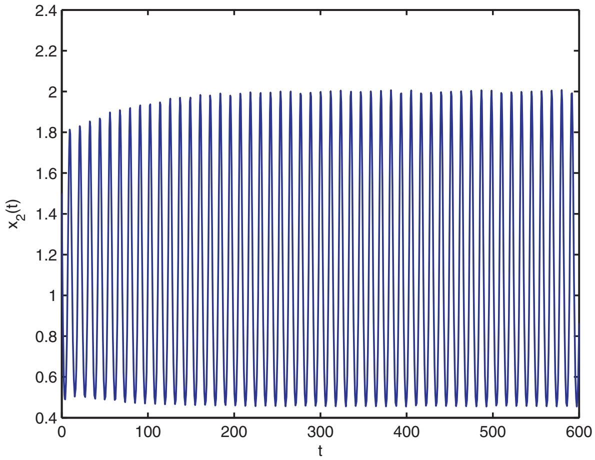

Fig. 6. Trajectory portrait and phase portrait of system (4.35) with $\tau = 2 < \tau_0 \approx 1.94$. Hopf bifurcation occurs from the positive equilibrium $E_*(0.32, 0.96)$. The initial value is (0.2, 1.5).

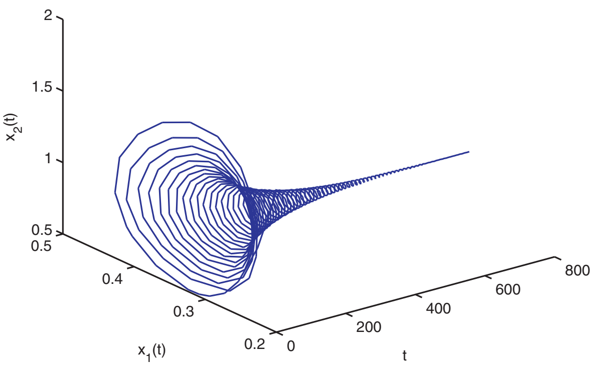

Fig. 4. Trajectory portrait and phase portrait of system (4.35) with $\tau = 1.9 < \tau_0 \approx 1.94$. The positive equilibrium $E_*(0.32, 0.96)$ is asymptotically stable. The initial value is (0.2, 1.5).

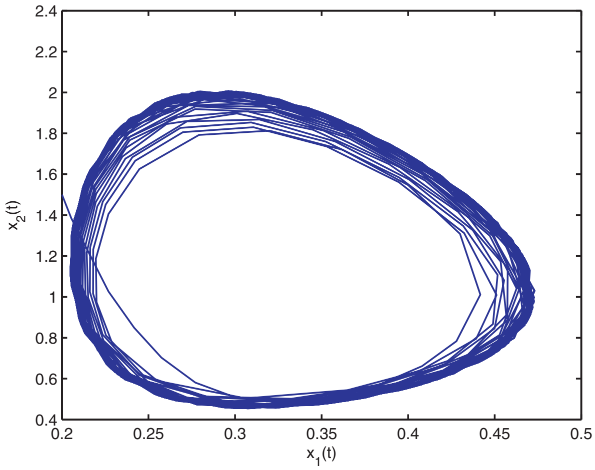

Fig. 7. Trajectory portrait and phase portrait of system (4.35) with $\tau = 2 < \tau_0 \approx 1.94$. Hopf bifurcation occurs from the positive equilibrium $E_*(0.32, 0.96)$. The initial value is (0.2, 1.5).

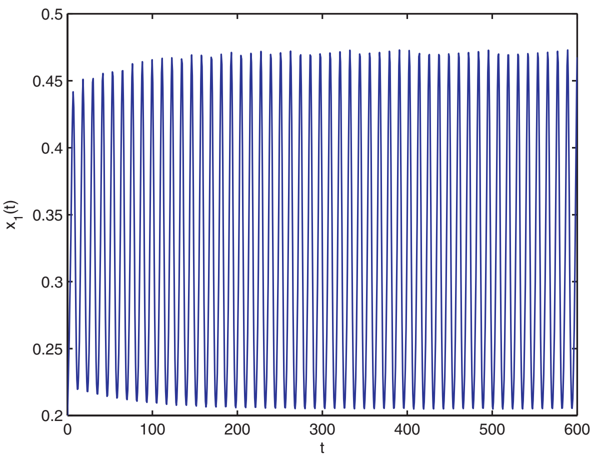

Fig. 5. Trajectory portrait and phase portrait of system (4.35) with $\tau = 2 < \tau_0 \approx 1.94$. Hopf bifurcation occurs from the positive equilibrium $E_*(0.32, 0.96)$. The initial value is (0.2, 1.5).

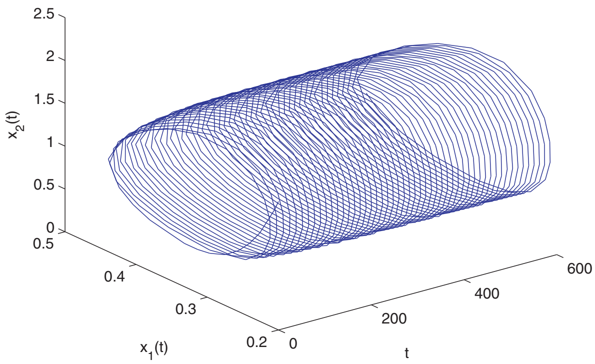

Fig. 8. Trajectory portrait and phase portrait of system (4.35) with $\tau = 2 < \tau_0 \approx 1.94$. Hopf bifurcation occurs from the positive equilibrium $E_*(0.32, 0.96)$. The initial value is (0.2, 1.5).

bifurcation of (4.35) with τ = 1.94 is orbitally asymptotically stable on the center. Moreover, all the roots of Eq. (2.3) with τ = 1.94, except $\pm 0.9254i$, have negative real parts. Thus, the center manifold theory implies that the stability of the periodic solutions projected on the center manifold coincides with the stability of the periodic solutions in the whole phase space.

5. Global continuation of local Hopf bifurcations

In the earlier sections we have established that the model system (1.6) undergoes a Hopf bifurcation at $E_*(x_1^*, x_2^*)$ for $\tau = \tau_k$ and also investigated the stability of bifurcating periodic solutions. We all know that periodic solutions can arise through the Hopf bifurcation in delay differential equations. However, these bifurcating periodic solutions are generally local, i.e., they only exist in a small neighborhood of the critical value. Therefore, we want to know that whether these nonconstant periodic solutions obtained through local Hopf bifurcations exist globally or not. In this section, we will consider the global continuation of periodic solutions bifurcating from the positive equilibrium $E_*(x_1^*, x_2^*)$ of system (1.6).

Throughout this section, we closely follow the notations in Wu [29]. For the simplification of notations, setting $z(t) = (z_1(t), z_2(t))^T = (x_1(t), x_2(t))^T$, we can rewrite system (1.6) as the following form

$$\dot{z}(t) = F(z_t, \tau, p), \tag{5.1}$$

where $z_t(\theta) = (z_{1t}(\theta), z_{2t}(\theta))^{\mathrm{T}} = (z_1(t+\theta), z_2(t+\theta)) \in \left([-\tau, 0], R^2\right)$. It is easy to see that if the condition (H_1) holds, system (5.1) has a positive equilibrium $E_*(x_1^*, x_2^*)$.

Following the work of Wu [29], we need to define:

$$X = C\left([-\tau, 0], R^2\right),$$

$$\Gamma = Cl\{(z, \tau, p) \in X \times R \times R^+; z \text{ is a nonconstant periodic solution to (5.1)}\},$$

$$N = \{(\bar{z}, \tau, p); F(\bar{z}, \tau, p) = 0\}.$$

Let $\bar{z}$ be the equilibrium of system (5.1). Then, the characteristic matrix of (5.1) at the equilibrium $\bar{z}$ takes the following form

$$\Delta(\bar{z}, \tau, p)(\lambda) = \lambda Id - DF(\bar{z}, \tau, p)\left(e^{\lambda} \cdot Id\right),$$

where Id is the identity matrix and $DF(\bar{z}, \tau, p)$ is the Fréchet derivative of F with respect to z_t evaluated at $(\bar{z}, \tau, p)$. From Wu [29], we know that $(\bar{z}, \tau, p)$ is called a center if $(\bar{z}, \tau, p) \in N$ and $\Delta(\bar{z}, \tau, p)(\lambda) = 0$. A center $(\bar{z}, \tau, p)$ is said to be isolated if it is the only center in some neighborhood of $(\bar{z}, \tau, p)$.

For the benefit of the readers, we first state the global Hopf bifurcation theory due to Wu [29] for functional differential equations.

Lemma 5.1. *Assume that $(\bar{z}, \tau, p)$ is an isolated center satisfying the hypotheses (A_1) – (A_4) in Wu [7]. Denote by $C(\bar{z}, \tau, p)$ the connected component of $(\bar{z}, \tau, p)$ in Γ. Then, either*

(i) $C(\bar{z}, \tau, p)$ is unbounded, or

(ii) $C(\bar{z}, \tau, p)$ is bounded, $C(\bar{z}, \tau, p) \cap \Gamma$ is finite and

$$\sum_{(\bar{z}, \tau, p) \in C(\bar{z}, \tau, p) \cap N} \gamma_m(\bar{z}, \tau, p) = 0$$

for all m = 1,2,..., where $\gamma_m(\bar{z}, \tau, p)$ is the m-th crossing number of $(\bar{z}, \tau, p)$ if $m \in J(\bar{z}, \tau, p)$, or it is zero if otherwise.

Obviously, if (ii) of Lemma 5.1 is not true, then $C(\bar{z}, \tau, p)$ is unbounded. Thus, if the projections of $C(\bar{z}, \tau, p)$ onto z-space and onto p-space are bounded, then the projections of $C(\bar{z}, \tau, p)$ onto τ-space is unbounded. Further, if we can show that the projections of $C(\bar{z}, \tau, p)$ onto τ-space is away from zero, then the projections of $C(\bar{z}, \tau, p)$ onto τ-space must include the interval $[\tau, \infty)$. Based on the idea, we can prove our results on the global continuation of the local Hopf bifurcation.

Lemma 5.2. *If τ is bounded, then all periodic solutions to (1.6) is uniformly bounded.*

Proof. Let $(x_1(t), x_2(t))$ be a nontrivial solution to (1.6) and define

$$x_1(\xi_1) = \min\{x_1(t)\}, x_1(\eta_1) = \max\{x_1(t)\}, x_2(\xi_2) = \min\{x_2(t)\}, x_2(\eta_2) = \max\{x_2(t)\}$$

with initial value

$$x_1(t) = \varphi_1(t) \geq 0, x_1(0) = \varphi_1(0) > 0; x_2(t) = \varphi_2(t) \geq 0, x_2(0) = \varphi_2(0) > 0,$$

where $t \in [-\tau, 0]$. Then, it follows from (1.6) that

$$\begin{cases} x_1(t) = x_1(0)\exp\left\{\int_0^t \left[a - bx_1(t-\tau) - \dfrac{cx_2(t-\tau)}{mx_2(t-\tau) + x_1(t-\tau)}\right] ds\right\}, \\ x_2(t) = x_2(0)\exp\left\{\int_0^t \left[-d + \dfrac{rx_1(t-\tau)}{mx_2(t-\tau) + x_1(t-\tau)}\right] ds\right\} \end{cases}$$

Thus, $x_1(t) > 0, x_2(t) > 0$, which implies that solutions to (1.6) cannot cross the x-axes and y-axes. Thus, the nonconstant periodic orbits must be located in the interior of the first quadrant. If $(x_1(t), x_2(t))$ is a solution to (1.6), then it follows from the first equation of (1.6) that

$$\frac{dx_1}{dt} < x_1[a - bx_1(t-\tau)] \tag{5.2}$$

Clearly, $x_1(t) < x_1(t-\tau)e^{a\tau}$ for $t > \tau$, which implies that $x_1(t-\tau) > x_1(t)e^{-a\tau}$. This, together with (5.2), leads to

$$\frac{dx_1}{dt} < x_1[a - b\ e^{-a\tau}x_1(t)],$$

for $t > \tau$. Then, we obtain that

$$\lim_{t\to\infty} \sup x_1(t) \leq \frac{a\ e^{a\tau}}{b}$$

Thus,

$$x_1(t) \leq \frac{a\ e^{a\tau}}{b} + \varepsilon := M$$

From the second equation of (1.6), we have

$$\frac{dx_2}{dt} < (r-d)x_2,$$

which leads to

$$x_2(t) < e^{(r-d)\tau}x_2(t-\tau)$$

Then,

$$x_2(t-\tau) > e^{(d-r)\tau} x_2(t)$$

Therefore,

$$x_2(\eta_2-\tau) > e^{(d-r)\tau} x_2(\eta_2)$$

Applying the second equation of (1.6), we get

$$-d + \frac{rx_1(\eta_2-\tau)}{mx_2(\eta_2-\tau)+x_1(\eta_2-\tau)} = 0,$$

i.e.

$$\frac{rx_1(\eta_2-\tau)}{mx_2(\eta_2-\tau)+x_1(\eta_2-\tau)} = d$$

It follows that

$$\frac{rM}{m\ e^{(d-r)\tau} x_2(\eta_2)+M} > d,$$

which leads to

$$x_2(\eta_2) < \frac{(r-d)M}{dm\ e^{(d-r)\tau}} := N$$

Thus, the possible periodic solutions lying in the first quadrant of (1.6) must be uniformly bounded. This completes the proof of Lemma 5.2.

Lemma 5.3. *If $a < d$, $r < c$, then system (1.6) has nonconstant periodic solution with period τ.*

Proof. Suppose for a contradiction that system (1.6) has a nonconstant periodic solution with period τ. Then, the following ordinary differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left[a - bx_1(t) - \dfrac{cx_2(t)}{mx_2(t)+x_1(t)}\right], \\ \dfrac{dx_2}{dt} = x_2\left[-d + \dfrac{rx_1(t)}{mx_2(t)+x_1(t)}\right] \end{cases} \tag{5.3}$$

has nonconstant periodic solution. System (5.3) has a boundary equilibrium $E_0(0, 0)$ and a positive equilibrium $E_*(x_1^*, x_2^*)$, where

$$x_1^* = \frac{am^2+cd-cr}{abm}, x_2^* = \frac{(r-d)(am^2+cd-cr)}{abdm^2}$$

Note that x-axis and y-axis are the invariable manifold of system (5.3) and the orbits of system (5.3) do not intersect each other. Thus, there are no solutions crossing the coordinate axes. On the other hand, consider that if system (5.3) has a periodic solution, then there must be an equilibrium in its interior, and that $E_0(0, 0)$ is located on the coordinate axis. Thus, we can conclude that the periodic orbit of system (5.3) must lie in the first quadrant.

In the sequel, we will prove that system (5.3) has no nonconstant periodic solution in the first quadrant.

Define

$$D = \left\{(x_1, x_2) \in R^2 | 0 \le x_1 \le M, 0 \le x_2 \le N\right\}.$$

It is easy to show that D is an ultimately bounded region (or absorbing and positively invariant set) of system (5.3). Let

$$P(x,y) = x_1\left[a - bx_1(t) - \frac{cx_2(t)}{mx_2(t)+x_1(t)}\right],$$
$$Q(x,y) = x_2\left[-d + \frac{rx_1(t)}{mx_2(t)+x_1(t)}\right]$$

Then, a direct computation show that, for $(x, y) \in D$ and $a < d$, $r < c$,

$$\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y} = a - d - 2bx_1 + \frac{(r-c)mx_2^2}{(mx_2+x_1)^2} < 0$$

Thus, the Bendixson–Dulac criterion, together with the fact that D is an ultimately bounded region of (5.3), implies that (5.3) has no nontrivial periodic solutions, leading to a contradiction. Thus, the proof is complete.

Theorem 5.4. *Assume that $a < d$, $r <$ c. Let ω_0 and τ_k (k = 0,1,2,...) be defined by (2.8) and (2.9), respectively. Then, for each $\tau > \tau_k$ ($k \ge 1$), system (1.6) has at least k periodic solutions.*

Proof. Obviously, $\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ is an isolated center of (1.6). Let

$C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ denote the connected component passing through $\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ in Γ. It follows from Theorem 2.4 that $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ is nonempty. It is only necessary to prove that the projection of $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ onto τ-space is $[\bar{\tau}, \infty)$ for each $j > 0$, where $\bar{\tau} \le \tau_k$.

By Lemma 2.2 and 2.3, there exists $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda: (\tau_k - \delta, \tau_k + \delta) \to C$ such that $\det \Delta(\lambda(\tau)) = 0$, $|\lambda(\tau) - i\omega_0| < \varepsilon$ for all $\tau \in [\tau_k - \delta, \tau_k + \delta]$, and $\lambda(\tau_k) = i\omega_0, \frac{d}{d\tau}\mathrm{Re}(\lambda(\tau))|_{\tau=\tau_k} > 0$

Let $\Omega_{\varepsilon,\frac{2\pi}{\omega_0}} = \left\{(\mu, p) : 0 < \mu < \varepsilon, |p - \frac{2\pi}{\omega_0}| < \varepsilon\right\}$. It is easy to verify that on $[\tau_k - \delta, \tau_k + \delta] \times \partial\Omega_{\varepsilon,\frac{2\pi}{\omega_0}}$, $\det \Delta_{(E_*,\tau_k,p)}\left(\mu + \frac{2\pi}{p}i\right) = 0$ if and only if $\mu = 0$, $\tau = \tau_k$ and $p = \frac{2\pi}{\omega_0}$. This verifies assumption (A_4) of Wu [29]. Moreover, if we put

$$H^{\pm}\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right) = \det\left(\Delta_{(E_*,\tau_k,p)}\left(\mu + i\frac{2\pi}{p}\right)\right),$$

then, we have the crossing number of isolated centers $\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ as follows:

$$\gamma_1\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right) = \deg_B\left(H^-\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right), \Omega_{\varepsilon,\frac{2\pi}{\omega_0}}\right) - \deg_B\left(H^+\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right), \Omega_{\varepsilon,\frac{2\pi}{\omega_0}}\right) = -1$$

By Theorem 3.3 of Wu [29], we conclude that the connected component $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ through $\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ in Σ is nonempty, and

$$\sum_{(\bar{z},\tau,p) \in C(\bar{z},\tau,p)} \gamma(\bar{z}, \tau, p) < 0$$

Thus, $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ is unbounded. From (2.9), one can show that, for $k \geq 1$,

$$\tau_k = \frac{1}{\omega_0}\left[\arcsin\frac{(m_1+n_2)\sqrt{m_1n_2-m_2n_1}}{2(m_1n_2-m_2n_1)} + 2k\pi\right], k = 1, 2, \cdots$$

Hence,

$$\frac{2\pi}{\omega_0} < \tau_k \tag{5.4}$$

Now, we are in position to prove that the projection of $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ onto τ-space is $[\tau, \infty)$, where $\tau \leq \tau_k$. Clearly, It follows from the proof of Lemma 5.3 that system (1.6) with $\tau = 0$ has no nontrivial periodic solution. Hence, the projection of $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ onto τ-space is away from zero.

For the sake of contradiction, we suppose that the projection of $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ onto τ-space is bounded. This means that the projection of $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ onto τ-space is included in an interval (0, τ^*). From (5.4) and applying Lemma 5.3, we get $0 < p < \tau^*$ for $(z_t, \tau, p) \in C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$. This implies that the projection of $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ onto p-space is also bounded. Thus, combining this with Lemma 5.2, we can get that the connected component $C\left(E_*, \tau_k, \frac{2\pi}{\omega_0}\right)$ is bounded. This contradiction completes the proof.

6. Conclusions and biological explanations

In this paper, we have investigated the local stability of the positive equilibrium $E_*(x_1^*, x_2^*)$ and the local Hopf bifurcation in a delayed predator–prey model with Hassell–Varley-type functional response. We have showed that if conditions (H_1)–(H_3) hold, the positive equilibrium $E_*(x_1^*, x_2^*)$ of system (1.6) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. This shows that, in this case, the population of preys and predators will tend to stabilization and still keep stable whenever the delay parameter lies in the range $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. We have also showed that, if conditions (H_1)–(H_3) hold, as the delay τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occur at $E_*(x_1^*, x_2^*)$, i.e., a family of periodic orbits bifurcate from the positive equilibrium $E_*(x_1^*, x_2^*)$. This means that the population of preys and predators may coexist and keep in an oscillatory mode. Moreover, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. Numerical simulations supporting our theoretical results are also included. Further, sufficient conditions ensuring the existence of global Hopf bifurcation are given, i.e., if $a < d$, $r < c$, then system (1.6) has at least k periodic solutions for $\tau > \tau_k (k \geq 1)$. It is shown that the population of preys and predators still keep in an oscillatory mode near the positive equilibrium $E_*(x_1^*, x_2^*)$ for $\tau > \tau_k (k \geq 1)$.

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