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Kelvin's article on the magnetic centre and Schmidt's one on the optimal eccentric dipole, revisited

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ABSTRACT

Thomson's formulas (1872) that give the position of the magnetic centre of a magnet and the first results on the global geomagnetic field were sufficient to show that the source of this field had a disymmetry in the equatorial plane. To confirm this with Schmidt's (1934) explicit formulas giving the position of the optimal eccentric dipole, numerical calculus was necessary. To obtain their relatively simple formulas, Thomson and Schmidt used lengthy algebra that was avoidable.

In his article (Lowes, 1994), F.J. Lowes examines the numerous methods that have been proposed to maximize the dipolar character of a magnetic field. All these procedures find their origin in W. Thomson's article (Thomson, 1872). In this article, Thomson generalized to magnetism the notion, so important in mechanics, of the centre of gravity, in introducing that of the magnetic centre of a magnet. It is however the notion of optimal eccentric dipole, proposed by A. Schmidt in his paper (Schmidt, 1934), that has been the most widely used to exploit the data of the International Geomagnetic Reference Formula. The terrestrial dynamo theory leaves to this notion only an interest for the history of sciences insofar it pointed, already 80 years ago, to the E–W core dissymmetry, recently demonstrated by seismology. This property has been recently invoked (Aubert et al., 2013) to develop a theory of the geomagnetic secular variation. Let us recall shortly Thomson's method.

Let us suppose know, at a point M, on a sphere of centre O, $OM = a$, the magnetic potential of a magnet completely contained in the sphere. At a point P, $OP = R >> a$, its first two terms may be written: $V = \mathbf{D}^{\mathbf{t}}\mathbf{OP}/R^3 = a \cdot \mathrm{tr}[\mathbf{QH}]$ (1), where $\mathbf{D}$ is the dipole moment, $\mathbf{H(X)}$ the harmonic matrix: $[3\mathbf{XX^t} - (\mathbf{X^tX})\mathbf{I}]/|\mathbf{X}|^5$ with which one may associate a 5×1 vector $\mathbf{h}$, the components of which are Schmidt's semi-normalized harmonic functions. $\mathbf{Q}$ is a symmetric, null-trace matrix with which may be associated the 5×1 vector $\mathbf{q}$ formed with the coefficients g_2^m of the geomagnetic mean field. The quadrupole term is then given by: $a\mathbf{h^tq}$ Thomson, admitting there are no monopoles, evaluates the moments $\mathbf{D,Q}$ as integrals of elementary dipoles. By ingenuous displacements of the system of reference: rotation $\boldsymbol{R_1}$ bringing the geographic polar vector on the dipolar moment, translations $\mathbf{OO_1}$, $\mathbf{O_1O'}$, lastly rotation $\boldsymbol{R_2}$ around the moment, Thomson reduces the quadrupole term to a single element, in P_2^2. Not foreseeing the development of the global geomagnetic measurements, he did not explicit the components of $\boldsymbol{R_i}$, leaving free room to Schmidt's formulas for the displacement $\mathbf{OO''}$. He gives however the coordinates of the projection of the centre on the geomagnetic equatorial plane as functions of the quadrupole coefficients $ag_2^1/D\sqrt{3}$. With the data of IGRF 2010, one finds a point with coordinates 330 and 372 km, about 132°E.

On his side, Schmidt develops the potential in spherical harmonics and takes integrals of magnetic masses with null sum. He then follows Thomson's method, moving the

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origin of the coordinates system in order to minimize the quadrupole energy. The apparent simplicity of his formulas, after several pages of algebra, may seem miraculous. A few lines suffice to explain it, starting from (1).

When the origin is moved from O to O′, $\mathbf{OO'} = \mathbf{X}$, $\mathbf{OP}/R^3$ is to be replaced by: $\mathbf{O'P}/R'^3 - \mathbf{H(O'P)X}/R'^5 + \ldots$ The change in the quadrupole term may be written: $\mathbf{h}^t\mathbf{NX}$, where $\mathbf{N}$ is a 5×3 matrix, the elements of which are deduced from the components of $\mathbf{D}$ (see Annexe). The position of the optimal dipole is found by searching for the minimum of the squared mean on a sphere $R = a$, of the difference: $\mathbf{h}^t(a\mathbf{q} - \mathbf{NX})$. Taking into account the orthonormality of the surface harmonics, this quantity reduces to: $(a\mathbf{q}^t - \mathbf{X}^t\mathbf{N}^t)\cdot(a\mathbf{q} - \mathbf{NX})$. The optimal dipole is thus given by: $\mathbf{X} = a(\mathbf{N}^t\mathbf{N})^{-1}\mathbf{N}^t\mathbf{q}$. But, if $\mathbf{d}$ is the unit vector of the dipole moment, $\mathbf{N}^t\mathbf{N} = 3D^2(\mathbf{I} + \mathbf{dd}^t/3)$, with: $(\mathbf{I} - \mathbf{dd}^t/4)/3D^2$ as inverse. This explains the simplicity of Thomson and Schmidt formulas.

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Annexe 1

Calculus of the matrix N

$$\mathbf{d}^t\mathbf{H}(\mathbf{OP}) = (d_1 d_2 d_3)\begin{pmatrix} (3x^2 - R^2) & 3xy & 3xz \\ 3yx & (3y^2 - R^2) & 3yz \\ 3zx & 3zy & (3z^2 - R^2) \end{pmatrix}/R^5 = \left(\left(3z^2 - R^2\right)/2\sqrt{3}zx\ \sqrt{3}zy\ \sqrt{3}(x^2 - y^2)/2\ \sqrt{3}xy\right)\mathbf{N}/DR^5$$

with:

$$\mathbf{N} = D\sqrt{3}\begin{pmatrix} -d_1/\sqrt{3} & -d_2/\sqrt{3} & 2d_3/\sqrt{3} \\ d_3 & 0 & d_1 \\ 0 & d_3 & d_4 \\ d_1 & -d_2 & 0 \\ d_2 & d_1 & 0 \end{pmatrix},\quad \mathbf{N}^t = D\sqrt{3}\begin{pmatrix} -d_1/\sqrt{3} & d_3 & 0 & d_1 & d_2 \\ -d_2/\sqrt{3} & 0 & d_3 & -d_2 & d_1 \\ 2d_3/\sqrt{3} & d_1 & d_2 & 0 & 0 \end{pmatrix}$$

$$\mathbf{N}^t\mathbf{N} = D^2\begin{pmatrix} 3 + d_1^2 & d_1d_2 & d_1d_3 \\ d_2d_1 & 3 + d_2^2 & d_2d_3 \\ d_3d_1 & d_3d_2 & 3 + d_3^2 \end{pmatrix} = D^2\left(3\mathbf{I} + \mathbf{dd}^t\right)$$ the inverse of which is: $(\mathbf{I} - \mathbf{dd}^t/4)/3D^2$

One has: $$\mathbf{N}^t\mathbf{g} = \mathbf{L} = D\sqrt{3}\begin{pmatrix} -d_1g_0/\sqrt{3} + d_3g_1 + d_1g_2 + d_2h_2 \\ -d_2g_0/\sqrt{3} + d_3h_1 - d_2g_2 + d_1h_2 \\ 2d_3g_0/\sqrt{3} + d_1g_1 + d_2h_2 \end{pmatrix}$$ and finally, with $\mathbf{d}^t\mathbf{L} = 4s$, $\mathbf{OO'} = a(\mathbf{L} \times d\mathbf{D})/3D^2$ equivalent to Schmidt's formulas (13), his parameters q_i $(i \neq 0)$ differing from ours, g/h, by the factor $\sqrt{3}$

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