Hafedh Khalfoun and Ismail Laraiedh

The linear $n(1|N)$–invariant differential operators and $n(1|N)$–relative cohomology


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The linear $\mathfrak{n}(1|N)$–invariant differential operators and $\mathfrak{n}(1|N)$–relative cohomology

Opérateurs différentiels linéaires $\mathfrak{n}(1|N)$–invariants et cohomology $\mathfrak{n}(1|N)$–relative

Hafedh Khalfoun$^{a, b}$ and Ismail Laraiedh$^{a, b}$

$^a$ Département de Mathématiques, Faculté des Sciences de Sfax, BP 802, 3038 Sfax, Tunisie

$^b$ Departement of Mathematics, College of Sciences and Humanities - Kowaiyia, Shaqra University, Kingdom of Saudi Arabia.

E-mails: hafedhkhalf@su.edu.sa, ismail.laraiedh@su.edu.sa.

Abstract. Over the $(1,N)$-dimensional supercircle $S^{1|N}$, we classify $\mathfrak{n}(1|N)$-invariant linear differential operators acting on the superspaces of weighted densities on $S^{1|N}$, where $\mathfrak{n}(1|N)$ is the Heisenberg Lie superalgebra. This result allows us to compute the first differential $\mathfrak{n}(1|N)$-relative cohomology of the Lie superalgebra $\mathcal{K}(N)$ of contact vector fields with coefficients in the superspace of weighted densities. For $N = 0, 1, 2$, we investigate the first $\mathfrak{n}(1|N)$-relative cohomology space associated with the embedding of $\mathcal{K}(N)$ in the superspace of the supercommutative algebra $\mathcal{SP}(S^{1|N})$ of pseudodifferential symbols on $S^{1|N}$ and in the Lie superalgebra $\mathcal{SPD}(S^{1|N})$ of superpseudodifferential operators with smooth coefficients. We explicitly give 1-cocycles spanning these cohomology spaces.

Résumé. Sur le supercercle $(1,N)$-dimensionnel $S^{1|N}$, nous classifions les opérateurs différentiels linéaires $\mathfrak{n}(1|N)$-invariants agissant sur les densités tensorielles sur $S^{1|N}$, où $\mathfrak{n}(1|N)$ est la superalgèbre de Lie de Heisenberg. Ce résultat permet de calculer le premier espace de cohomologie différentiels $\mathfrak{n}(1|N)$-relative de la superalgèbre de Lie des champs de vecteurs de contact $\mathcal{K}(N)$ à coefficients dans le superespace des densités tensorielles. Pour $N = 0, 1, 2$, nous étudions le premier espace de cohomologie $\mathfrak{n}(1|N)$-relative de $\mathcal{K}(N)$ dans le superespace de l’algèbre supercommutative $\mathcal{SP}(N)$ des symboles pseudodifférentiels sur $S^{1|N}$ et dans la superalgèbre de Lie $\mathcal{SPD}(S^{1|N})$ des opérateurs superpseudodifférentiels. Nous donnons explicitement les 1-cocycles engendrent ces espaces de cohomologie.


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1. Introduction

Let $\text{Vect}(S^1)$ is the Lie algebra of smooth vector fields on the circle $S^1$. Consider the 1-parameter deformation of the $\text{Vect}(S^1)$-action on $C_C^\infty(S^1)$:

$$L^\lambda_{\frac{dX}{dx}}(f) = Xf' + \lambda Xf,$$

where $X, f \in C_C^\infty(S^1)$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\text{Vect}(S^1)$-module structure on $C_C^\infty(S^1)$ defined by $L^\lambda$ for a fixed $\lambda$. Geometrically, $\mathcal{F}_\lambda = \{f \frac{dX}{dx} | f \in C_C^\infty(S^1)\}$ is the space of weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1, 0$ and 1, respectively.

Denote by $D_{\lambda, \mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the $\text{Vect}(S^1)$-module of linear differential operators with the natural $\text{Vect}(S^1)$-action denoted $L^\lambda_X(A)$. Each module $D_{\lambda, \mu}$ has a natural filtration by the order of differential operators; the graded module $\mathcal{F}_{\lambda, \mu} := \text{gr}D_{\lambda, \mu}$ is called the space of symbols. The quotient-module $D^{k}_{\lambda, \mu}/D^{k-1}_{\lambda, \mu}$ is isomorphic to the module of weighted densities $\mathcal{F}_{\mu - \lambda - k}$; the isomorphism is provided by the principal symbol map $\sigma_r$ defined by:

$$A = \sum_{i=0}^{k} a_i(x) \left( \frac{\partial}{\partial x} \right)^i \mapsto \sigma_{pr}(A) = a_k(x)(dx)^{\mu - \lambda - k},$$

We study the classification of $n(1|N)$-invariant linear differential operators on $S^{1|N}$ acting in the spaces $\mathcal{S}_N^N$. Ovsienko and Roger [11] calculated the space $H^1(\text{Vect}(S^1), \Psi\mathcal{D}\mathcal{O}(S^1))$, where $\text{Vect}(S^1)$ is the Lie algebra of smooth vector fields on the circle $S^1$ and $\Psi\mathcal{D}\mathcal{O}(S^1)$ is the space of pseudodifferential operators. The action is given by the natural embedding of $\text{Vect}(S^1)$ in $\Psi\mathcal{D}\mathcal{O}(S^1)$. They used the results of D. B. Fuks [5] on the cohomology of $\text{Vect}(S^1)$ with coefficients in tensor densities to determine the cohomology with coefficients in the graded module $\text{Grad}(\Psi\mathcal{D}\mathcal{O}(S^1))$, namely $H^1(\text{Vect}(S^1), \text{Grad}^p(\Psi\mathcal{D}\mathcal{O}(S^1)))$; here $\text{Grad}^p(\Psi\mathcal{D}\mathcal{O}(S^1))$ is isomorphic, as $\text{Vect}(S^1)$-module, to the space of tensor densities $\mathcal{F}_p$ of degree $p$ on $S^1$. To compute $H^1(\text{Vect}(S^1), \Psi\mathcal{D}\mathcal{O}(S^1))$, V. Ovsienko and C. Roger applied the theory of spectral sequences to a filtered module over a Lie algebra.

In this paper we consider the superspace $S^{1|N}$ equipped with the contact structure determined by a 1-form $\alpha_N$, and the Lie superalgebra $\mathcal{K}(N)$ of contact vector fields on $S^{1|N}$. We introduce the $\mathcal{K}(N)$-module $\mathcal{S}_N$ of $\lambda$-densities on $S^{1|N}$ and the $\mathcal{K}(N)$-module of linear differential operators, $\mathcal{D}_{\lambda, \mu} := \text{Hom}_{\text{diff}}(\mathcal{S}_\lambda, \mathcal{S}_\mu)$, which are super analogues of the spaces $\mathcal{F}_{\lambda, \mu}$ and $D_{\lambda, \mu}$, respectively. We classify all $n(1|N)$-invariant linear differential operators on $S^{1|N}$ acting in the spaces $\mathcal{S}_N$. We use the result to compute $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathcal{S}_N^N)$. We show that, the non-zero cohomology only appear for resonant values of weights. Moreover, we give explicit bases of these cohomology spaces.

2. Definitions and notations

In this section, we recall the main definitions and facts related to the geometry of the superspace $S^{1|N}$; for more details, see [6, 7, 8, 9, 10].
2.1. The Lie superalgebra of contact vector fields on $S^{1|N}$

We define the supercircle $S^{1|N}$ in terms of its superalgebra of functions, denoted by $C^{\infty}_{\mathbb{C}}(S^{1|N})$ and consisting of elements of the form:

$$F = \sum_{s=0}^{N} \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq N} f_{i_1 i_2 \ldots i_s}(x) \theta_{i_1} \cdots \theta_{i_s},$$

where $f_{i_1 i_2 \ldots i_s} \in C^{\infty}_{\mathbb{C}}(S^1)$, and where $x$ is the even indeterminate, $\theta_1, \ldots, \theta_N$ are the odd indeterminates, i.e., $\theta_i \theta_j = -\theta_j \theta_i$. Consider the standard contact structure given by the following 1-form:

$$\alpha_N = dx + \sum_{i=1}^{N} \theta_i d\theta_i.$$

On the space $C^{\infty}_{\mathbb{C}}(S^{1|N})$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2} \sum_{i=1}^{N} (-1)^{p(F)} \bar{\eta}_i(F) \cdot \bar{\eta}_i(G),$$

where $\bar{\eta}_i = \frac{\partial}{\partial \theta_i} - \theta_i \frac{\partial}{\partial x}$ and $p(F)$ is the parity of $F$. Let $\text{Vect}_{\mathbb{C}}(S^{1|N})$ be the superspace of vector fields on $S^{1|N}$:

$$\text{Vect}_{\mathbb{C}}(S^{1|N}) = \left\{ F_0 \partial_x + \sum_{i=1}^{N} F_i \partial_i \bigg| F_i \in C^{\infty}_{\mathbb{C}}(S^{1|N}) \right\},$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(N)$ of contact vector fields on $S^{1|N}$:

$$\mathcal{K}(N) = \{ X \in \text{Vect}_{\mathbb{C}}(S^{1|N}) \big| \text{there exists } F \in C^{\infty}_{\mathbb{C}}(S^{1|N}) \text{ such that } \mathcal{L}_X(\alpha_N) = F \alpha_N \}.$$

The Lie superalgebra $\mathcal{K}(N)$ is spanned by the fields of the form:

$$X_F = F \partial_x - \frac{1}{2} \sum_{i=1}^{N} (-1)^{p(F)} \bar{\eta}_i(F) \bar{\eta}_i,$$

where $F \in C^{\infty}_{\mathbb{C}}(S^{1|N})$.

In particular, we have $\mathcal{K}(0) = \text{Vect}_{\mathbb{C}}(S^1)$. The bracket in $\mathcal{K}(N)$ can be written as:

$$[X_F, X_G] = X_{[F,G]}.$$

The Lie superalgebra $\mathcal{K}(N - 1)$ can be realized as a subalgebra of $\mathcal{K}(N)$:

$$\mathcal{K}(N - 1) = \{ X_F \in \mathcal{K}(N) \big| \partial_N F = 0 \}.$$

Note also that, for any $i$ in $\{1, 2, \ldots, N\}$, $\mathcal{K}(N - 1)$ is isomorphic to

$$\mathcal{K}(N - 1)^i = \{ X_F \in \mathcal{K}(N) \big| \partial_i F = 0 \}.$$

2.2. The Heisenberg subalgebra $n(1|N)$

The Heisenberg Lie superalgebra $n(1|N)$ can be realized as a subalgebra of $\mathcal{K}(N)$:

$$n(1|N) = \text{Span} \left\{ X_1, X_0 \right\}, \quad 1 \leq i \leq N.$$

We easily see that $n(1|N - 1)$ is a subalgebra of $n(1|N)$:

$$n(1|N - 1) = \{ X_F \in n(1|N - 1) \big| \partial_N F = 0 \}.$$

Note also that, for any $i$ in $\{1, 2, \ldots, N - 1\}$, $n(1|N - 1)$ is isomorphic to

$$n(1|N - 1)^i = \{ X_F \in n(1|N - 1) \big| \partial_i F = 0 \}.$$
2.3. Modules of weighted densities

For every contact vector field $X_F$, define a one-parameter family of first-order differential operators on $C^\infty(S^{1|N})$:

$$\mathcal{L}^\lambda_{X_F} = X_F + \lambda F', \quad \lambda \in \mathbb{C}.$$  

We easily check that

$$[\mathcal{L}^\lambda_{X_F}, \mathcal{L}^\lambda_{X_G}] = \mathcal{L}^\lambda_{[X_F, X_G]}.$$  

We thus obtain a one-parameter family of $\mathcal{K}(N)$-modules on $C^\infty(S^{1|N})$ that we denote $\tilde{\mathfrak{d}}^N_\lambda$, the space of all weighted densities on $S^{1|N}$ of weight $\lambda$ with respect to $\alpha_N$:

$$\tilde{\mathfrak{d}}^N_\lambda = \left\{ F \alpha^\lambda_N \, | \, F \in C^\infty(S^{1|N}) \right\}.$$  

2.4. Differential operators on weighted densities

A differential operator on $S^{1|N}$ is an operator on $C^\infty(S^{1|N})$ of the form:

$$A = \sum_{k=0}^M \sum_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)} a_{k,\varepsilon}(x, \theta) \partial^{k} \theta^{\varepsilon_1}_1 \cdots \partial^{\varepsilon_N}_N; \quad \varepsilon_i = 0, 1; \quad M \in \mathbb{N}.$$  

Of course any differential operator defines a linear mapping $F \alpha^\lambda_N \mapsto (AF) \alpha^\mu_N$ from $\tilde{\mathfrak{d}}^N_\lambda$ to $\tilde{\mathfrak{d}}^N_\mu$ for any $\lambda, \mu \in \mathbb{R}$, thus the space of differential operators becomes a family of $\mathcal{K}(N)$-modules $\mathcal{D}^N_{\lambda,\mu}$ for the natural action:

$$X_F \cdot A = \Sigma^\mu_{X_F} \circ A - (-1)^{p(A)p(F)} A \circ \Sigma^\lambda_{X_F}.$$  

Every differential operator $A \in \mathcal{D}^N_{\lambda,\mu}$ can be expressed in the form

$$A(F \alpha^\lambda_N) = \sum_{\ell = (\ell_1, \ldots, \ell_N)} a_f(x, \theta) \eta^{\ell_1}_1 \cdots \eta^{\ell_N}_N(F) \alpha^\mu_N,$$

where the coefficients $a_f(x, \theta)$ are arbitrary functions.

**Lemma 1** ([2]). As a $\mathcal{K}(N-1)$-module, we have

$$\mathcal{D}^N_{\lambda,\mu} \cong \mathcal{D}^{N-1}_{\lambda,\mu} \oplus \mathcal{D}^{N-1}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}} \oplus \Pi \left( \mathcal{D}^{N-1}_{\lambda,\mu+\frac{1}{2}} \oplus \mathcal{D}^{N-1}_{\lambda+\frac{1}{2},\mu} \right),$$

where $\Pi$ is the change of parity operator.

2.5. Pseudodifferential operators on $S^{1|N}$

Let $T^*S^{1|N}$ be the cotangent bundle on $S^{1|N}$ with local coordinates $(x, \theta_1, \ldots, \theta_N, \xi, \tilde{\theta}_1, \ldots, \tilde{\theta}_N)$, where $p(\tilde{\theta}_i) = 1$. The superspace of the supercommutative algebra $\mathcal{SP}(N)$ of pseudodifferential symbols on $S^{1|N}$ with its natural multiplication is spanned by the series

$$\mathcal{SP}(N) = \left\{ \sum_{k=-\infty}^\infty \sum_{\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)} a_{k,\varepsilon}(x, \theta) \xi^{-k} \tilde{\theta}^{\varepsilon_1}_1 \cdots \tilde{\theta}^{\varepsilon_N}_N \alpha^\varepsilon_N \in C^\infty(S^{1|N}); \quad \varepsilon_i = 0, 1; \quad M \in \mathbb{N} \right\}.$$  

This space has a structure of the Poisson Lie superalgebra given by the following bracket:

$$[A, B] = \partial_x A \partial_A B - \partial_x A \partial_B B - (-1)^{p(A)} \sum_{i=1}^N \left[ \partial_i A \partial_{\tilde{\theta}_i} B + \partial_{\tilde{\theta}_i} A \partial_i B \right],$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_\xi = \frac{\partial}{\partial \xi}$, $\partial_i = \frac{\partial}{\partial \theta_i}$, and $\partial_{\tilde{\theta}_i} = \frac{\partial}{\partial \tilde{\theta}_i}$. Of course $\mathcal{SP}(0)$ is the classical space of symbols, usually denoted $\mathcal{P}$.
The associative superalgebra of pseudodifferential operators $\mathcal{S}\Psi\mathcal{D}(S^{1|N})$ on $S^{1|N}$ has the same underlying vector space as $\mathcal{S}\mathcal{P}(N)$, but the multiplication is now defined by the following rule:

$$A \circ B = \sum_{a \geq 0, v_i = 0, 1} \frac{(-1)^{p(A)+1}}{a!}\left(\partial_x^a \partial_{\tilde{\theta}_i} A \right) \left( \partial_x^a \partial_{\tilde{\theta}_i} B \right).$$

Denote by $\mathcal{S}\Psi\mathcal{D}(S^{1|N})_S$ the Lie superalgebra with the same superspace as $\mathcal{S}\Psi\mathcal{D}(S^{1|N})$ and the supercommutator defined on homogeneous elements by:

$$[A, B] = A \circ B - (-1)^{p(A)p(B)} B \circ A.$$

In particular, we have $\mathcal{S}\Psi\mathcal{D}(S^{1|0}) = \Psi\mathcal{D}(S^1)$.

3. The structure of $\mathcal{S}\mathcal{P}(N)$ as a $\mathcal{K}(N)$–module

The natural embedding of $\mathcal{K}(N)$ into $\mathcal{S}\mathcal{P}(N)$ defined by

$$\pi(X_F) = F \xi - \frac{(-1)^{p(F)}}{2} \sum_{i=1}^{N} \tilde{\eta}_i(F) \zeta_i,$$

where $\zeta_i = \tilde{\theta}_i - \theta_i \xi$,

induces a $\mathcal{K}(N)$-module structure on $\mathcal{S}\mathcal{P}(N)$.

Setting $\deg x = \deg \theta_i = 0$, $\deg \xi = \deg \tilde{\theta}_i = 1$ for all $i$, we endow the Poisson superalgebra $\mathcal{S}\mathcal{P}(N)$ with a $\mathbb{Z}$-grading:

$$\mathcal{S}\mathcal{P}(N) = \bigoplus_{n \in \mathbb{Z}} \mathcal{S}\mathcal{P}_n(N),$$

where $\mathbf{\tilde{\Theta}}_{n \in \mathbb{Z}} = (\mathbf{\Theta}_{n < 0}) \oplus \prod_{n \geq 0}$ and

$$\mathcal{S}\mathcal{P}_n(N) = \{ F \xi^n + G_1 \xi^{n-1} \tilde{\theta}_1 + G_2 \xi^{n-2} \tilde{\theta}_2 + \cdots + H_{1,2} \xi^{-n-2} \tilde{\theta}_1 \tilde{\theta}_2 + \cdots | F, G_i, H_{i,j} \in C^\infty(S^{1|N}) \}$$

is the homogeneous subspace of degree $-n$.

Note that each element of $\mathcal{S}\Psi\mathcal{D}(S^{1|N})$ can be expressed as

$$A = \sum_{k \in \mathbb{Z}} (F_k + G_k^1 \xi^{-1} \tilde{\theta}_1 + \cdots + H_k^{1,2} \xi^{-2} \tilde{\theta}_1 \tilde{\theta}_2 + \cdots) \xi^{-k},$$

where $F_k, G_k^i, H_k^{1,2} \in C^\infty(S^{1|N})$. We define the order of $A$ to be

$$\text{ord}(A) = \sup \left\{ k \mid F_k \neq 0 \text{ or } G_k^i \neq 0 \text{ or } H_k^{1,2} \neq 0 \right\}.$$

This definition of order equips $\mathcal{S}\Psi\mathcal{D}(S^{1|N})$ with a decreasing filtration as follows: set

$$\mathcal{F}_n = \{ A \in \mathcal{S}\Psi\mathcal{D}(S^{1|N}) \mid \text{ord}(A) \leq -n \},$$

where $n \in \mathbb{Z}$. So we have

$$\cdots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \cdots$$

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for $A \in \mathcal{F}_n$ and $B \in \mathcal{F}_p$, one has $AB \in \mathcal{F}_{n+p}$ and $(A, B) \in \mathcal{F}_{n+p-1}$. This filtration makes $\mathcal{S}\Psi\mathcal{D}(S^{1|N})$ an associative filtered superalgebra. Moreover, this filtration is compatible with the natural $\mathcal{K}(N)$-action on $\mathcal{S}\Psi\mathcal{D}(S^{1|N})$. Indeed,

$$X_F(A) = [X_F, A] \in \mathcal{F}_n \text{ for any } X_F \in \mathcal{K}(N) \text{ and } A \in \mathcal{F}_n.$$
4. $n(1|N)$–invariant linear differential operators

Now, we describe the spaces of $n(1|N)$-invariant linear differential operators $\mathfrak{g}_\Lambda^N \rightarrow \mathfrak{g}_\mu^N$ for $N \in \mathbb{N}$. Our main result of this section is the following:

**Theorem 2.** Let $\mathcal{N}_{\Lambda,\mu}^N : \mathfrak{g}_\Lambda^N \rightarrow \mathfrak{g}_\mu^N$, $(F\alpha_N) \mapsto \mathcal{N}_{\Lambda,\mu}^N(F)\alpha_N$ be a non-zero $\mathcal{N}(1|N)$-invariant linear differential operator. Then, up to a scalar factor, the map $\mathcal{N}_{\Lambda,\mu}^N$ is given by:

$$
\mathcal{N}_{\Lambda,\mu}^N(F) = \begin{cases} 
\sum_{k=0}^\infty \gamma_k F^{(k)}, & \text{for } N \in \mathbb{N} \\
\sum_{k=0}^\infty \gamma_k \eta_1 \eta_2 \ldots \eta_N(F^{(k)}), & \text{for } N \geq 1,
\end{cases}
$$

(2)

where $\gamma_k \in \mathbb{R}$.

**Proof.** (i). For $N = 0$, the generic form of any such a differential operator is 

$$
\mathcal{N}_{\Lambda,\mu}^0 : \mathfrak{g}_\Lambda^0 \rightarrow \mathfrak{g}_\mu^0, Fdx^\Lambda \mapsto \sum_{i=0}^m \gamma_i F^{(i)} dx^\mu,
$$

where $\gamma_i \in C^\infty(S^1)$ are arbitrary functions and $F^{(i)}$ stands for $\frac{d^iF}{dx^i}$. The invariance property with respect to the vector field $X = \frac{d}{dx}$ implies that $\frac{d^iF}{dx^i} = 0$.

(ii). By induction on $N$. For $N = 1$, let $\mathcal{N}_{\Lambda,\mu}^1 : \mathfrak{g}_\Lambda^1 \rightarrow \mathfrak{g}_\mu^1$ be an $n(1|1)$-invariant linear differential operator. The $n(1|1)$-invariance of $\mathcal{N}_{\Lambda,\mu}^1$ is equivalent to invariance with respect just to the subalgebra $n(1|0)$ and the vector fields $X_{\theta_1}$. Using the fact that, as $\text{vect}(S^1)$-modules,

$$
\mathfrak{g}_\Lambda^1 \cong \mathfrak{g}_\Lambda^0 \oplus \Pi \left( \mathfrak{g}_\Lambda^0 \right),
$$

(3)

we can deduce, by induction hypothesis, the restriction of $\mathcal{N}_{\Lambda,\mu}^1$ to each component of the right-hand side of (3). The invariance of $\mathcal{N}_{\Lambda,\mu}^1$ with respect $X_{\theta_1}$ determine thus completely the space of $n(1|1)$-invariant linear differential operator $\mathfrak{g}_\Lambda^1 \rightarrow \mathfrak{g}_\mu^1$.

Now, assume that the result holds for $N > 1$. Observe that the $n(1|N)$-invariance of any linear differential operators $\mathcal{N}_{\Lambda,\mu}^N : \mathfrak{g}_\Lambda^N \rightarrow \mathfrak{g}_\mu^N$ is equivalent to invariance with respect just to the subalgebras $n(1|N - 1)$ and $n(1|N - 1)^i$, $i = 1, \ldots, N - 1$, and that $\mathcal{N}_{\Lambda,\mu}^N$ is decomposed into four $n(1|N - 1)$-invariant maps:

$$
\Pi^i \left( \mathfrak{g}_\Lambda^{N-1} \right) \rightarrow \Pi^j \left( \mathfrak{g}_\mu^{N-1} \right), \quad i, j = 0, 1.
$$

(4)

Thus, by induction assumption, we exhibit the $n(1|N - 1)$-invariant linear differential operators $\mathfrak{g}_\Lambda^N \rightarrow \mathfrak{g}_\mu^N$. More precisely, any $n(1|N - 1)$-invariant binary differential operators $\mathcal{N}_{\Lambda,\mu}^{N} : \mathfrak{g}_\Lambda^N \rightarrow \mathfrak{g}_\mu^N$ can be expressed as:

$$
\mathcal{N}_{\Lambda,\mu}^{N}(F) = \Xi_{\Lambda,\mu}(1 - \theta_{N,\mu})\Xi_{\Lambda,\mu}^{N-1} - \Theta_{\Lambda,\mu}(-1)^{p(F)} \partial_{\theta_{N,\mu}} \Xi_{\Lambda,\mu}^{N-1} \theta_{N},
$$

$$
\mathcal{N}_{\Lambda,\mu}^{N}(F) = (-1)^{p(F)} \Omega_{\Lambda,\mu}(1 - \theta_{\mu,\mu})\Xi_{\Lambda,\mu}^{N-1} \theta_{N} + \Gamma_{\Lambda,\mu}(\partial_{\theta_{\mu}} \Xi_{\Lambda,\mu}^{N-1}),
$$

where the coefficients $\Omega_{\Lambda,\mu}, \Gamma_{\Lambda,\mu}, \Xi_{\Lambda,\mu}$ and $\Theta_{\Lambda,\mu}$ are, a priori, arbitrary constants, but the invariance of $\mathcal{N}_{\Lambda,\mu}^{N}$ with respect $n(1|N - 1)^i$, $i = 1, \ldots, N - 1$, shows that

$$
\Gamma_{\Lambda,\mu} = -\Omega_{\Lambda,\mu}, \quad \Xi_{\Lambda,\mu} = \Theta_{\Lambda,\mu}.
$$

Therefore, we easily check that $\mathcal{N}_{\Lambda,\mu}^{N}$ is expressed as in Theorem 2. This completes the proof of Theorem 2. \qed
5. Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [4]). Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra acting on a superspace \( V = V_0 \oplus V_1 \) and let \( h \) be a subalgebra of \( g \). (If \( h \) is omitted it assumed to be \( \{0\} \)). The space of \( h \)-relative \( n \)-cochains of \( g \) with values in \( V \) is the \( g \)-module

\[
C^n(g, h; V) := \text{Hom}_h(\Lambda^n(g/h); V).
\]

The coboundary operator \( \delta : C^n(g, h; V) \to C^{n+1}(g, h; V) \) is a \( g \)-map satisfying \( \delta \circ \delta_{n-1} = 0 \). The kernel of \( \delta_n \), denoted \( Z^n(g, h; V) \), is the space of \( h \)-relative \( n \)-cocycles, among them, the elements in the range of \( \delta_{n-1} \) are called \( h \)-relative \( n \)-coboundaries. We denote \( B^n(g, h; V) \) the space of \( n \)-coboundaries.

By definition, the \( n^{th} \) \( h \)-relative cohomology space is the quotient space

\[
H^n(g, h; V) = Z^n(g, h; V)/B^n(g, h; V).
\]

5.1. The spaces \( H^1_{\text{diff}}(\mathcal{N}(N), n(1|N), \mathfrak{F}^N_A) \)

In this subsection, we will compute the first differential cohomology spaces \( H^1_{\text{diff}}(\mathcal{N}(N), n(1|N), \mathfrak{F}^N_A) \). Our main result is the following:

**Theorem 3.** The space \( H^1_{\text{diff}}(\mathcal{N}(N), n(1|N), \mathfrak{F}^N_A) \) has the following structure:

\[
H^1_{\text{diff}}(\mathcal{N}(N), n(1|N), \mathfrak{F}^N_A) = \begin{cases} \mathbb{R}^2 & \text{if } N = 2 \text{ and } \lambda = 0 \\ \mathbb{R} & \text{if } N = 0 \text{ and } \lambda = 0, 1, 2 \\ \mathbb{R} & \text{if } N = 1 \text{ and } \lambda = 0, 1, 2 \\ 0 & \text{if } N = 2 \text{ and } \lambda = 1 \\ 0 & \text{if } N = 3 \text{ and } \lambda = 0, 1 \\ 0 & \text{if } N \geq 4 \text{ and } \lambda = 0 \\ \text{otherwise.} \end{cases}
\]

The following 1-cocycles \( Y^N_X \) span the corresponding cohomology spaces:

\[
\begin{align*}
Y^0_0(X_F) &= F'; \ N \in \mathbb{N}, & Y^1_0(X_F) &= \hat{\eta}_1(F') \alpha^1_1, \\
Y^0_1(X_F) &= F'' \mathrm{d}x^1, & Y^2_0(X_F) &= \hat{\eta}_1 \hat{\eta}_2(F) \alpha_2, \\
Y^0_2(X_F) &= F''' \mathrm{d}x^2, & Y^2_1(X_F) &= \hat{\eta}_1 \hat{\eta}_2(F') \alpha_2, & (5) \\
Y^1_1(X_F) &= \hat{\eta}_1(F') \alpha^1_1, & Y^3_1(X_F) &= \hat{\eta}_1 \hat{\eta}_2 \hat{\eta}_3(F) \alpha^1_3.
\end{align*}
\]

The proof of Theorem 3 will be the subject of subsection 5.2. In fact, we need first the following classical fact:

**Lemma 4 ([3]).** Any 1-cocycle \( Y \) on \( \mathcal{N}(N) \) vanishing on \( n(1|N) \), with values in \( \mathfrak{F}^N_A \), the linear differential operator \( \mathcal{N} : \mathcal{N}(N) \to \mathfrak{F}^N_A \) defined by

\[
\mathcal{N}(X) = Y(X),
\]

is \( n(1|N) \)-invariant.
5.2. Proof of the Theorem 3

Let $\gamma_{1,\mu}^N$ be a 1-cocycle on $\mathcal{K}(N)$ vanishing on $n(1|N)$, with values in $\mathfrak{f}_1^N$. By Lemma 4, up to a scalar factor, $\gamma_{1,\mu}^N$ is a linear differential operator $n(1|N)$-invariant $\mathcal{N}_{-1,\mu}^N: \mathfrak{f}_1^N \to \mathfrak{f}_1^N$. Thus, by Theorem 2, we get the explicit formulae for $\mathcal{N}_{-1,\mu}^N$:

For $N = 0$, \[
\mathcal{N}_{-1,\mu}^0(X_F) = \sum_{k \geq 0} \gamma_k F^{(k)}(X_F) d^\mu
\]

For $N \geq 1$, \[
\begin{align*}
\mathcal{N}_{-1,\mu}^N(X_F) &= \sum_{k \geq 0} \gamma_k F^{(k)}(X_F) d^\mu_N \\
\mathcal{N}_{-1,\mu}^N(X_F) &= \sum_{k \geq 0} \gamma_k \bar{t}_1 \bar{t}_2 \cdots \bar{t}_N (F^{(k)}(X_F)) d^\mu_N.
\end{align*}
\]

Now let us check if each of the maps $\mathcal{N}_{-1,\mu}^N$ are 1-cocycles. If the maps $\mathcal{N}_{-1,\mu}^N$ are 1-cocycles one has to check the 1-cocycle relation. It reads as follows:

$$
\delta(\mathcal{N}_{-1,\mu}^N) = (-1)^{p(X)p(\mathcal{A}_{-1,\mu}^N)} \sum_{X} \mathcal{N}_{\mathcal{A}_{-1,\mu}^N}(Y) - (-1)^{p(Y)(p(X)+p(\mathcal{A}_{-1,\mu}^N))} \sum_{Y} \mathcal{N}_{\mathcal{N}_{-1,\mu}^N}(X) - \mathcal{A}_{-1,\mu}^N([X,Y]) = 0,
$$

where $X, Y \in \mathcal{K}(N)$. By direct computation, we can see that only the operators $\mathcal{N}_{-1,\mu}^N = \gamma_{1,\mu}^N$ expressed as in (5) are 1-cocycles vanishing on $n(1|N)$.

Finally, we study the non-triviality of these 1-cocycles $\mathcal{N}_{-1,\mu}^N$. For instance, assume that the 1-cocycle $\mathcal{N}_{0,2}^0$ is trivial, then there exists a density $\varphi(x) dx^2 \in \mathfrak{f}_2^0$ such that

$$
\mathcal{N}_{0,2}^0(X_F) = \int_{X_F} \varphi(x) dx^2.
$$

The coefficient of $F''$ is zero in the expression of the coboundary and the coefficient of $F'''$ is 1 in the expression of 1-cocycle $\mathcal{N}_{0,2}^0$. Thus, the relation (6) implies $1 = 0$ which is absurd. With the same arguments, we prove the non-triviality of 1-cocycles $\mathcal{N}_{-1,0}^N, \mathcal{N}_{-1,1}^N, \mathcal{N}_{0,2}^N, \mathcal{N}_{-1,1}^{-1}, \mathcal{N}_{-1,1}^{-2}, \mathcal{N}_{-1,1}^{-3}$ and $\mathcal{N}_{1,1}^{-1}$. Therefore, we easily check that $\gamma_{1,\mu}^N$ is expressed as in (5). This completes the proof of Theorem 3.

6. $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{I} \mathcal{P}_n(N))$ and $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{I} \Psi \mathcal{O}(S^1|N))$

6.1. The space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{I} \mathcal{P}_n(N))$

The space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{I} \mathcal{P}_n(N))$ inherits the grading (3) of $\mathcal{I} \mathcal{P}_n(N)$, so it suffices to compute it in each degree. The main result of this section for $N = 0, 1, 2$, is the following.

**Theorem 5.** The space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{I} \mathcal{P}_n(N))$ has the following structure:

$$
H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{I} \mathcal{P}_n(N)) \simeq \begin{cases} 
\mathbb{R} & \text{if } \begin{cases} N = 2 \text{ and } n = 1 \\
N = 0 \text{ and } n = 0, 1, 2 \\
N = 1 \text{ and } n = 1
\end{cases} \\
\mathbb{R}^2 & \text{if } \begin{cases} N = 2 \text{ and } n = -1 \\
N = 1 \text{ and } n = 0
\end{cases} \\
\mathbb{R}^5 & \text{if } N = 2 \text{ and } n = 0 \\
0 & \text{otherwise.}
\end{cases}
$$

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The following 1-cocycles $\chi^N_n$ span the corresponding cohomology spaces:

\[
\begin{align*}
\chi_0^N &= F', \text{ for } N = 0, 2, \\
\chi_1^0 &= F''\xi^{-1}, \\
\chi_2^0 &= F'\xi^{-2}, \\
\chi_0^1 &= (1 + (-1)^p(F'))F' + \overline{\eta}_1(F')\xi^{-1}\xi_1, \\
\chi_1^1 &= \overline{\eta}_1(F')\xi^{-2}\xi_1 - 2\theta_1\overline{\eta}_1(F'), \\
\chi_2^1 &= \overline{\eta}_1(F'')\xi^{-1}\xi_1 - 2\theta_1\overline{\eta}_1(F''), \\
\chi_0^2 &= \frac{2}{3}F^{(3)}\xi^{-3}\xi_1\xi_2 - (1)^p(F)(\overline{\eta}_2(F'')\xi_1 - \overline{\eta}_1(F'')\xi_2)\xi^{-2} + 2\overline{\eta}_1\overline{\eta}_2(F')\xi^{-1}.
\end{align*}
\]

**Proof.** The case where $N = 0$. In this case, we can see that the map $\phi : \mathcal{H}_n \to \mathcal{D}_n$ defined by $\phi(F)dx^n = F\xi^{-n}$ provides us with an isomorphism of $\text{Vect}(S^1)$-modules. So, we can deduce the structure of $H^1_{\text{diff}}(\text{Vect}(S^1), n(1)|0; \mathcal{D}_n)$ from $H^1_{\text{diff}}(\text{Vect}(S^1), n(1)|0; \mathcal{H}_n)$ given in Theorem 3.

**The case where $N = 1$.** In this case, as a $\mathcal{K}(1)$-module, we have

$$\mathcal{I}\mathcal{D}_n(1) = \mathcal{I}\mathcal{D}^1_n \oplus \mathcal{I}\mathcal{D}^2_n,$$

where

$$\begin{align*}
\mathcal{I}\mathcal{D}^1_n &= \{1 + (-1)^p(F)\xi^{-n} + \overline{\eta}_1(F)\xi^{-n-1}\xi_1, \ F \in C^{\infty}_C(S^{11})\}, \\
\mathcal{I}\mathcal{D}^2_n &= \{F\xi^{-n-1}\xi_1 - 2\theta_1F\xi^{-n}, \ F \in C^{\infty}_C(S^{11})\}.
\end{align*}$$

The natural maps

$$\begin{align*}
\varphi_1 : \mathfrak{S}^1_n &\to \mathcal{I}\mathcal{D}^1_n, \\
F\alpha^n_1 &\mapsto (1 + (-1)^p(F)\xi^{-n} + \overline{\eta}_1(F)\xi^{-n-1}\xi_1), \\
\varphi_2 : \Pi\left(\mathfrak{S}^{n+1}_n\right) &\to \mathcal{I}\mathcal{D}^2_n, \\
\Pi(F\alpha^{n+1}_1) &\mapsto F\xi^{-n-1}\xi_1 - 2\theta_1F\xi^{-n},
\end{align*}$$

provide us with isomorphisms of $\mathcal{K}(1)$-modules. Hence, as $\mathcal{K}(1)$-modules, we have $\mathcal{I}\mathcal{D}_n(1) \simeq \mathfrak{S}^1_n \oplus \Pi(\mathfrak{S}^{n+1}_n)$. This isomorphism induces the following isomorphism between cohomology spaces:

$$H^1_{\text{diff}}(\mathcal{K}(1), n(1)|1; \mathcal{I}\mathcal{D}_n(1)) \simeq H^1_{\text{diff}}(\mathcal{K}(1), n(1)|1; \mathfrak{S}^1_n) \oplus H^1_{\text{diff}}\left(\mathcal{K}(1), n(1)|1; \Pi\left(\mathfrak{S}^{n+1}_n\right)\right).$$

We deduce from this isomorphism and Theorem 3, the 1-cocycles (8).

**The case where $N = 2$.** To prove Theorem 5 in this case, we need first the following:

**Proposition 6.** The space $H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1)|1)^i, \mathfrak{S}^2_{\lambda})$ has the following structure:

$$H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1)|1)^i, \mathfrak{S}^2_{\lambda}) = \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0 \\
\mathbb{R} & \text{if } \lambda = -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2} \\
0 & \text{otherwise.}
\end{cases}$$

The following 1-cocycles $\gamma^i_\lambda$ span the corresponding cohomology spaces:

\[
\begin{align*}
\gamma^i_0 &= F', \\
\gamma^i_\frac{1}{2} &= \overline{\eta}_1(F''), \\
\gamma^i_{-\frac{1}{2}} &= \eta_1(F'), \\
\overline{\gamma}^i_0 &= (-1)^p(F)\overline{\eta}_{3-i}(F')\theta_i, \\
\gamma^i_1 &= (-1)^p(F)\overline{\eta}_{3-i}(F'')\theta_i, \\
\gamma^i_{-1} &= F'\theta_i.
\end{align*}
\]
Proof of Proposition 6. Let \( F\alpha^i_2 = (f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2)\alpha^i_2 \in \mathfrak{S}^2_{\lambda_i} \). The map
\[
\Phi: \mathfrak{S}^2_{\lambda_i} \rightarrow \mathfrak{S}^{1,i}_{\lambda_i} \oplus \Pi\left(\mathfrak{S}^{1,i}_{\lambda_i + \frac{1}{2}}\right)
\]
\[
F\alpha^i_2 \rightarrow \left(1 - \theta_i\partial\theta\right)(F)\alpha^i_2 \oplus \Pi\left((-1)^{\partial(F) + 1}\partial\theta\left(F\alpha^i_2\right)\right)
\]
provides us with an isomorphism of \( \mathcal{K}(1)^i \)-modules. This map induces the following isomorphism between cohomology spaces:
\[
H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \mathfrak{S}^2_{\lambda_i}\right) = H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \mathfrak{S}^{1,i}_{\lambda_i}\right) \oplus H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \Pi\left(\mathfrak{S}^{1,i}_{\lambda_i + \frac{1}{2}}\right)\right).
\]
(10)
Of course, we can deduce the structure of
\[
H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \mathfrak{S}^2_{\lambda_i}\right) \text{ from } H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \mathfrak{S}^{1,i}_{\lambda_i}\right).
\]
Indeed, to any \( Y \in H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \mathfrak{S}^{1,i}_{\lambda_i}\right) \) corresponds \( \tilde{Y} \in H^1_{_{diff}}\left(\mathcal{K}(1)^i, n(1|1) ; \Pi\left(\mathfrak{S}^{1,i}_{\lambda_i + \frac{1}{2}}\right)\right) \) where \( \tilde{Y}(X_F) = \Pi(\sigma \circ Y(X_F)) \) with \( \sigma(F) = (-1)^{\partial(F)} F \). Obviously, \( \tilde{Y} \) is a coboundary if and only if \( \tilde{Y} \) is a coboundary. We deduce from isomorphism (10) and formula (5), the 1-cocycles (9).

Lemma 7. For \( n \in \mathbb{Z} \), any element of \( Z^1(\mathcal{K}(2), n(2|2); \mathcal{S}\mathcal{P}_n(2)) \) is a \( n(1|2) \)-relative coboundary over \( \mathcal{K}(2) \) if and only if its restriction to the subalgebra \( \mathcal{K}(1)^i \) is a \( n(1|1)^i \)-relative coboundary for \( i = 1 \) and 2.

Proof of Lemma 7. It is easy to see that if \( C \) is a \( n(1|2) \)-relative coboundary over \( \mathcal{K}(2) \), then \( \mathcal{C}_i, \mathcal{K}(1)^i \) is a \( n(1|1)^i \)-relative coboundary of \( \mathcal{K}(1)^i \). Now, assume that \( \mathcal{C}_i, \mathcal{K}(1)^i \) is a \( n(1|1)^i \)-relative coboundary of \( \mathcal{K}(1)^i \) for \( i = 1 \) and 2. Using the condition of a 1-cocycle, we prove that there exists an element \( n(1|1)^i \)-invariant \( G \in \mathcal{S}\mathcal{P}_n(2) \) such that
\[
\mathcal{C} \left( X_{f_0 + f_1\theta_1} \right) = \left\{ \rho_0 \left( X_{f_0 + f_1\theta_1} \right), G \right\} \text{ for any } f_0, f_1 \in C^\infty_C(S^1), \quad i = 1, 2
\]
\[
\mathcal{C} \left( X_{f_2 + f_2\theta_2} \right) = \left\{ \rho_0 \left( X_{f_2 + f_2\theta_2} \right), G \right\} \text{ for any } f_2 \in C^\infty_C(S^1).
\]
We deduce that \( \mathcal{C}(X_F) = \left\{ \rho_0(X_F), G \right\} \), for any \( F \in C^\infty_C(S^{1|2}) \), and therefore \( \mathcal{C} \) is a \( n(1|2) \) -relative coboundary of \( \mathcal{K}(2) \).

We also need the following:

Proposition 8 (11).

1. As a \( \mathcal{K}(1)^i \)-module, \( i = 1, 2 \), we have
\[
\mathcal{S}\mathcal{P}_n(2) \simeq \mathfrak{S}^2_n \oplus \Pi\left(\mathfrak{S}^2_{n+\frac{1}{2}} \oplus \mathfrak{S}^2_{n+\frac{1}{2}}\right) \oplus \mathfrak{S}^2_{n+1}, \text{ for } n = 0, -1.
\]
(11)

2. For \( n \neq 0, -1 \):
   (a) The following subspace of \( \mathcal{S}\mathcal{P}_n(2) \):
   \[
   \mathcal{S}\mathcal{P}_{n,i} = \left\{ B_{F}^{(n,i)} = F\theta_{i}^{\frac{1}{2}}\xi^{n-1} + \theta_{i}^{\frac{1}{2}}\left(\frac{1}{2} - \frac{1}{2} \eta_{3-i}\right) F\left(\xi^{n-2}\right) \mid F \in C^\infty_C(S^{1|2}) \right\}
   \]
   (12)
   is a \( \mathcal{K}(1)^i \)-module, \( i = 1, 2 \), isomorphic to \( \mathfrak{S}^2_{n+1} \).
   (b) As a \( \mathcal{K}(1)^i \)-module we have
   \[
   \mathcal{S}\mathcal{P}_n(2) / \mathcal{S}\mathcal{P}_{n,i} \simeq \mathfrak{S}^2_n \oplus \Pi\left(\mathfrak{S}^2_{n+\frac{1}{2}} \oplus \mathfrak{S}^2_{n+\frac{1}{2}}\right), \quad i = 1, 2.
   \]
(13)
Moreover, in [1] it was proved that the natural maps
\[
\psi_{n,0}^i: \mathfrak{S}^2_n \longrightarrow \mathcal{F}(\frac{1}{2},1), \\
F\alpha_n^2 \longrightarrow A_{\frac{1}{2}}(0,1),
\]
\[
\psi_{n,1}^i: \mathfrak{S}^2_{n+1} \longrightarrow \mathcal{F}(\frac{1}{2},1), \\
F\alpha_{n+1}^2 \longrightarrow A_{\frac{1}{2}}(1,1)
\]
provide us with isomorphisms of \(\mathcal{K}(1)\)-modules.

Now, according to Lemma 7, the restriction of any nontrivial \(n(1|2)\)-relative 1-cocycle of \(\mathcal{K}(2)\) with coefficients in \(\mathcal{F}(n|2)\) to \(\mathcal{K}(1)\) is a nontrivial \(n(1|1)\)-relative 1-cocycle. Using Proposition 6 and Propositions 8, we obtain:
\[
H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{F}(n|2)) \simeq \begin{cases} 
\mathbb{R}^4 & \text{if } n = -1 \\
\mathbb{R}^5 & \text{if } n = 0 \\
\mathbb{R}^3 & \text{if } n = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

In the case \(n = -1\), the space \(H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{F}(n|2))\) is spanned by the following 1-cocycles:
\[
C_{-1}^{1,i}(X_F) = \psi_{-1,1}^i \circ \gamma_{0,0}^i(X_F), \\
C_{-1}^{3,i}(X_F) = \psi_{-1,1}^i \circ \Pi \left( \gamma_{-1,2}^i(X_F) \right), \\
C_{-1}^{2,i}(X_F) = \psi_{-1,1}^i \circ \gamma_{0,0}^i(X_F), \\
C_{-1}^{4,i}(X_F) = \psi_{-1,1}^i \circ \Pi \left( \gamma_{-1,2}^i(X_F) \right).
\]

In the case \(n = 0\), the space \(H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{F}(n|2))\) is spanned by the following 1-cocycles:
\[
C_{0}^{1,i}(X_F) = \psi_{0,1}^i \circ \gamma_{0,0}^i(X_F), \\
C_{0}^{4,i}(X_F) = \psi_{0,1}^i \circ \Pi \left( \gamma_{1,2}^i(X_F) \right), \\
C_{0}^{2,i}(X_F) = \psi_{0,1}^i \circ \gamma_{0,0}^i(X_F), \\
C_{0}^{3,i}(X_F) = \psi_{0,1}^i \circ \Pi \left( \gamma_{1,2}^i(X_F) \right), \\
C_{0}^{5,i}(X_F) = \psi_{0,1}^i \circ \gamma_{1,2}^i(X_F).
\]

In the case \(n = 1\), the space \(H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{F}(n|2))\) is spanned by the following 1-cocycles:
\[
C_{1}^{1,i}(X_F) = \psi_{1,1}^i \circ \gamma_{1,1}^i(X_F), \\
C_{1}^{2,i}(X_F) = \psi_{1,1}^i \circ \Pi \left( \gamma_{1,2}^i(X_F) \right), \\
C_{1}^{3,i}(X_F) = \psi_{1,1}^i \circ \gamma_{1,1}^i(X_F), \\
C_{1}^{4,i}(X_F) = \psi_{1,1}^i \circ \Pi \left( \gamma_{1,2}^i(X_F) \right),
\]
where the cocycles \(\gamma_{0,0}^i, \gamma_{0,0}^i, \gamma_{0,1}^i, \gamma_{0,2}^i, \gamma_{1,0}^i\) and \(\gamma_{1,1}^i\) are defined by the formulae (9) and \(\psi_{n,j}^i, \bar{\psi}_{n,j}^i\) are as in (14).

Now, note that any nontrivial \(n(1|2)\)-relative 1-cocycle of \(\mathcal{K}(2)\) with coefficients in \(\mathcal{F}(n|2)\) should retain the following general form \(Y = Y^1 + Y^2 + Y^3 + Y^4\), where
\[
\begin{align*}
Y^1 : \mathfrak{S}^1(1) & \longrightarrow \mathcal{F}(n|2), \\
Y^2, Y^3 : \mathfrak{F}_{n|2} & \longrightarrow \mathcal{F}(n|2), \\
Y^4 : \mathcal{F}_0 & \longrightarrow \mathcal{F}(n|2),
\end{align*}
\]
are linear maps. The space \(H^1_{\text{diff}}(\mathcal{K}(1)^i, n(1|1)^i; \mathcal{F}(n|2))\), \(i = 1, 2\), determines the linear maps \(Y^1, Y^2\) and \(Y^3\). The 1-cocycle conditions determines \(Y^4\). More precisely, we get:
For $n = -1$, the space $H^1_{\text{diff}}(\mathcal{K}(2), n(1|2), \mathcal{F} \mathcal{P}^{-1}(2))$ is generated by the nontrivial $n(1|2)$-relative cocycles $\chi^{2,1}$ and $\chi^{2}_2$, corresponding to the $n(1|1)^i$-relative cocycles $C^{2,i}_{-1}$ and $C^{3,i}_{-1}$ respectively, via their restrictions to $\mathcal{K}(1)^i$.

For $n = 0$, the space $H^1_{\text{diff}}(\mathcal{K}(2), n(1|2), \mathcal{F} \mathcal{P}^0(2))$ is generated by the nontrivial $n(1|2)$-relative cocycles $\chi^{2,2}_0, \chi^{2,2}_{-1}, \chi^{2,2}_0$, and $\chi^{2,2}_{-1}$, corresponding to the $n(1|1)^i$-relative cocycles $C^{1,0}_0, C^{2,1}_0, C^{3,1}_0, C^{4,1}_0$ and $C^{5,1}_0$ respectively, via their restrictions to $\mathcal{K}(1)^i$.

For $n = 1$, the space $H^1_{\text{diff}}(\mathcal{K}(2), n(1|2), \mathcal{F} \mathcal{P}^1(2))$ is generated by the nontrivial $n(1|2)$-relative cocycles $\chi^{2,1}$ corresponding to the $n(1|1)^i$-relative cocycles $C^{1,1}_1$, via their restrictions to $\mathcal{K}(1)^i$. Theorem 5 is proved.

### 6.2. The spectral sequence for a filtered module over a Lie (super)algebra

The reader should refer to [12], for the details of the homological algebra used to construct spectral sequences. We will merely quote the results for a filtered module $M$ with decreasing filtration $\{M_n\}_{n \in \mathbb{Z}}$ over a Lie (super)algebra $g$ so that $M_{n+1} \subset M_n \cup_{n \in \mathbb{Z}} M_n = M$ and $gM_n \subset M_n$.

Consider the natural filtration induced on the space of cochains by setting:

$$F^n(C^*(g, M)) = C^*(g, M_n),$$

then we have:

$$dF^n(C^*(g, M)) \subset F^n(C^*(g, M)) \quad \text{(i.e., the filtration is preserved by $d$);}$$

$$F^{n+1}(C^*(g, M)) \subset F^n(C^*(g, M)) \quad \text{(i.e. the filtration is decreasing).}$$

Then there is a spectral sequence $(E^{r,s}_r, d_r)$ for $r \in \mathbb{N}$ with $d_r$ of degree $(r, 1 - r)$ and

$$E^{0,q}_r = F^p(C^0(q + d)(g, M)) / F^{p+1}(C^{p+q}(g, M)) \quad \text{and} \quad E^{p,q}_1 = H^p(q, \text{Gr}^p(M)).$$

To simplify the notations, we have to replace $F^n(C^*(g, M))$ by $F^n C^*$. We define

$$Z^{p,q}_r = F^p C^{p+q} \cap d^{-1}(F^{p+r} C^{p+q+1}),$$

$$B^{p,q}_r = F^p C^{p+q} \cap d(F^{p-r} C^{p+q-1}),$$

$$E^{p,q}_r = Z^{p,q}_r / (Z^{p+1,q-1}_r + B^{p,q}_r).$$

The differential $d$ maps $Z^{p,q}_r$ into $Z^{p+r,q-r+1}_r$, and hence includes a homomorphism

$$d_r : E^{p,q}_r \longrightarrow E^{p+r,q-r+1}_r$$

The spectral sequence converges to $H^*(C, d)$, that is

$$E^{p,q}_\infty = F^p H^{p+q}(C, d) / F^{p+1} H^{p+q}(C, d),$$

where $F^p H^*(C, d)$ is the image of the map $H^*(F^p C, d) \to H^*(C, d)$ induced by the inclusion $F^p C \to C$.

### 6.3. Computing $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathcal{F} \mathcal{P} \mathcal{O}(S^{1|N}))$

Since the cohomology space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N), \mathcal{F} \mathcal{P} \mathcal{O}(S^{1|N}))$ is upper bounded by cohomology space $H^1_{\text{diff}}(\mathcal{K}(N), n(1|N); \mathcal{F} \mathcal{P} \mathcal{O}(S^{1|N}))$, we can check the behavior of the cocycles with values in $\mathcal{F} \mathcal{P} \mathcal{O}(S^{1|N})$ under the successive differentials of the spectral sequence. More precisely we consider a cocycle with values in $\mathcal{F} \mathcal{P} \mathcal{O}(N)$; but we compute its boundary as it was in $\mathcal{F} \mathcal{P} \mathcal{O}(S^{1|N})$ for $N = 0, 1, 2$, and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one. We iterate this procedure, we establish a recurrence formula between successive terms. A straightforward computations leads to the following result:
Theorem 9. The space $H^1_{\text{diff}}(\mathcal{H}(N), n(1|N); \mathcal{D}(S^{1|N}))$ has the following structure:

$$H^1_{\text{diff}}(\mathcal{H}(N), n(1|N); \mathcal{D}(S^{1|N})) = \begin{cases} \mathbb{R}^3 & \text{if } N = 0, 1 \\ \mathbb{R}^8 & \text{if } N = 2 \\ 0 & \text{otherwise} \end{cases}$$

(16)

The following 1-cocycles $\Xi^N_i$ span the corresponding cohomology spaces:

- $\Xi^1_N(X_F) = F'$, for $N = 0, 1, 2$,
- $\Xi^2_2(X_F) = \eta_1 \eta_2(F)$,
- $\Xi^3_3(X_F) = F' \xi^{-1} \zeta_2$,
- $\Xi^0_2(X_F) = \sum_{n=2}^{\infty} (-1)^{n-1} \frac{2(n-3)n}{n} F(n)(x) \xi^{-n+1}$,
- $\Xi^3_3(X_F) = \eta_1 \eta_2(F) \xi^{-1} \zeta_2$,
- $\Xi^0_3(X_F) = \sum_{n=2}^{\infty} (-1)^{n} \frac{3(n-1)n}{n+1} F(n+1)(x) \xi^{-n}$,
- $\Xi^1_2(X_F) = \sum_{n=1}^{\infty} (-1)^{n} \left( \frac{n-2}{n} \eta_1(F(n)) \xi^{-n} \eta_1 - \frac{n-3}{n+1} F(n+1) \xi^{-n} \right)$,
- $\Xi^2_3(X_F) = \sum_{n=2}^{\infty} (-1)^{n} \left( \frac{n-1}{n} \eta_1(F(n)) \xi^{-n} \eta_1 - \frac{n-1}{n+1} F(n+1) \xi^{-n} \right)$,
- $\Xi^3_3(X_F) = \sum_{n=0}^{\infty} (-1)^{p(F)+n} \left( \eta_1(F(n+1)) \xi^{-n} \eta_1 + \eta_2(F(n+1)) \xi^{-n} \right)$
  $$+ \sum_{n=0}^{\infty} \frac{2(-1)^{n} F(n+2)}{n+2} \xi^{-n-1},$$
- $\Xi^2_6(X_F) = \sum_{n=0}^{\infty} (-1)^n \eta_1 \eta_2(F(n+1)) \xi^{-n-2} \zeta_2 + \sum_{n=1}^{\infty} (-1)^n \eta_1 \eta_2(F(n)) \xi^{-n}$,
- $\Xi^3_7(X_F) = \sum_{n=0}^{\infty} (-1)^{n} \eta_1 \eta_2(F(n+1)) \xi^{-n-2} \zeta_2$ 
  $$+ \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1} \left( \eta_1(F(n+1)) \xi^{-n} \eta_1 + \eta_2(F(n+1)) \xi^{-n} \right) \xi^{-n-1},$$
- $\Xi^2_6(X_F) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n+2} F(n+2) \xi^{-n-2} \zeta_2$ 
  $$+ \sum_{n=1}^{\infty} (-1)^{p(F)+n} \frac{2n}{n+1} \eta_1(F(n+1)) \xi^{-n-1} \zeta_2,$$
- $\Xi^3_7(X_F) = \sum_{n=1}^{\infty} (-1)^{p(F)+n} \frac{2n}{n+1} \eta_2(F(n+1)) \xi^{-n-1} \zeta_1$ 
  $$+ \sum_{n=1}^{\infty} 2(-1)^{n+1} \eta_1 \eta_2(F(n)) \xi^{-n}.$$ 

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References


