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BV-operators and the secondary Hochschild complex

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Abstract. We introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}$ on a homotopy $G$-algebra $V^\bullet$ such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by $\Delta$ in a manner similar to the BV-formalism. As an application, we produce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra $A$ over a commutative algebra $B$. In this case, we also show that the operator $\Delta^*$ corresponds to Connes’ operator.

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1. Introduction

A Gerstenhaber algebra (see [3]) consists of a graded vector space $W^\bullet = \bigoplus_{n \geq 0} W^n$ equipped with the following two structures:

(a) A dot product $x \cdot y$ of degree zero making $W^\bullet$ into an associative graded commutative algebra.

(b) A bracket $[x, y]$ of degree $-1$ making $W^\bullet$ into a graded Lie algebra satisfying the compatibility property that

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(\deg(x)-1)\deg(y)} y \cdot [x, z].$$

Gerstenhaber algebra structures appear in a variety of situations, from Hochschild cohomology of algebras to the exterior algebra of a Lie algebra and the algebra of differential forms on a Poisson manifold.
An operator \( \partial = \{ \partial^n : W^n \rightarrow W^{n-1} \}_{n \geq 0} \) on \( W^* \) of degree \(-1\) is said to generate the Gerstenhaber bracket (see Koszul [7, §2] and also [6, Definition 3.2]) if it satisfies

\[
[x, y] = (-1)^{(\deg(x)-1)(\deg(y))} (\partial(x) \cdot y + (-1)^{\deg(x)} x \cdot \partial(y) - \partial(x \cdot y))
\]

In particular, a Batalin–Vilkovisky algebra (or BV-algebra) consists of a Gerstenhaber algebra along with a generator \( \partial \) for the bracket such that \( \partial^2 = 0 \).

In [4], [5], Gerstenhaber and Voronov introduced the notion of a homotopy G-algebra, which is a brace algebra equipped with a differential of degree 1 and a dot product of degree 0 satisfying certain conditions. In particular, the cohomology groups \( H(V^*) \) of a homotopy G-algebra \( V^* \) carry the structure of a Gerstenhaber algebra.

In this paper, we introduce the notion of a BV-operator \( \Delta = \{ \Delta^n : V^n \rightarrow V^{n-1} \}_{n \geq 0} \) on a homotopy G-algebra \( V^* \) such that the Gerstenhaber bracket on \( H(V^*) \) is determined by \( \Delta \) in a manner similar to the BV-formalism. More explicitly, for classes \( \overline{f} \in H^0(V^*) \) and \( \overline{g} \in H^m(V^*) \), we have

\[
\langle \overline{f}, \overline{g} \rangle = (-1)^{(m-1)n}(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in H^{m+n-1}(V^*)
\]

where \( f \in Z^n(V^*) \), \( g \in Z^m(V^*) \) are cocycles representing \( \overline{f} \) and \( \overline{g} \) respectively. We note that \( \Delta \) need not be a morphism of cochain complexes and therefore may not induce any operator on \( H(V^*) \). As such, \( \Delta \) may not descend to a generator for the Gerstenhaber bracket on \( H(V^*) \).

Our motivation is to introduce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra \( A \) over a commutative algebra \( B \). For a datum \((A, B, \varepsilon)\) consisting of an algebra \( A \), a commutative algebra \( B \) and an extension of rings \( \varepsilon : B \rightarrow A \) such that \( e(B) \subseteq Z(A) \), the secondary Hochschild cohomology \( H^*(A, B, \varepsilon) \) was introduced by Staic [9] in order to study deformations of algebras \( A[[t]] \) having a \( B \)-algebra structure. More generally, Staic [9] introduced the secondary Hochschild complex \( C^*((A, B, \varepsilon); M) \) with coefficients in an \( A \)-bimodule \( M \).

In [10], Staic and Stancu showed that the secondary Hochschild complex \( C^*((A, B, \varepsilon); A) \) with coefficients in \( A \) is a non-symmetric operad with multiplication, giving it the structure of a homotopy G-algebra. Hence, the secondary cohomology \( H^*(A, B, \varepsilon) \) is equipped with a graded commutative cup product and a Lie bracket which makes it a Gerstenhaber algebra. For more on the secondary cohomology, the reader may see, for instance, [1], Corrigan-Salter and Staic [2], Laubacher, Staic and Stancu [8].

Let \( k \) be a field. It is well known (see Tradler [11]) that the Hochschild cohomology of a finite dimensional \( k \)-algebra \( A \) equipped with a symmetric, non-degenerate, invariant bilinear form \( \langle \cdot, \cdot \rangle : A \times A \rightarrow k \) carries the structure of a BV-algebra. For the terms \( C^n((A, B, \varepsilon)) = \text{Hom}_k(A^n \otimes B^{\otimes n}, A) \) in the secondary Hochschild complex, we define the BV-operator \( \Delta = \sum_{i=1}^n (-1)^i \Delta_i : C^{n+1}(A, B, \varepsilon) \rightarrow C^n(A, B, \varepsilon) \) by the condition (see Section 3)

\[
\Delta_i f = \left\langle \Delta_i f \otimes \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1 \\
1 \\
\end{array} \right)
, \left( \begin{array}{c}
b_{1,1} \ b_{1,3} \ldots \ b_{1,n} \\
b_{2,1} \ b_{2,3} \ldots \ b_{2,n} \\
\vdots \\
b_{n-1,1} \ b_{n-1,3} \ldots \ b_{n-1,n} \\
b_n \\
\end{array} \right) \right) , a_{n+1}
\right]\text{,}
\]

\[
= \left\langle f \otimes \left( \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_n \\
a_{n+1} \\
\end{array} \right)
, \left( \begin{array}{c}
1 \\
1 \\
\vdots \\
1 \\
1 \\
\end{array} \right) \right) , 1 \right\rangle.
\]
We then show that the Gerstenhaber bracket on the secondary Hochschild cohomology of $(A,B,e)$ is determined by $\Lambda$ in a manner similar to the BV-formalism.

From Tradler [11], we also know that the operator $\Delta^* : C^*(A,A) \to C^{*-1}(A,A)$ on usual Hochschild cochains inducing the BV-structure on $H^*(A,A)$ corresponds to the operator $N$s on duals of Hochschild chains, where $N$ is the “norm operator” and $s$ is the “extra degeneracy” (see (16)). The isomorphism between the two complexes is induced by the $k$-module isomorphism $A^* \cong A$ determined by the non-degenerate bilinear form $\langle \cdot, \cdot \rangle : A \times A \to k$. If we pass to the cohomology and take the normalized Hochschild complex which is a quasi-isomorphic subcomplex of $C^*(A,A)$, it follows that Tradler’s $\Delta^*$ operator corresponds to Connes’ operator on Hochschild cohomology with coefficients in $A$.

However, in the case of secondary cohomology, we have mentioned that the operator $\Delta^*$ defined in (1) is not a morphism of complexes and we cannot pass to cohomology. Accordingly, we show that the operator $\Delta^*$ defined in (1) fits into a commutative diagram (see Theorem 10)

$$
\begin{array}{ccc}
\overline{C}^*(A,B,e) & \xrightarrow{B} & \overline{C}^{*-1}(A,B,e) \\
\downarrow & & \downarrow \\
C^*((A,B,e); A) & \xrightarrow{\Delta^*} & C^{*-1}((A,B,e); A)
\end{array}
$$

(2)

where $B$ is Connes’ operator. Here, $\overline{C}^*(A,B,e)$ is the normalization of the co-simplicial module $C^*(A,B,e)$ introduced by Laubacher, Staic and Stancu [8], which is used to compute the secondary Hochschild cohomology associated to the triple $(A,B,e)$. It should be noted (see [8, Remark 4.7]) that despite similar names, the complex $\overline{C}^*(A,B,e)$ cannot be expressed as a secondary Hochschild complex with coefficients in some $A$-bimodule. The vertical morphisms in (2) are induced by composing the canonical morphisms $(A \otimes B^{\otimes n})^* \to A^*$ for each $n \geq 0$, the isomorphism $A^* \cong A$ as well as the inclusion of the quasi-isomorphic subcomplex $\overline{C}^*(A,B,e) \to \overline{C}^*(A,B,e)$.

2. Main Result: BV-operator on homotopy G-algebra

We begin by recalling the notion of a homotopy $G$-algebra from [5]. A brace algebra (see [5, Definition 1]) is a graded vector space $V = \bigoplus_{n \geq 0} V^n$ with a collection of multilinear operators (braces) $x_{[1,\ldots,x_n]}$ satisfying the following conditions (with $|x|$ understood to be $x$):

1. $\deg(x_{[x_1,\ldots,x_n]}) = \deg(x) + \sum_{i=1}^n \deg(x_i) - n$
2. For homogeneous elements $x, x_1, \ldots, x_m, y_1, \ldots, y_n$, we have

$$
x_{[x_1,\ldots,x_m]} \{y_1,\ldots,y_n\} = \sum_{0 \leq i_1 \leq j_1 \leq i_2 \leq j_2 \leq \cdots \leq i_m \leq j_m \leq n} (-1)^{\varepsilon} x_{[y_1,\ldots,y_{i_1},x_1[y_{i_1+1},\ldots,x_1\{y_{j_1+1},\ldots,y_{j_1}\}],\ldots,y_{i_m}]}\{y_{j_m+1},\ldots,y_{j_m}\}
$$

where $\varepsilon = \sum_{p=1}^m |x_p| (\sum_{q=1}^p |y_q|)$ and $|x| := \deg(x) - 1$.

**Definition 1 (see [5, Definition 2]).** A homotopy $G$-algebra consists of the following data:

1. A brace algebra $V = \bigoplus_{n \geq 0} V^n$.
2. A dot product of degree zero

$$
V^m \otimes V^n \longrightarrow V^{m+n} \quad x \otimes y \longrightarrow x \cdot y
$$

for all $m, n \geq 0$.
3. A differential $d : V^* \to V^{*+1}$ of degree one making $V$ into a DG-algebra with respect to the dot product.
(4) The dot product satisfies the following compatibility conditions
\[
(x_1 \cdot x_2)\{y_1, \ldots, y_n\} = \sum_{k=0}^{n} (-1)^{e_k} (x_1\{y_1, \ldots, y_k\}) \cdot (x_2\{y_{k+1}, \ldots, y_n\})
\]
where \(e_k = |x_2| \sum_{p=1}^{k} |y_p|\) and
\[
d(x[x_1, \ldots, x_{n+1}]) - (dx)[x_1, \ldots, x_{n+1}] = (-1)^{|x|} \sum_{i=1}^{n+1} (-1)^{|x_1|+\cdots+|x_{i-1}|} x[x_1, \ldots, dx_i, \ldots, x_{n+1}]
\]
\[
\sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_{i-1}|} x[x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_{n+1}] - x[x_1, \ldots, x_n] \cdot x_{n+1}
\]

In particular, a homotopy \(G\)-algebra is equipped with a graded Lie bracket which descends to the cohomology of the corresponding cochain complex \((V^*, d)\) (see [5])
\[
\{ \cdot, \cdot \} : H^m(V^*) \otimes H^n(V^*) \rightarrow H^{m+n-1}(V^*)
\]
The dot product also descends to the cohomology and the bracket with an element becomes a graded derivation for the induced dot product on \(H(V^*) = \bigoplus_{n \geq 0} H^n(V^*)\). In other words, the cohomology \((H(V^*), \{ \cdot, \cdot \})\) of a homotopy \(G\)-algebra \(V^*\) is canonically equipped with the structure of a Gerstenhaber algebra.

We now introduce the notion of a BV-operator on a homotopy \(G\)-algebra.

**Definition 2.** Let \(V^* = \bigoplus_{n \geq 0} V^n\) be a homotopy \(G\)-algebra, let \(d : V^* \rightarrow V^{*+1}\) be its differential and let \(\{ \cdot, \cdot \} : V^* \otimes V^* \rightarrow V^{m+n-1}\) be its Lie bracket. We will say that a family \(\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}\) is a BV-operator on \(V^*\) if it satisfies
\[
[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g) \in d(V^{m+n-2})
\]
for any cocycles \(f \in Z^n(V^*), g \in Z^m(V^*)\).

If \(V^*\) is a homotopy \(G\)-algebra equipped with a BV-operator \(\Delta\), we now show that the bracket on the Gerstenhaber algebra \(H(V^*)\) is determined by \(\Delta\) in a manner similar to the BV-formalism.

**Theorem 3.** Let \(V^* = \bigoplus_{n \geq 0} V^n\) be a homotopy \(G\)-algebra equipped with a BV-operator \(\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}\). Consider \(\overline{f} \in H^n(V^*)\) and \(\overline{g} \in H^m(V^*)\) and choose cocycles \(f \in Z^n(V^*)\) and \(g \in Z^m(V^*)\) corresponding respectively to \(\overline{f}\) and \(\overline{g}\). Then, we have
\[
(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in Z^{m+n-1}(V^*)
\]
The Gerstenhaber bracket on the cohomology of \(V^*\) is now determined by
\[
[\overline{f}, \overline{g}] = (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in H^{m+n-1}(V^*)
\]
In particular, the right hand side does not depend on the choice of representatives \(f\) and \(g\).

**Proof.** We know that \(f \in Z^n(V^*)\) and \(g \in Z^m(V^*)\). Since the bracket \(\{ \cdot, \cdot \} : V^n \otimes V^m \rightarrow V^{m+n-1}\) descends to a bracket on the cohomology, it follows that \([f, g] \in Z^{m+n-1}(V^*)\). Since \(\Delta = \{\Delta^n : V^n \rightarrow V^{n-1}\}_{n \geq 0}\) is a BV-operator, it follows from Definition 2 that
\[
[f, g] - (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in d(V^{m+n-2})
\]
Let us put \(z_1 = [f, g]\) and \(z_2 = (-1)^{(n-1)m} (\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g))\). Since \(z_1 - z_2 \in d(V^{m+n-2})\), we must have \(z_1 - z_2 \in Z^{m+n-1}(V^*)\). We have already seen that \(z_1 \in Z^{m+n-1}(V^*)\). Hence, \(z_2 \in Z^{m+n-1}(V^*)\). By (4), we know that \(z_1 - z_2\) is a coboundary and hence the cohomology classes \(\overline{z_1} = \overline{z_2}\). The result is now clear. \(\square\)
3. Application: BV-operator on secondary Hochschild cohomology

Let $k$ be a field and $A$ be an algebra over $k$. Let $B$ be a commutative $k$-algebra and $\epsilon : B \to A$ be a morphism of $k$-algebras such that $\epsilon(B) \subseteq Z(A)$, where $Z(A)$ denotes the center of $A$. Let $M$ be an $A$-bimodule such that $\epsilon(b)m = me(b)$ for all $b \in B$ and $m \in M$. Following [9, §4.2], we consider the complex $(C^*((A,B,\epsilon); M), \delta^*)$ whose terms are given by

$$C^n((A,B,\epsilon); M) = \text{Hom}_k \left( A^{\otimes n} \otimes B^\otimes \frac{n(n-1)}{2}, M \right)$$

An element in $A^{\otimes n} \otimes B^\otimes \frac{n(n-1)}{2}$ will be expressed as a “tensor matrix” of the form

$$\left( \begin{array}{cccc} a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & b_{2,3} & \cdots & b_{2,n-1} & b_{2,n} \\ 1 & 1 & a_3 & \cdots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \cdots & 1 & a_n \end{array} \right)$$

where $a_i \in A$ and $b_{i,j} \in B$. The differentials

$$\delta^n : C^n((A,B,\epsilon); M) \to C^{n+1}((A,B,\epsilon); M)$$

may be described as follows

$$\delta^n(f) = a_1 \epsilon(b_{1,2}b_{1,3} \cdots b_{1,n+1}) f \left( \begin{array}{cccc} a_2 & b_{2,2} & b_{2,3} & \cdots & b_{2,n+1} \\ 1 & a_3 & b_{3,3} & \cdots & b_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & a_n & b_{n,n+1} \\ 1 & 1 & \cdots & 1 & a_{n+1} \end{array} \right)$$

$$+ \sum_{i=1}^n (-1)^i f \left( \begin{array}{cccc} a_1 & b_{1,i} & b_{1,i+1} & \cdots & b_{1,n} & b_{1,n+1} \\ 1 & a_2 & b_{2,i} & \cdots & b_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & e(b_{i,i+1})a_i a_{i+1} & b_{i,n} & b_{i,n+1}b_{i+1,n+1} \\ 1 & 1 & \cdots & \cdots & a_n & b_{n,n+1} \\ 1 & 1 & \cdots & \cdots & 1 & a_{n+1} \end{array} \right)$$

$$+ (-1)^{n+1} f \left( \begin{array}{cccc} a_1 & b_{1,2} & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \cdots & 1 \end{array} \right) \epsilon(b_{1,n+1} b_{2,n+1} \cdots b_{n,n+1}) a_{n+1}$$
for \( f \in C^n((A,B,\varepsilon); M) \), \( a_j \in A, b_{i,j} \in B \). The cohomology groups of \( (C^*((A,B,\varepsilon); M), \delta^*) \) are known as the secondary Hochschild cohomologies \( H^n((A,B,\varepsilon); M) \) of the triple \((A,B,\varepsilon)\) with coefficients in \( M \) (see [9]).

From [10, Proposition 3.1], we know that the secondary Hochschild complex \( C^*((A,B,\varepsilon); A) \) carries the structure of a homotopy \( G \)-algebra. This induces a graded Lie bracket

\[
\langle \cdot, \cdot \rangle : H^n(A, B, \varepsilon) \otimes H^n(A, B, \varepsilon) \longrightarrow H^{n+1}(A, B, \varepsilon)
\]

on the secondary cohomology. It follows (see [10, Corollary 3.2]) that the secondary cohomology \( H^*(A, B, \varepsilon) \) carries the structure of a Gerstenhaber algebra in the sense of [3].

From now onwards, we always let \( A \) be a finite dimensional \( k \)-algebra equipped with a symmetric, non-degenerate, invariant bilinear form \( \langle \cdot, \cdot \rangle : A \times A \rightarrow k \). In particular, \( \langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle \), \( \langle a_1 a_2, a_3 \rangle = \langle a_1, a_2 a_3 \rangle \) for any \( a_1, a_2, a_3 \in A \). For \( i \in \{1, \ldots, n+1\} \), we define the maps \( \Delta_i : C^{n+1}(A, B, \varepsilon) \rightarrow C^n(A, B, \varepsilon) \) as follows:

\[
\langle \Delta_i f \otimes \left( \begin{array}{cccc}
    a_1 & b_{i,1} & b_{i,2} & \ldots & b_{i,n} \\
    1 & a_2 & b_{2,3} & \ldots & b_{2,n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & 1 & \ldots & b_{n-1,n} \\
    1 & 1 & 1 & \ldots & a_n
\end{array} \right), a_{n+1} \rangle
\]

\[
= \left\langle f \otimes \left( \begin{array}{cccc}
    a_1 & b_{i,1} & b_{i,1} & \ldots & b_{i,n} \\
    1 & a_1 & b_{1,i+1} & \ldots & b_{1,n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & 1 & \ldots & a_n \\
    1 & 1 & 1 & \ldots & 1
\end{array} \right), a_{n+1} \right\rangle
\]

To clarify the above operator, let us express

\[
\otimes \left( \begin{array}{cccc}
    a_1 & b_{1,2} & b_{1,3} & \ldots & b_{1,n} \\
    1 & a_2 & b_{2,3} & \ldots & b_{2,n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & 1 & \ldots & b_{n-1,n} \\
    1 & 1 & 1 & \ldots & a_n
\end{array} \right) = \left( \begin{array}{cc}
    U(i-1) & X_{12} \\
    1 & U(n-i-1)
\end{array} \right)
\]

where \( U(k) \) is a square matrix of dimension \( k \). Then, we have

\[
\langle \Delta_i f \left( \begin{array}{cc}
    U(i-1) & X_{12} \\
    1 & U(n-i-1)
\end{array} \right), a_{n+1} \rangle = \left\langle f \left( \begin{array}{cc}
    U(n-i-1) & X_{12}^t \\
    1 & U(i)
\end{array} \right), 1 \right\rangle
\]

where \( X_{12}^t \) denotes the transpose of \( X_{12} \). The operator \( \Delta : C^{n+1}(A, B, \varepsilon) \rightarrow C^n(A, B, \varepsilon) \) is then defined as

\[
\Delta := \sum_{i=1}^{n+1} (-1)^i \Delta_i.
\]

Following [10, §3], we know that the complex \( C^*((A,B,\varepsilon) \) carries a dot product of degree 0, i.e., for \( f \in C^n(A,B,\varepsilon), g \in C^m(A,B,\varepsilon), \) we have \( f \cdot g \in C^{n+m}(A,B,\varepsilon) \). We also consider the operations

\[
\circ_1 : C^n(A,B,\varepsilon) \otimes C^m(A,B,\varepsilon) \rightarrow C^{n+m-1}(A,B,\varepsilon)
\]
and set \( f \circ g := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f \circ_i g \) as in \([10, \S 3]\). We also set

\[
\rho^1, \rho^2 : C^n(A, B, \varepsilon) \otimes C^m(A, B, \varepsilon) \to C^{n+m-1}(A, B, \varepsilon)
\]

\[
\rho^1(f \otimes g) := \sum_{i=1}^{m} (-1)^{(i+m-1)} \Delta_i(f \cdot g) \quad \rho^2(f \otimes g) := \sum_{i=m+1}^{m+n} (-1)^{(i+m-1)} \Delta_i(f \cdot g)
\]

for \( f \in C^n(A, B, \varepsilon) \), \( g \in C^m(A, B, \varepsilon) \). It is clear that \( \rho^1(f \otimes g) + \rho^2(f \otimes g) = \Delta(f \cdot g) \).

**Lemma 4.** \( \rho^1(f \otimes g) = (-1)^{nm} \rho^2(g \otimes f) \) for all \( f \in C^n(A, B, \varepsilon) \) and \( g \in C^m(A, B, \varepsilon) \).

**Proof.** This may be verified by direct computation. \( \square \)

**Lemma 5.** Let \( f \in Z^n(A, B, \varepsilon) \), \( g \in Z^m(A, B, \varepsilon) \). Then \( f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) \) is a coboundary. In fact, if we define \( H \)

\[
H = \sum_{i,j=1, i+j \leq n} (-1)^{(j-1)(m-1)+i(j+m)} \Delta_i(f \circ_j g),
\]

then,

\[
\delta H = f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g).
\]

**Proof.** We set, for \( k \geq 0, p \geq 0 \):

\[
T^k_{k+p} = \otimes \left( \begin{array}{l}
1 \\
... \\
1 ... a_{k+p}
\end{array} \right)
\]

We see that

\[
\left\langle \delta(\Delta_i(f \circ_j g)) \otimes \begin{pmatrix}
a_1 
b_{1,2} & b_{1,3} & \cdots & b_{1,n+m-1} 
1 & a_2 & b_{2,3} & \cdots & b_{2,n+m-1} 
\vdots & \vdots & \vdots & \ddots & \vdots 
1 & 1 & 1 & \cdots & b_{n+m-2,n+m-1} 
1 & 1 & 1 & \cdots & a_{n+m-1}
\end{pmatrix}, a_{n+m} \right\rangle
\]

\[
\left( \begin{array}{l}
a_{i+1} 
\vdots 
1 
\vdots 
1 
\vdots 
1 
1 
1 
1 \
\vdots 
1 
1 
1 
1 
a_i
\end{array} \right)
\]

\[
+ \sum_{\lambda=1}^{i-1} (-1)\lambda \left\langle f \right. \left( \begin{array}{l}
a_{i+1} 
\vdots 
1 
\vdots 
1 
\vdots 
1 
1 
1 
1 
\vdots 
1 
1 
1 
1 
\vdots 
1 
1 
1
\end{array} \right), 1 \right\rangle
\]
where $\alpha := \varepsilon(b_{1,2} \ldots b_{1,n+m-1})a_{n+m}a_1$, $\gamma := \varepsilon(b_{1,n+m-1} \ldots b_{n+m-2,n+m-1})a_{n+m}a_{n+m-1}$, $\beta_\lambda := \varepsilon(b_{\lambda,\lambda+1})a_\lambda a_{\lambda+1}$ for $1 \leq \lambda \leq n + m - 2$ and 

$$
\sqrt{i+j-2}^\text{i+j+m-2} := \left(\begin{array}{cccc}
\alpha & b_{i+1} & \ldots & b_{i+j-1} \\
1 & a_{i+1} & \ldots & b_{i+j-1} \\
1 & a_{i+j-1} & b_{i+1} & \ldots \\
\end{array}\right),
\end{equation}$$
We write the entire expression of (6) as

$$\left< \delta(\Delta_i(f \circ g)) \otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \ldots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \ldots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \ldots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right> = E_1 + E_2 + E_3 + E_4 + E_5 + E_6,$$

where $E_k$ denotes the $k$-th term in the expression.

We set for $i, j \geq 1$ and $i + j \leq n$,

$$A_{i,j} := (-1)^{i+1} \left< a'_i f \right>$$

$$+ E_3 + (-1)^{i+j-1} \left< f \right>$$

$$\left( \begin{array}{cccccccc} a_1 & b_{i+1,i+j-1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \ldots & 1 & \ldots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & a_{i+j-1} & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \ldots & 1 & \ldots & b_{i-1,i+j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a_{n+m} & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & \ldots & a_{n+m} \end{array} \right), 1$$

where $a'_i = a_i \varepsilon (b_{i,i+1} \ldots b_{i,n+m-1} b_{1,i} \ldots b_{1-1,j})$. We also set

$$B_{i,j} := (-1)^{i+j+m-2} \left< f \right>$$

$$\left( \begin{array}{cccccccc} a_1 & b_{i+j-2} & \prod_{k=0}^{m-2} b_{i+j+k} & \ldots & 1 & \ldots & b_{i-1,i+j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & a_{i+j-2} & \prod_{k=0}^{m-2} b_{i+j-2,i+j+k} & \ldots & 1 & \ldots & b_{i-1,i+j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a_{n+m} & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & \ldots & a_{n+m} \end{array} \right), 1$$

$$+ E_5 + E_6$$

where

$$\eta = a_{i+j-1} \varepsilon (b_{i+1,i+j-1} \ldots b_{i+j-1,n+m-1}) g \left( T_{i+j-1}^{i+j-1} \right)$$

$$\zeta = g \left( T_{i+j+m-2}^{i+j-2} \right) \varepsilon \left( \prod_{k=1}^{m-2} b_{i+j+k,i+j+m-1} \right),$$
The first term of $A_{i,j}$ and that of $C_{i,j}$ are the same modulo a sign. Using the fact that $\delta g = 0$, the third term of $A_{i,j}$ and the first term of $B_{i,j}$ add up to give $E_4$. Thus, we have

\[
\left\langle (\bar{\delta}(f \circ j \circ g)) \prod_j \left( \begin{array}{cccc}
1 & a_1 & b_{1,2} & b_{1,3} \\
1 & a_2 & b_{2,3} & b_{2,4} \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & b_{n+m-2,n+m-1} & a_{n+m-1}
\end{array} \right), a_{n+m} \right\rangle = A_{i,j} + B_{i,j} + C_{i,j} \tag{7}
\]

It may be verified that

\[
(-1)^{i+1} A_{i,j-1} + (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i-1,j}
\]

\[
= \left\langle \Delta_i((\bar{\delta} f) \circ j \circ g) \prod_j \left( \begin{array}{cccc}
1 & a_1 & b_{1,2} & b_{1,3} \\
1 & a_2 & b_{2,3} & b_{2,4} \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & b_{n+m-2,n+m-1} & a_{n+m-1}
\end{array} \right), a_{n+m} \right\rangle = 0 \tag{8}
\]

for $2 \leq i \leq n, 2 \leq j \leq n-1$ and $i+j \leq n$. The second equality in (8) uses the fact that $\delta f = 0$.

For $i, j \in \{1, \ldots, n+1\}$, we define

\[
A_{i,0} := (-1)^{i+1} \left\langle g(T_{i+1,m-1}^\top) f \prod_k \left( \begin{array}{cccc}
a_{i+m} & b_{i+m,n+m-1} & \ldots & b_{i+m,n+m-1} \\
1 & a_{i+m+1} & \ldots & b_{i+m+1,n+m-1} \\
\vdots & \ddots & \ddots & \ddots \\
1 & 1 & \ldots & a_{n+m-1}
\end{array} \right), 1 \right\rangle,
\]

\[
C_{0,j} = \left\langle f \prod_k \left( \begin{array}{cccc}
a_1 & b_{j-1,j} & b_{j-1,m} & b_{1,n+m-1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & a_{j-1} & \ldots & \ldots \\
1 & 1 & \ldots & a_{j+m}
\end{array} \right), a_{n+m}, 1 \right\rangle.
\]
and for $i \in \{1, \ldots, n\}$, define

$$B_{i,n-i+1} := (-1)^{n+m+1} \left< f \left( \begin{array}{cccccc}
   a_i & \ldots & b_{i,n-1} & \prod_{k=1}^{m} b_{i,n-1+k} & b_{1,i} & \ldots & b_{i-1,i} \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   1 & \ldots & a_{n-1} & \prod_{k=1}^{m} b_{n-1,n-1+k} & b_{1,n-1} & \ldots & b_{1,n-1} \\
   1 & \ldots & 1 & g(T_{n+m-1}^{m-1}) \cdot a_{n+m} & \prod_{k=0}^{m-1} b_{1,n+k} & \ldots & \prod_{k=0}^{m-1} b_{1,n+k} \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   1 & \ldots & \ldots & \ldots & a_{n+m} & \ldots & a_{n+m} \\
   \end{array} \right), 1 \right>.$$ 

Thus, $A_{i,j}$, $B_{i,j+1}$, $C_{i-1,j}$ are defined for all the values of $i, j$ with $i, j \geq 1$ and $i + j \leq n + 1$. Moreover, it may be verified that

$$A_{1,j} + (-1)^{m+1} B_{1,j} + (-1)^{n+1} C_{0,j}$$

$$= \left< (\delta f) \left( \begin{array}{cccccc}
   a_1 & \ldots & b_{1,j-1} & \prod_{k=0}^{m-1} b_{1,j+k} & b_{1,j+m} & \ldots & b_{1,n+m-1} & 1 \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\
   1 & \ldots & a_{1-j} & \prod_{k=0}^{m-1} b_{1-j,k+j+k} & b_{1-j,m+j} & \ldots & b_{1-j,n+m-1} & 1 \\
   1 & \ldots & 1 & g(T_{j+m-1}^{j-1}) \cdot \prod_{k=0}^{m-1} b_{j+k,j+m} & \ldots & \prod_{k=0}^{m-1} b_{j+k,n+m-1} & 1 \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\
   1 & \ldots & \ldots & \ldots & a_{n+m} & \ldots & a_{n+m} & 1 \\
   1 & \ldots & 1 & 1 & \ldots & \ldots & \ldots & a_{1-1} \\
   \end{array} \right), 1 \right> = 0$$

We also have

$$(-1)^{i+1} A_{i,0} + (-1)^{i+m} B_{i,1} + (-1)^{i+n} C_{i-1,1}$$

$$= \left< (\delta f) \left( \begin{array}{ccccccc}
   g(T_{i+m-1}^{i-1}) \cdot \prod_{k=0}^{m-1} b_{i+k,i+m} & \ldots & \prod_{k=0}^{m-1} b_{i+k,n+m-1} & 1 & \prod_{k=0}^{m-1} b_{i,i+k} & \ldots & \prod_{k=0}^{m-1} b_{i-1,i+k} \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   a_{i+m} & \ldots & b_{i+m,n+m-1} & 1 & b_{1,i+m} & \ldots & \prod_{k=0}^{m-1} b_{i-1,i+m} \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \end{array} \right), 1 \right> = 0$$

and

$$(-1)^{i+1} A_{i,n-i} + (-1)^{i+m} B_{i,n-i+1} + (-1)^{i+n} C_{i-1,n-i+1}$$

$$= \left< (\delta f) \left( \begin{array}{cccccccc}
   a_i & \ldots & b_{i,n-1} & \prod_{k=0}^{m-1} b_{i,n+k} & 1 & b_{1,i} & \ldots & b_{i-1,i} \\
   1 & a_{i+1} & \ldots & b_{i+1,n-1} & \prod_{k=0}^{m-1} b_{i+1,n+k} & 1 & b_{1,i+1} & \ldots & b_{1,i+1} \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   1 & \ldots & a_{n-1} & \prod_{k=0}^{m-1} b_{n-1,n+k} & 1 & \ldots & \ldots & b_{1,n-1} & \ldots & b_{1,n-1} \\
   1 & \ldots & \ldots & \ldots & g(T_{n+m-1}^{n-1}) & \prod_{k=0}^{m-1} b_{n+k,n+m-1} & \ldots & \prod_{k=0}^{m-1} b_{1,n+k} & \ldots & \prod_{k=0}^{m-1} b_{1,n+k} \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
   \end{array} \right), 1 \right>
Thus, we obtain

\[
0 = \sum_{1 \leq i, j \leq s, 1 \leq n, i + j \leq n + 1} (-1)^{(j-1)(m-1) + i(n + m - 1)} \left( (-1)^{i+1} A_{i,j} - (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i,j} \right) \quad (9)
\]

Rearranging the terms in the above sum, and using equation (7), we get,

\[
0 = \sum_{1 \leq i, j \leq s, 1 \leq n, i + j \leq n} (-1)^{(j-1)(m-1) + i(n + m) + 1} \left( A_{i,j} + B_{i,j} + C_{i,j} \right) - \sum_{i=1}^{n} (-1)^{(j-1)(m-1) + i(n + m) + 1} A_{i,0} - \sum_{i=1}^{n} (-1)^{(j-1)(m-1) + i(n + m) + 1} B_{i,n-i+1} - \sum_{j=1}^{n} (-1)^{(j-1)(m-1) + i(n + m) + 1} C_{0,j}
\]

Since \( \rho \) is a coboundary, applying Lemma 4, it follows from (11) that

\[
-\sum_{i=1}^{n} (-1)^{m(n+1)} \left( \rho^2(f \otimes g) \right) \left( \begin{array}{c|c} a_{1} & b_{1,2} b_{1,3} \cdots & b_{1,n,m+1} \\
1 & a_{2} & b_{2,3} \cdots & b_{2,n,m+1} & b_{n,m+2,n,m+1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & b_{n,m+2,n,m+1} \\
1 & 1 & \cdots & a_{n,m+1} \\
\end{array} \right) \left( a_{n,m} \right) = \left( -1 \right)^{m(n+1)} \left( \Delta(f) \cdot g \right) \left( \begin{array}{c|c} a_{1} & b_{1,2} b_{1,3} \cdots & b_{1,n,m+1} \\
1 & a_{2} & b_{2,3} \cdots & b_{2,n,m+1} & b_{n,m+2,n,m+1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & b_{n,m+2,n,m+1} \\
1 & 1 & \cdots & a_{n,m+1} \\
\end{array} \right) \left( a_{n,m} \right)
\]

\[
-\left( f \circ g \right) \left( \begin{array}{c|c} a_{1} & b_{1,2} b_{1,3} \cdots & b_{1,n,m+1} \\
1 & a_{2} & b_{2,3} \cdots & b_{2,n,m+1} & b_{n,m+2,n,m+1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & b_{n,m+2,n,m+1} \\
1 & 1 & \cdots & a_{n,m+1} \\
\end{array} \right) \left( a_{n,m} \right)
\]

Proposition 6. The family \( \Delta = \{ \Delta^* : C^*(A, B, \varepsilon) \to C^{*-1}(A, B, \varepsilon) \} \) determines a BV-operator on the homotopy G-algebra \( C^*(A, B, \varepsilon) \).

Proof. We consider \( f \in Z^n(A, B, \varepsilon) \) and \( g \in Z^m(A, B, \varepsilon) \). By definition (see [10, §3]), we know that

\[
[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f \in C^{m+n-1}(A, B, \varepsilon)
\]

(10)

Applying Lemma 5, we know that the cochains

\[
f \circ g - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) \]
\[
g \circ f - (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{(m-1)n} \rho^2(g \otimes f)
\]

are coboundaries. From (10), it now follows that

\[
[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) + (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{(m-1)n} \rho^2(g \otimes f)
\]

(11)

is a coboundary. Applying Lemma 4, it follows from (11) that

\[
[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(n-1)m} \rho^2(f \otimes g) + (-1)^{(m-1)n} \Delta(g) \cdot f + (-1)^{(m-1)n} \rho^2(g \otimes f)
\]

is a coboundary. Since \( \rho^1(f \otimes g) + \rho^2(f \otimes g) = \Delta(f \cdot g) \), we get

\[
[f, g] - (-1)^{(n-1)m} \Delta(f) \cdot g + (-1)^{(m-1)n} \Delta(f \cdot g) + (-1)^{(m-1)n} \Delta(g) \cdot f
\]

C. R. Mathématique, 2020, 358, no 11-12, 1239-1258
is a coboundary. Using the fact that the dot product is graded commutative, we can put \( \Delta(g) \cdot f = (-1)^{n(n-1)} f \cdot \Delta(g) \). The result is now clear. \( \square \)

**Theorem 7.** For secondary cohomology classes \( \bar{f} \in H^n(A, B, \varepsilon) \) and \( \bar{g} \in H^m(A, B, \varepsilon) \), the Gerstenhaber bracket is determined by

\[
[\bar{f}, \bar{g}] = (-1)^{(n-1)m}(\Delta(f) \cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f \cdot g)) \in H^{m+n-1}(A, B, \varepsilon)
\]

Here \( f \) and \( g \) are any cocycles representing the classes \( \bar{f} \) and \( \bar{g} \) respectively.

**Proof.** This follows directly by applying Theorem 3 and Proposition 6. \( \square \)

It is natural to ask whether the BV-operator determined by \( \Delta = [\Delta^*: C^*(A, B, \varepsilon) \to C^{*-1}(A, B, \varepsilon)] \) induces a BV-algebra structure on the secondary Hochschild cohomology \( H^*(A, B, \varepsilon) \). In the special case of \( B = k \), we are reduced to ordinary Hochschild cohomology and hence \( \Delta \) determines a BV-algebra structure on \( H^*(A, A) \). However, this is not true in general because \( \Delta^*: C^*(A, B, \varepsilon) \to C^{*-1}(A, B, \varepsilon) \) does not commute with the differentials. For instance, we take \( n = 2 \) and ask when the following diagram commutes:

\[
\begin{array}{ccc}
C^2(A, B, \varepsilon) & \xrightarrow{\delta^2} & C^3(A, B, \varepsilon) \\
\downarrow{\Delta^2} & & \downarrow{\Delta^3} \\
C^1(A, B, \varepsilon) & \xrightarrow{\delta^1} & C^2(A, B, \varepsilon)
\end{array}
\]

(12)

For any \( a_1, a_2, a_3 \in A, b_{1,2} \in B \) and \( f \in C^2(A, B, \varepsilon) \), we have

\[
\langle (-\Delta^3 \delta^2) f \rangle \left( \begin{array}{c} a_1 \\ b_{1,2} \\ a_2 \\ a_3 \end{array} \right)
= -\langle (\delta^2 f) \left( \begin{array}{c} a_1 \\ 1 \\ a_2 \\ a_3 \end{array} \right), 1 \rangle - \langle (\delta^2 f) \left( \begin{array}{c} a_2 \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle - \langle (\delta^2 f) \left( \begin{array}{c} a_3 \\ 1 \\ a_1 \\ a_2 \end{array} \right), 1 \rangle
\]

\[
= -\langle a_1 \varepsilon(b_{1,2}) f \left( \begin{array}{c} a_2 \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle + \langle f \left( \begin{array}{c} a_1 a_2 \varepsilon(b_{1,2}) \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle - \langle f \left( \begin{array}{c} a_2 \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle
\]

\[
- \langle a_2 \varepsilon(b_{1,2}) f \left( \begin{array}{c} a_3 \\ 1 \\ a_1 \\ a_2 \end{array} \right), 1 \rangle + \langle f \left( \begin{array}{c} a_2 a_3 \varepsilon(b_{1,2}) \\ 1 \\ a_1 \\ a_2 \end{array} \right), 1 \rangle - \langle f \left( \begin{array}{c} a_3 \\ 1 \\ a_1 \\ a_2 \end{array} \right), 1 \rangle + \langle f \left( \begin{array}{c} a_3 \\ 1 \\ a_1 \end{array} \right), 1 \rangle a_2 \varepsilon(b_{1,2}), 1 \rangle
\]

\[
= -\langle a_1 \varepsilon(b_{1,2}) f \left( \begin{array}{c} a_2 \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle + \langle f \left( \begin{array}{c} a_1 a_2 \varepsilon(b_{1,2}) \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle - \langle f \left( \begin{array}{c} a_2 \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle
\]

Now, using the properties of the inner product \( \langle \cdot, \cdot \rangle \) on \( A \) we see that the first term cancels with the eighth, the fourth term cancels with the ninth, the fifth term cancels with the twelfth. Thus, the above expression reduces to

\[
\langle f \left( \begin{array}{c} a_1 a_2 \varepsilon(b_{1,2}) \\ 1 \\ a_3 \\ a_1 \end{array} \right), 1 \rangle - \langle f \left( \begin{array}{c} a_1 \\ b_{1,2} \\ a_2 a_3 \\ 1 \end{array} \right), 1 \rangle + \langle f \left( \begin{array}{c} a_2 a_3 \\ b_{1,2} \\ a_1 a_2 \end{array} \right), 1 \rangle - \langle f \left( \begin{array}{c} a_2 \\ b_{1,2} \\ a_3 a_1 \end{array} \right), 1 \rangle
\]

\[
+ \langle f \left( \begin{array}{c} a_3 a_1 \\ b_{1,2} \\ a_2 \\ a_1 \end{array} \right), 1 \rangle - \langle f \left( \begin{array}{c} a_3 \\ b_{1,2} \\ a_1 a_2 \varepsilon(b_{1,2}) \\ 1 \end{array} \right), 1 \rangle
\]

(13)
On the other hand, we have

\[ \langle \delta^1 \Delta^2 f \rangle \left[ \begin{array}{c} a_1 \ b_{1,2} \\ a_2 \end{array} \right], a_3 \] = \langle a_1 \varepsilon(b_{1,2}) \Delta^2 f(a_2), a_3 \rangle - \langle \Delta^2 f(a_1 a_2 \varepsilon(b_{1,2})), a_3 \rangle + \langle \Delta^2 f(a_1 a_2 \varepsilon(b_{1,2})), a_3 \rangle = \langle \Delta^2 f(a_2), a_3 a_1 \varepsilon(b_{1,2}) \rangle - \langle \Delta^2 f(a_1 a_2 \varepsilon(b_{1,2})), a_3 \rangle + \langle \Delta^2 f(a_1), a_2 a_3 \varepsilon(b_{1,2}) \rangle

\[ = -\langle f \left( a_2 \begin{array}{c} 1 \\ a_3 a_1 \varepsilon(b_{1,2}) \end{array} \right), 1 \rangle + \langle f \left( a_1 a_2 \varepsilon(b_{1,2}) \begin{array}{c} 1 \\ a_2 \end{array} \right), 1 \rangle + \langle f \left( a_1 a_2 \varepsilon(b_{1,2}) \begin{array}{c} 1 \\ a_3 \end{array} \right), 1 \rangle \]

From the expressions in (13) and (14), it is clear that the diagram (12) does not commute in general, even if we take \( A \) to be commutative and \( B = A \).

### 4. Relation with extra degeneracy and norm operator

We continue with \( A \) being a finite dimensional \( k \)-algebra equipped with a symmetric, non-degenerate and invariant bilinear form \( \langle \cdot, \cdot \rangle : A \times A \rightarrow k \) and \( B \) a commutative \( k \)-algebra with a morphism of \( k \)-algebras \( \varepsilon : B \rightarrow A \) such that \( \varepsilon(B) \subseteq Z(A) \). In particular, the non-degenerate pairing on \( A \) induces mutually inverse isomorphisms

\[ \phi : A^* \xrightarrow{\cong} A \quad \phi^{-1} : A \xrightarrow{\cong} A^* \]  

We let \( C^*(A, M) \) denote the ordinary Hochschild complex of \( A \) with coefficients in an \( A \)-bimodule \( M \) and its cohomology by \( H^*(A, M) \). In particular, we may set \( M = A^* \) equipped with the \( A \)-bimodule structure \( (a'f) : (a''a'd') \) for \( f \in A^* = \text{Hom}(A, k) \) and \( a, a', a'' \in A \). In that case, the terms \( C^n(A, A^*) \) in the Hochschild complex \( C^*(A, A^*) \) may also be written as \( C^n(A, A^*) \cong \text{Hom}(A^{n+1}, k) \). We denote by \( \tilde{C}^*(A) \) the corresponding complex defined by setting \( \tilde{C}^n(A) := \text{Hom}(A^{n+1}, k) \) for \( n \geq 0 \).

From Tradler [11], we know that the operator \( \Delta^* : C^*(A, A) \rightarrow C^*-1(A, A) \) on Hochschild cochains inducing the BV-structure on \( H^*(A, A) \) fits into the following commutative diagram with the duals of Hochschild chains

\[ \begin{array}{ccc}
\tilde{C}^{n+1}(A) & \xrightarrow{Ns} & \tilde{C}^n(A) \\
\equiv & & \equiv \\
C^*(A, A^*) & \cong & C^*-1(A, A^*) \\
\phi^* & \equiv & \phi^{-1} \\
C^*(A, A) & \xrightarrow{\Delta^*} & C^*-1(A, A) \\
\end{array} \]  

(16)

Here, each \( \phi^* : C^*(A, A^*) \rightarrow C^*(A, A) \) is the isomorphism induced by \( \phi : A^* \xrightarrow{\cong} A \), while \( s \) and \( N \) respectively are the usual extra degeneracy and norm operators given by

\[ s : \tilde{C}^{n+1}(A) \rightarrow \tilde{C}^n(A) \quad (sf)(a_1, \ldots, a_n) := f(a_1, \ldots, a_n, 1) \]

\[ N := 1 + \lambda + \cdots + \lambda^n : \tilde{C}^n(A) \rightarrow \tilde{C}^n(A) \quad (\lambda \cdot f)(a_0, \ldots, a_n) := (-1)^n f(a_n, a_0, \ldots, a_{n-1}) \]  

(17)

If we pass to the normalized Hochschild complex which is a quasi-isomorphic subcomplex of \( C^*(A, A^*) \), then (16) induces the following commutative diagram

\[ \begin{array}{ccc}
H^*(A, A^*) & \xrightarrow{B^*} & H^{*-1}(A, A^*) \\
\phi^* & \equiv & \phi^{-1} \\
H^*(A, A) & \xrightarrow{\Delta^*} & H^{*-1}(A, A) \\
\end{array} \]  

(18)
where $B^* : H^*(A, A^*) \rightarrow H^{*+1}(A, A^*)$ is the standard Connes operator.

In the case of secondary Hochschild cohomology, we have shown in Section 3 that $\Delta^*$ is not in general a morphism of complexes, i.e., it does not descend to cohomology. We will now show that the operator $\Delta^*$ on secondary Hochschild cohomology $H^*(A, B, \epsilon)$ fits into a commutative diagram similar to (18).

In [8], Laubacher, Staic and Stancu have introduced a co-simplicial module $\tilde{\mathcal{C}}^*(A, B, \epsilon)$ which is used to compute the secondary Hochschild cohomology associated to the triple $(A, B, \epsilon)$. The terms of this co-simplicial module are given by

$$
\tilde{\mathcal{C}}^*(A, B, \epsilon) := \left\{ \text{Hom}\left( A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k) \right) \right\}_{n \geq 0}
$$

It is important to note (see [8, Remark 4.7]) that despite the similar names, the complex $\tilde{\mathcal{C}}^*(A, B, \epsilon)$ cannot be expressed as the secondary Hochschild complex of $(A, B, \epsilon)$ with coefficients in some $A$-bimodule. This is because the “coefficient module” $\text{Hom}(A \otimes B^n, k)$ appearing in (19) varies with $n$.

In addition, the cosimplicial module $\tilde{\mathcal{C}}^*(A, B, \epsilon)$ is equipped with a cyclic operator, which can be used to compute the secondary cyclic cohomology associated to the triple $(A, B, \epsilon)$. Using the isomorphisms

$$
\Psi^n : \text{Hom}\left( A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k) \right) \cong \text{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{(n+1)(n)}{2}}, k)
$$

given by

$$
(\Psi^n g) = \left( \begin{array}{cccccc}
  a_0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-1} & b_{0,n} \\
  1 & a_1 & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n} \\
  1 & 1 & a_2 & \cdots & b_{2,n-1} & b_{2,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\
  1 & 1 & 1 & \cdots & 1 & a_n
\end{array} \right)
$$

we first transfer the cyclic operator from [8] to a complex $\tilde{\mathcal{C}}^*(A, B, \epsilon)$ whose terms are given by

$$
\tilde{\mathcal{C}}^n(A, B, \epsilon) := \text{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{(n+1)(n)}{2}}, k)
$$

\textbf{Lemma 8.} For each $n \geq 0$, there is an action of the cyclic group $\mathbb{Z}_{n+1} = \langle \lambda \rangle$ on the $k$-space $\tilde{\mathcal{C}}^n(A, B, \epsilon) = \text{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{(n+1)(n)}{2}}, k)$ given by

$$
(\lambda \cdot f) = \left( \begin{array}{cccccc}
  a_0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-1} & b_{0,n} \\
  1 & a_1 & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n} \\
  1 & 1 & a_2 & \cdots & b_{2,n-1} & b_{2,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\
  1 & 1 & 1 & \cdots & 1 & a_n
\end{array} \right) = (-1)^n f
$$

for any $f \in \tilde{\mathcal{C}}^n(A, B, \epsilon)$, $a_i \in A$ and $b_{i,j} \in B$.

\textbf{Proof.} This is clear from the definition in [8, §4.2] and the isomorphisms in (20). \hfill \square
The norm operator \( N : \tilde{C}^n(A, B, \epsilon) \to \tilde{C}^{n-1}(A, B, \epsilon) \) is then defined as \( N = 1 + \lambda + \cdots + \lambda^n \). The extra degeneracy \( s : \tilde{C}^n(A, B, \epsilon) \to \tilde{C}^{n-1}(A, B, \epsilon) \) is given by

\[
(sf) \begin{pmatrix}
(a_0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-2} & b_{0,n-1} \\
1 & a_1 & b_{1,2} & \cdots & b_{1,n-2} & b_{1,n-1} \\
& & & & & \\
& & & & & \\
1 & 1 & 1 & \cdots & a_{n-2} & b_{n-2,n-1} \\
1 & 1 & 1 & \cdots & 1 & a_{n-1}
\end{pmatrix} = f \begin{pmatrix}
(a_0 & b_{0,1} & b_{0,2} & \cdots & b_{0,n-1}) \\
1 & a_1 & b_{1,2} & \cdots & b_{1,n-1} \\
& & & & \\
& & & & \\
1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\]

for any \( f \in \tilde{C}^n(A, B, \epsilon) \), \( a_i \in A \) and \( b_{i,j} \in B \).

For \( n \geq 0 \), let \( a^n : (A \otimes B^{\otimes n})^* \to A^* \) be the map defined by \( p \mapsto \varepsilon : a^n(p) = p(a \otimes 1_B \otimes \cdots \otimes 1_B) \). We denote by \( \alpha^* : \tilde{C}^n(A, B, \epsilon) \to C^*((A, B, \epsilon); A^*) \) the induced map.

We also let \( \Phi^* : C^*((A, B, \epsilon); A^*) \to C^*((A, B, \epsilon); A) \) be the map induced by the isomorphism \( \phi : A^* \to A \) and \( \Phi^* : C^*((A, B, \epsilon); A) \to C^*((A, B, \epsilon); A^*) \) be the map induced by \( \phi^{-1} \). It may also be verified that the inverse \( \Psi^n : \text{Hom}(A^{(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) \to \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \text{Hom}(A \otimes B^n, k)) \) of the map in (21) is given by

\[
(\Psi^n f) \begin{pmatrix}
(a_1 & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n} \\
1 & a_2 & \cdots & b_{2,n} & b_{2,n+1} \\
& & & & \\
& & & & \\
1 & 1 & \cdots & a_{n-1} & b_{n-1,n+1} \\
1 & 1 & \cdots & 1 & a_{n}
\end{pmatrix} = f \begin{pmatrix}
(a_1 & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n}) \\
1 & a_2 & \cdots & b_{2,n} & b_{2,n+1} \\
& & & & \\
& & & & \\
1 & 1 & \cdots & a_{n-1} & b_{n-1,n+1} \\
1 & 1 & \cdots & 1 & a_{n}
\end{pmatrix}
\]

**Proposition 9.** Let \( A \) be a finite dimensional \( k \)-algebra equipped with a symmetric, non-degenerate and invariant bilinear form \( \langle \cdot, \cdot \rangle : A \times A \to k \) and \( B \) be a commutative \( k \)-algebra with a morphism of \( k \)-algebras \( \epsilon : B \to A \) such that \( \epsilon(B) \subseteq Z(A) \). Then, the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{C}^*(A, B, \epsilon) & \xrightarrow{Ns} & \tilde{C}^{*,-1}(A, B, \epsilon) \\
\Phi^* \circ \alpha^* & \downarrow & \Phi^* \circ \alpha^* \\
C^*((A, B, \epsilon); A) & \xrightarrow{\Delta^*} & C^*((A, B, \epsilon); A)
\end{array}
\]

**Proof.** We will show that for any \( n \geq 0 \), the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}(A^{(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) & \xrightarrow{Ns} & \text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, k) \\
\Psi^n & \downarrow & \alpha^n \\
\text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, (A \otimes B^{\otimes n})^*) & \xrightarrow{\Phi^*} & \text{Hom}(A^{\otimes (n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, (A \otimes B^{\otimes (n-1)})^*) \\
\phi^n & \downarrow & \phi^n \\
\text{Hom}(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, A^*) & \xrightarrow{\Delta} & \text{Hom}(A^{\otimes (n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A^*)
\end{array}
\]
Since $\Phi^{n-1}$ is an isomorphism, it suffices to check that this diagram is commutative when composed with $\Phi^{n-1} : \text{Hom} \left( A^\otimes(n-1) \otimes B^{\otimes \frac{\nu(n-1)}{2}}, A \right) \to \text{Hom} \left( A^\otimes(n-1) \otimes B^{\otimes \frac{\nu(n-1)-2}{2}}, A^* \right)$. Let $f \in \text{Hom}(A^\otimes(n+1) \otimes B^{\otimes \frac{\nu(n+1)}{2}}, k)$. Then, for $a_{ij} \in A$ and $b_{i,j} \in B$, we have

$$
\left\langle (\Delta \circ \Phi^n \circ \mathcal{A}^n \circ \Psi^m(f)) \right\rangle \left( \begin{array}{cccccc}
 a_1 & b_{1,2} & b_{1,3} & \cdots & b_{1,n-2} & b_{1,n-1} \\
 1 & a_2 & b_{2,3} & \cdots & b_{2,n-2} & b_{2,n-1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & 1 & 1 & \cdots & a_{n-2} & b_{n-2,n-1} \\
 1 & 1 & 1 & \cdots & 1 & a_{n-1} \\
 \end{array} \right), a_n
$$

$$
= \sum_{i=1}^n (-1)^{(n-1)i} \left\langle \Phi^n \mathcal{A}^n \Psi^m(f) \right\rangle \left( \begin{array}{ccccccc}
 a_i & b_{i,1+1} & b_{i,1+2} & \cdots & b_{i,n-1} & b_{i,n-1} \\
 1 & a_{i+1} & b_{i+1,2} & \cdots & b_{i+1,n-1} & b_{i+1,n-1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & 1 & 1 & \cdots & a_{n-1} & b_{i-1,n-1} \\
 1 & 1 & 1 & \cdots & 1 & a_{n-1} \\
 \end{array} \right), 1
$$

$$
= \sum_{i=1}^n (-1)^{(n-1)i} \left\langle \mathcal{A}^n \Psi^m(f) \right\rangle \left( \begin{array}{ccccccc}
 a_i & b_{i,1+1} & b_{i,1+2} & \cdots & b_{i,n-1} & b_{i,n-1} \\
 1 & a_{i+1} & b_{i+1,2} & \cdots & b_{i+1,n-1} & b_{i+1,n-1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & 1 & 1 & \cdots & a_{n-1} & b_{i-1,n-1} \\
 1 & 1 & 1 & \cdots & 1 & a_{n-1} \\
 \end{array} \right), (1)
$$

$$
= \sum_{i=1}^n (-1)^{(n-1)i} \left\langle \mathcal{A}^n \Psi^m(f) \right\rangle \left( \begin{array}{ccccccc}
 a_i & b_{i,1+1} & b_{i,1+2} & \cdots & b_{i,n-1} & b_{i,n-1} \\
 1 & a_{i+1} & b_{i+1,2} & \cdots & b_{i+1,n-1} & b_{i+1,n-1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & 1 & 1 & \cdots & a_{n-1} & b_{i-1,n-1} \\
 1 & 1 & 1 & \cdots & 1 & a_{n-1} \\
 \end{array} \right), (1)
$$

$$
= \sum_{i=1}^n (-1)^{(n-1)i} \left\langle \mathcal{A}^n \Psi^m(f) \right\rangle \left( \begin{array}{ccccccc}
 a_i & b_{i,1+1} & b_{i,1+2} & \cdots & b_{i,n-1} & b_{i,n-1} \\
 1 & a_{i+1} & b_{i+1,2} & \cdots & b_{i+1,n-1} & b_{i+1,n-1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & 1 & 1 & \cdots & a_{n-1} & b_{i-1,n-1} \\
 1 & 1 & 1 & \cdots & 1 & a_{n-1} \\
 \end{array} \right), (1)
$$

C. R. Mathématique, 2020, 358, no 11-12, 1239-1258
\[
= \sum_{i=1}^{n} (-1)^{(n-1)i} (\Psi^m(f)) \left( \begin{array}{cccccccc}
\begin{array}{cccccccc}
(1, \ldots, 1) & (1, \ldots, 1) & \cdots & (1, \ldots, 1) \\
& & & & & & & & \vdots \\
& & & & & & & & a_{i-1}
\end{array}
\end{array} \right)
\]

On the other hand, let \( g := (Ns)(f) \). Then, we have

\[
(a^{n-1} \circ \Psi^{n-1}(g)) \left( \begin{array}{cccccccc}
\begin{array}{cccccccc}
(1, \ldots, 1) & (1, \ldots, 1) & \cdots & (1, \ldots, 1) \\
& & & & & & & & \vdots \\
& & & & & & & & a_{n-1}
\end{array}
\end{array} \right)
\]

This proves the result.
We now let $\overline{C}^*(A, B, \varepsilon)$ denote the normalized complex associated to the cosimplicial module $\overline{C}^*(A, B, \varepsilon)$ given in (19). Again, using the isomorphisms in (20), the complex $\overline{C}^*(A, B, \varepsilon)$ becomes isomorphic to the normalized complex $\overline{C}^*(A, B, \varepsilon)$ whose terms are given by

$$\overline{C}^n(A, B, \varepsilon) := \text{Ker} \left( \overline{C}^n(A, B, \varepsilon) \xrightarrow{j} \bigoplus_{j=0}^{n-1} \overline{C}^{n-1}(A, B, \varepsilon) \right)$$

where $s_j : \overline{C}^n(A, B, \varepsilon) \to \overline{C}^{n-1}(A, B, \varepsilon)$ for $0 \leq j \leq n - 1$ is the degeneracy

$$s_j(f) \begin{pmatrix} a_1 & b_{1,2} & \cdots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \cdots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \cdots & 1 & a_n \end{pmatrix} = f \begin{pmatrix} a_1 & b_{1,2} & \cdots & b_{1,j} & 1 & b_{1,j+1} & \cdots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \cdots & b_{2,j} & 1 & b_{2,j+1} & \cdots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & a_j & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & a_j+1 & \cdots & b_{j+1,n} & b_{j+1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \cdots & 1 & \cdots & 1 & a_n \end{pmatrix}$$

**Theorem 10.** Let $A$ be a finite dimensional $k$-algebra equipped with a symmetric, non-degenerate and invariant bilinear form $(\cdot, \cdot) : A \times A \to k$ and $B$ be a commutative $k$-algebra with a morphism of $k$-algebras $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:

$$\begin{array}{ccc} \overline{C}^*(A, B, \varepsilon) & \xrightarrow{B} & \overline{C}^{*-1}(A, B, \varepsilon) \\ \downarrow \overline{C}^*(A, B, \varepsilon) & & \downarrow \overline{C}^{*-1}(A, B, \varepsilon) \\ \overline{C}^*(A, B, \varepsilon) & \xrightarrow{N_s} & \overline{C}^{*-1}(A, B, \varepsilon) \\ \phi^* \circ \alpha^* & & \phi^{*-1} \circ \alpha^{*-1} \\ C^*((A, B, \varepsilon); A) & \xrightarrow{\Delta^*} & C^{*-1}((A, B, \varepsilon); A) \end{array}$$

where $B$ is Connes’ operator.

**Proof.** The commutativity of the lower square has already been shown in Proposition 9. By definition, Connes’ operator on the complex $\overline{C}^*(A, B, \varepsilon)$ is given by $B = N_s(1 - \lambda)$ which reduces to $N_s$ on the normalized complex $\overline{C}^*(A, B, \varepsilon)$. Hence, it may be directly verified that the upper diagram commutes. \qed

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**References**


