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Group Theory / *Théorie des groupes*

Influence of the number of Sylow subgroups on solvability of finite groups

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Abstract. Let G be a finite group. We prove that if the number of Sylow 3-subgroups of G is at most 7 and the number of Sylow 5-subgroups of G is at most 1455, then G is solvable. This is a strong form of a recent conjecture of Robati.

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1. Introduction

Given a prime p and a finite group G , we write $v_p(G)$ to denote the number of Sylow p -subgroups of G . The study of Sylow numbers has played an important role in group theory since the beginning of the work on the classification of finite simple groups by Burnside and others. It was proved in the main theorem of the recent paper [5], that if G is a finite non-solvable group and $v_p(G) \leq p^2 + 1$ for every prime p , then $G/K = A_5 \times N$ with $K = O_2(G) \times O_3(G) \times O_5(G)$ and $\pi(N) \cap \{2, 3, 5\} = \emptyset$. The proof of this result is unassumingly complicated and relies on the classification of finite simple groups. We start with a slightly stronger result that just depends on Feit–Thompson theorem.

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Theorem A. *Let G be a non-solvable finite group and let R be the solvable radical of G . If K is a Hall $\{2, 3, 5\}$ -subgroup of R , then the following are equivalent:*

- (1) $v_p(G) \leq p^2 + 1$ for $p \in \{2, 3, 5\}$;
- (2) K is nilpotent, normal in G and $G/K = A_5 \times R$.

Robati conjectured in [5] that “if G is a finite group and $v_p(G) \leq p^2 - p + 1$ for each odd prime p , then G is solvable”. We prove that it suffices to assume this hypothesis for the primes $p = 3$ and $p = 5$.

Theorem B. *Let G be a finite group. Assume that $v_p(G) \leq p^2 - p + 1$ for $p \in \{3, 5\}$. Then G is solvable.*

In fact, we can obtain an even stronger theorem (see Theorem 4 below). Our proof of Theorem B does not depend on the classification of finite simple groups, but it relies on Thompson's classification of minimal simple groups. Our results suggest that the Sylow numbers for small primes are relevant to determine solvability and even to determine some simple groups. For instance, we have observed that, as a consequence of Guralnick's [1] classification of maximal subgroups of prime power index of simple groups (see also [4]), the number of Sylow 2-subgroups of a simple group G is a prime p if and only if $G = \text{PSL}(2, p - 1)$, where p is a Fermat prime. We have not pursued this further, however.

2. Proofs

We start by recalling some standard group theory exercises that will sometimes be used without further explicit mention. See also [2] for a strong form of Lemma 2 (iii), that we will not need here. Lemma 1 (ii) can be checked with GAP [7].

Lemma 1. *Let G be a nonabelian finite simple group and $H < G$ with $|G : H| = n$. Then G is isomorphic to a subgroup of A_n . In particular, $n \geq 5$ and we have the following:*

- (i) if $n = 5$, then $G \cong A_5$;
- (ii) if $n = 7$, then $G \cong A_5, A_6, A_7$ or $\text{PSL}(2, 7)$;
- (iii) $v_p(G) \geq 5$ for every prime p that divides $|G|$. Moreover, if $v_p(G) = 5$ for some prime p , then $p = 2$ and $G \cong A_5$.

Lemma 2. *Let G be a finite group, and let p be a prime. The following hold:*

- (i) if $G = H \times K$, then $v_p(G) = v_p(H) \times v_p(K)$;
- (ii) if $H \leq G$, then $v_p(H) \leq v_p(G)$;
- (iii) if $N \trianglelefteq G$, then $v_p(G/N)v_p(N)$ divides $v_p(G)$;
- (iv) In particular, if $N \trianglelefteq G$ and $N \leq H \leq G$, then $v_p(H/N) \leq v_p(G)$.

Recall that if G is a finite group then the generalized Fitting subgroup of G is $F^*(G) = F(G)E(G)$, where $F(G)$ is the Fitting subgroup and $E(G)$ is the subgroup generated by the components of G . We will use that $C(F^*(G)) \leq F^*(G)$ (see [3, Theorem 6.5.8], for instance). Notice that if $F(G) = 1$, then $F^*(G) = E(G)$ coincides with the socle of G . Now, we are ready to prove Theorem A.

Proof of Theorem A. It is clear that (ii) implies (i), so we assume (i) and want to prove (ii). Let $P \in \text{Syl}_2(G)$. Let F/R be the generalized Fitting subgroup of G/R , which coincides with the socle of G/R , so it is a direct product of nonabelian simple groups. We have that

$$5 \geq v_2(G) \geq v_2(F/R)$$

and it follows from Lemma 2 (i) and Lemma 1 (iii) that $F/R = A_5$. Since $C_{G/R}(F/R) \leq F/R$, we deduce that $C_{G/R}(F/R) = 1$ so

$$A_5 = F/R \leq G/R \leq \text{Aut}(F/R) = S_5$$

and we deduce that $G/R \cong A_5$ or S_5 . In the later case, $v_2(G/R) = 15 > 5$, a contradiction. It follows that $G/R \cong A_5$. By Lemma 2 (iii), $v_p(G/R)v_p(R)$ divides $v_p(G) \leq p^2 + 1$. Since $v_p(G/R) = v_p(A_5) = p^2 + 1$ for $p = 2$ and $p = 3$, this implies that $v_2(R) = v_3(R) = 1$. Also, when $p = 5$ we have $6v_5(R)$ divides $v_5(G) \leq 26$. Since $v_5(R) \equiv 1 \pmod{5}$, this clearly implies that $v_5(R) = 1$, so R has normal Sylow 2, 3 and 5-subgroups. Hence, we have seen that K is nilpotent and normal in G .

Now we work in $\bar{G} = G/K$ and use the bar convention. Note that \bar{R} is the normal Hall $\{2, 3, 5\}'$ -subgroup of \bar{G} and let $\bar{H} \cong A_5$ be a Hall $\{2, 3, 5\}$ -subgroup of \bar{G} , which exists by the Schur-Zassenhaus theorem. It remains to see that the action of \bar{H} on \bar{R} is trivial. Recall that

$$5 = v_2(\bar{G}) = \left| \bar{G} : N_{\bar{G}}(\bar{P}) \right|$$

so $\bar{R} \leq N_{\bar{G}}(\bar{P})$. Thus \bar{P} acts trivially on \bar{R} . Since \bar{H} is generated by conjugates of \bar{P} , the result follows. □

Note that an analysis of the proof shows that we just need to assume that $v_5(G) < 36$ for this argument to work. Next, we recall Thompson's classification of *minimal simple groups*; i.e., finite non-abelian simple groups G such that all proper subgroups of G are solvable.

Theorem 3. [8, Corollary 1] *Let G be a minimal simple group. Then G is one of the following groups:*

- (1) $\text{PSL}(2, p)$ with $p \geq 7$ an odd prime and $p \equiv \pm 2 \pmod{5}$;
- (2) $\text{PSL}(2, 2^p)$ with p a prime;
- (3) $\text{PSL}(2, 3^p)$ with p an odd prime;
- (4) $\text{PSL}(3, 3)$;
- (5) $\text{Sz}(2^p)$ with p an odd prime.

In particular, if G is a minimal simple group and 3 does not divide $|G|$, then G is isomorphic to a Suzuki simple group.

We will just use the statement on minimal simple $3'$ -groups. Finally, we obtain the promised strong version of Theorem B.

Theorem 4. *Let G be a finite group. Assume that $v_3(G) \leq 7$ and $v_5(G) \leq 1455$. Then G is solvable.*

Proof. By way of contradiction, assume that G is not solvable. Let $S = K/L$ be a nonabelian composition factor of G . Assume first that 3 divides $|S|$. Then $v_3(S) = 7$ and S is isomorphic to a subgroup of A_7 . Looking at the number of Sylow 3-subgroups for the groups in Lemma 1 (ii), we obtain a contradiction.

Let $H \leq S$ be a (nonabelian) minimal simple subgroup of G . Since 3 does not divide $|H|$, using Theorem 3, we deduce that H is isomorphic to a Suzuki simple group. Now it follows from [6, Theorem 9] that the index of a Sylow 5-normalizer is, at least, 1456. In particular, $v_5(H) > 1455$. This is the final contradiction. □

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