Ji-Cai Liu

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“Elementary” Number Theory / Théorie “élémentaire” des nombres

On a congruence involving $q$-Catalan numbers

Sur une congruence impliquant des $q$-nombres de Catalan

Ji-Cai Liu$^a$

$^a$ Department of Mathematics, Wenzhou University, Wenzhou 325035, PR China. E-mail: jcliu2016@gmail.com.

Abstract. Based on a $q$-congruence of the author and Petrov, we set up a $q$-analogue of Sun–Tauraso’s congruence for sums of Catalan numbers, which extends a $q$-congruence due to Tauraso.

Résumé. À partir d’une $q$-congruence de l’auteur et Petrov, nous établissons un $q$-analogue de la congruence de Sun–Tauraso pour des sommes de nombres de Catalan, qui étend la $q$-congruence due à Tauraso.

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1. Introduction

In combinatorics, the Catalan numbers are a sequence of natural numbers, which play an important role in various counting problems. The $n$th Catalan number is given by the following binomial coefficient:

$$C_n = \binom{2n}{n} \frac{1}{n+1} = \frac{2n}{n} - \frac{2n}{2n+1}.$$

Closely related numbers are the central binomial coefficients $\binom{2n}{n}$ for $n \geq 0$.

Both Catalan numbers and central binomial coefficients satisfy many interesting congruences (see, for instance, [7, 9–11]). In 2011, Sun and Tauraso [11] proved that for primes $p \geq 5$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left( \frac{p}{3} \right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left( \frac{p}{3} \right) - \frac{1}{2} \pmod{p^2},$$

(1)  (2)
where \( \left( \frac{a}{p} \right) \) denotes the Legendre symbol.

In the past few years, \( q \)-analogues of congruences (\( q \)-congruences) for indefinite sums of binomial coefficients as well as hypergeometric series attracted many experts’ attention (see, for example, [2–6, 8, 12, 13]). It is worth mentioning that Guo and Zudilin [6] developed an interesting microscoping method to prove many \( q \)-congruences.

In order to discuss \( q \)-congruences, we first recall some \( q \)-series notation. The \( q \)-binomial coefficients are defined as

\[
\binom{n}{k}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise}, \end{cases}
\]

where the \( q \)-shifted factorial is given by \( (a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \) for \( n \geq 1 \) and \( (a; q)_0 = 1 \). Moreover, the \( q \)-integers are defined by \( [n]_q = (1-q^n)/(1-q) \), and the \( n \)th cyclotomic polynomial is given by

\[
\Phi_n(q) = \prod_{1 \leq k \leq n, (n,k)=1} (q-e^{2\pi i/n}).
\]

Recently, the author and Petrov [8] established a \( q \)-analogue for (1) as follows:

\[
\sum_{k=0}^{n-1} q^k \binom{2k}{k}_q \equiv (\frac{n}{3}) q^{2n-1} / (\Phi_n(q))^2 \pmod{\Phi_n(q)},
\]

which was originally conjectured by Guo [2] and generalises a \( q \)-congruence of Tauraso [12].

There are several natural \( q \)-analogues of Catalan numbers (see [1]). Here and throughout the paper, we consider the following \( q \)-analogue of Catalan numbers:

\[
C_n(q) = \frac{1}{[n+1]_q} \binom{2n}{n}_q = \binom{2n}{n}_q - q \binom{2n}{n+1}_q.
\]

In 2012, Tauraso [12] obtained a weak \( q \)-version of (2) as follows:

\[
\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} q^{\lfloor n/3 \rfloor} & \text{if } n \equiv 0,1 \pmod{3} \\ -1 - q^{\lfloor 2n-1 \rfloor/3} & \text{if } n \equiv 2 \pmod{3} \end{cases} \pmod{\Phi_n(q)},
\]

where \( \lfloor x \rfloor \) denotes the integral part of real \( x \). In this note, we aim to set up a \( q \)-analogue of (2) as well as another related \( q \)-congruence for sums of binomial coefficients.

**Theorem 1.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q)^2 \):

\[
\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} -q \frac{n^2-1}{3} - q^{n(2n-1)/3} & \text{if } n \equiv 2 \pmod{3} \\ q \frac{n^2-1}{3} - \frac{n-1}{3} (q^n - 1) & \text{if } n \equiv 1 \pmod{3} \end{cases}
\]

In order to prove (5), we shall establish the following \( q \)-congruence.

**Theorem 2.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q)^2 \):

\[
\sum_{k=0}^{n-1} q^{k+1} \binom{2k}{k+1}_q \equiv \begin{cases} q^{\lfloor n/3 \rfloor} & \text{if } n \equiv 2 \pmod{3}, \\ \frac{n-1}{3} (q^n - 1) & \text{if } n \equiv 1 \pmod{3} \end{cases}
\]

It is clear that (5) can be directly deduced from (3), (4) and (6). The remainder of the paper is organized as follows. We first set up a preliminary result in the next section, and prove Theorem 2 in Section 3.
2. An auxiliary result

Lemma 3. For any positive integer \( n \), the following holds modulo \( \Phi_n(q) \):

\[
\sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) (\frac{k^2-k}{3}) \frac{q^{k-1} - 1}{1 - q^k} = \begin{cases} 0 & \text{if } n \equiv 2 \pmod{3}, \\ \frac{n-1}{6} & \text{if } n \equiv 1 \pmod{3}. \end{cases}
\]  

(7)

Proof. Note that

\[
\sum_{k=1}^{n-1} (-1)^k \left( \frac{k-1}{3} \right) q^{k+1} (\frac{k^2-k+1}{3}) \frac{1}{1 - q^k} = | \sum_{k=0}^{n-3} (-1)^k q^{\frac{k(k+1)(3k+2)}{2}} | - \sum_{k=1}^{n-1} (-1)^k q^{\frac{k(3k+5)}{2}} - \sum_{k=1}^{n-1} (-1)^k q^{\frac{k(3k+5)}{2}}.
\]

We shall distinguish two cases to prove (7).

Case 1. \( n \equiv 2 \pmod{3} \). This case is equivalent to

\[
\sum_{k=1}^{n-1} (-1)^k q^{\frac{k(k+1)(3k+2)}{2}} - \sum_{k=1}^{n-1} (-1)^k q^{\frac{k(3k+5)}{2}} = 0 \pmod{\Phi_{3n+2}(q)}.
\]  

(8)

Let \( \omega \) be a primitive \((3n+2)\)th root of unity. Letting \( k \to n - k \) in the following sum gives

\[
\sum_{k=1}^{n-1} (-1)^k \omega^{\frac{k(k+1)(3k+2)}{2}} = \sum_{k=1}^{n-1} (-1)^{n-k} \omega^{\frac{(n-k)(k+1)(3n-k+2)}{2}} = \sum_{k=1}^{n-1} (-1)^{n-k} \omega^{\frac{k(3k-1)}{2} + \frac{(3n+2)(n+1)}{2} - (3n+2)k} = \sum_{k=1}^{n-1} (-1)^k \omega^{\frac{k(3k+5)}{2}},
\]

where we have used the fact that \( \omega^{\frac{(3n+2)(n+1)}{2}} = (-1)^{n+1} \). Thus,

\[
\sum_{k=0}^{n-1} (-1)^k \omega^{\frac{k(k+1)(3k+2)}{2}} - \sum_{k=1}^{n-1} (-1)^k \omega^{\frac{k(3k+5)}{2}} = 0,
\]

which is equivalent to (8).

Case 2. \( n \equiv 1 \pmod{3} \). Let \( \zeta \) be a primitive \((3n+1)\)th root of unity. It suffices to show that

\[
\sum_{k=0}^{n-1} (-1)^k \zeta^{\frac{k(k+1)(3k+2)}{2}} - \sum_{k=1}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = \frac{n}{2}.
\]  

(9)

Note that

\[
\sum_{k=0}^{n-1} (-1)^k \zeta^{\frac{k(k+1)(3k+2)}{2}} - \sum_{k=1}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = 2 \sum_{k=n+1}^{n-1} (-1)^{2n-k} \zeta^{\frac{(2n-k+1)(6n-k+2)}{2}} - \sum_{k=n+1}^{n-1} (-1)^k \zeta^{\frac{k(3k-1)}{2}} + (3n+1)(2n-2k+1) = 2 \sum_{k=n+1}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} - \sum_{k=n+1}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = 0,
\]

where we replace \( k \) by \( 2n - k \) in the first step. Thus,

\[
\sum_{k=0}^{n-1} (-1)^k \zeta^{\frac{k(k+1)(3k+2)}{2}} - \sum_{k=1}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}} = \sum_{k=1}^{n-1} (-1)^k \zeta^{\frac{k(3k+5)}{2}}.
\]  

(10)
Furthermore, letting \( k \to 2n + 1 - k \) on the right-hand side of (10) gives
\[
\sum_{k=0}^{n-1} \frac{(-1)^k \zeta^{(k+1)(3k+2)/2}}{1 - \zeta^{3k+2}} - \sum_{k=1}^{n} \frac{(-1)^k \zeta^{(3k+5)/2}}{1 - \zeta^{3k+2}} = -\sum_{k=1}^{2n} \frac{(-1)^{2n+1-k} \zeta^{(2n+1-k)(6n-3k+8)/2}}{1 - \zeta^{3(2n+1-k)}}
\]
\[
= -\sum_{k=1}^{2n} \frac{(-1)^{1-k} \zeta^{(2k-1)(k-2)/2} + (3n+1)(2n+3-2k)}{1 - \zeta^{1-3k}}
\]
\[
= -\sum_{k=1}^{2n} \frac{(-1)^k \zeta^{(3k-1)/2}}{1 - \zeta^{3k-1}}. \tag{11}
\]

An identity due to the author and Petrov [8, (2.4)] says
\[
\sum_{k=1}^{2n} \frac{(-1)^k \zeta^{(3k-1)/2}}{1 - \zeta^{3k-1}} = -\frac{n}{2}. \tag{12}
\]

Then the proof of (9) follows from (11) and (12). \(\square\)

3. Proof of Theorem 2

Now we are in a position to prove Theorem 2. We recall the following identity:
\[
\sum_{k=0}^{n-1} q^k \left\lfloor \frac{2k}{k+1} \right\rfloor = \sum_{k=0}^{n-1} \left( \frac{n-k-1}{3} \right) q^{(2(n-k)^2-(n-k)(\frac{n-k-1}{3})-3)} \left\lfloor \frac{2n}{k} \right\rfloor, \tag{13}
\]
which was proved by Tauraso in a more general form (see [12, Theorem 4.2]). Since \( 1 - q^n \equiv 0 \) (mod \( \Phi_n(q) \)), we have
\[
1 - q^{2n} = (1 + q^n)(1 - q^n) \equiv 2(1 - q^n) \quad \text{(mod \( \Phi_n(q)^2 \))}
\]
It follows that for \( 1 \leq k \leq n-1 \),
\[
\left\lfloor \frac{2n}{k} \right\rfloor = \frac{(1 - q^{2n})(1 - q^{2n-1}) \cdots (1 - q^{2n-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)}
\]
\[
\equiv 2(1 - q^n) \frac{(1 - q^{-1}) \cdots (1 - q^{-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)} \quad \text{(mod \( \Phi_n(q)^2 \))}
\]
\[
= 2(q^n - 1) \frac{(-1)^k q^{-\frac{1}{3}(k-1)-1}}{1 - q^{-k}} \tag{14}
\]

Multiplying both sides of (13) by \( q \) and substituting (14) into the right-hand side of (13), we arrive at
\[
\sum_{k=0}^{n-1} q^k \left\lfloor \frac{2k}{k+1} \right\rfloor = \left( \frac{n-1}{3} \right) q^{\frac{1}{3}(2n^2-n(n+1))} + \sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) q^{\frac{1}{3}(2(n-k)^2-(n-k)(\frac{n-k-1}{3})-3)} \left\lfloor \frac{2n}{k} \right\rfloor
\]
\[
\equiv \left( \frac{n-1}{3} \right) q^{\frac{1}{3}(2n^2-n(n+1))} + 2(q^n - 1) \sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) \left( -1 \right)^k q^{\frac{1}{3}(2(n-k)^2-(n-k)(\frac{n-k-1}{3})-\frac{1}{3}(k-1)-1)} \left( \frac{n-k-1}{3} \right)^{-1} \quad \text{(mod \( \Phi_n(q)^2 \))}. \tag{15}
\]
Furthermore,
\[
\sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) \frac{(-1)^k q^{\frac{1}{3}(2(n-k)^2-(n-k)(a-k-\frac{a-1}{3}))}}{1-q^k} \equiv \sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) \frac{(-1)^k q^{\frac{1}{3}(2k^2-k(\frac{k+1}{3}))}}{1-q^k} \quad \text{(mod } \Phi_n(q))
\]
where we set \( k = n - k \) in the first step. Thus,
\[
\sum_{k=0}^{n-1} q^{k+1} \left[ \frac{2k}{k+1} \right] \equiv \left( \frac{n-1}{3} \right) q^{\frac{1}{3}(2n^2-n(\frac{n-1}{3}))} + 2(q^n-1) \sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) \frac{(-1)^k q^{\frac{1}{3}(2k^2-k(\frac{k+1}{3}))}}{1-q^k} \quad \text{(mod } \Phi_n(q)) \quad (16)
\]
We complete the proof of (6) by combining (7) and (16).

References