



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Daniel Álvarez

Leaves of stacky Lie algebroids

Volume 358, issue 2 (2020), p. 217-226

Published online: 15 June 2020

<https://doi.org/10.5802/crmath.37>



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569



Differential Geometry / *Géométrie différentielle*

Leaves of stacky Lie algebroids

Feuilles des algébroides de Lie sur des champs différentiels

Daniel Álvarez^a

^a Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina, 110, Jardim Botânico, CEP 22460320, Rio de Janeiro, Brasil.

E-mail: uerbum@impa.br.

Abstract. We show that the leaves of an LA-groupoid which pass through the unit manifold are, modulo a connectedness issue, Lie groupoids. We illustrate this phenomenon by considering the cotangent Lie algebroids of Poisson groupoids thus obtaining an interesting class of symplectic groupoids coming from their symplectic foliations. In particular, we show that for a (strict) Lie 2-group the coadjoint orbits of the units in the dual of its Lie 2-algebra are symplectic groupoids, meaning that the classical Kostant–Kirillov–Souriau symplectic forms on these special coadjoint orbits are multiplicative.

Résumé. Nous montrons que les feuilles d'un groupoïde en algébroïde de Lie qui passent par la variété unité sont elles mêmes des groupoïdes de Lie, sous une condition de connexité. Ce résultat est appliqué aux algébroides de Lie cotangents des groupoïdes de Poisson. On obtient ainsi une classe intéressante de groupoïdes symplectiques associés à leurs feuilletages symplectiques. Nous montrons en particulier que pour un 2-groupe de Lie strict, les orbites coadjointes des unités dans le dual de sa 2-algèbre de Lie sont des groupoïdes symplectiques, en ce sens que les formes symplectiques classiques de Kostant–Kirillov–Souriau sur ces orbites coadjointes spéciales sont multiplicatives.

Manuscript received 17th September 2019, revised 20th February 2020, accepted 10th March 2020.

1. Introduction

Lie algebroids are a unifying concept in differential geometry: they allow us to describe foliations, Lie algebra actions, connections on principal bundles and Poisson brackets among many other examples. Lie algebroids can also be seen as the infinitesimal objects associated to Lie groupoids, generalizing the correspondence between Lie groups and Lie algebras. Lie groupoids, on the other hand, provide geometric models for singular spaces when regarded as atlases for differentiable stacks, see [3]. Lie algebroids over differentiable stacks can be described by *LA-groupoids* [15] which are essentially groupoid objects in the category of Lie algebroids and so they can be called “stacky Lie algebroids”, see [21] for a precise statement. To every Lie algebroid

is associated a singular foliation on its base manifold. We shall see that, at least up to a connectedness issue, the leaves of the singular foliation associated to an LA-groupoid which pass through the unit manifold inherit themselves a Lie groupoid structure, see Theorem 6.

In order to illustrate Theorem 6 we shall focus on Poisson structures. A Poisson structure on a manifold determines a singular symplectic foliation which completely characterizes it [22]. From the viewpoint of Lie theory, Poisson structures arise as the infinitesimal counterparts of Lie groupoids endowed with a compatible symplectic structure, which are called *symplectic groupoids* [11, 23]. Whenever a Poisson structure is induced by a symplectic groupoid, it is called *integrable*; but unlike Lie algebras, not every Poisson manifold is integrable. Symplectic groupoids are important tools in the study of Poisson manifolds but in general they are difficult to construct, see e.g. [9].

A *Poisson groupoid* [24] is a Lie groupoid endowed with a Poisson structure which is compatible with the groupoid multiplication: these objects generalize on the one hand symplectic groupoids and, on the other, they generalize Poisson–Lie groups [10]. We shall see in Proposition 12 that those leaves of the symplectic foliation associated to a Poisson groupoid structure which pass through the unit manifold provide examples of symplectic groupoids.

We exemplify this observation by means of (strict) Lie 2-groups [2], which are group objects in the category of Lie groupoids. Lie 2-groups equipped with a Poisson structure compatible with both the group and groupoid multiplications are called *Poisson 2-groups* [6]. The symplectic leaves of Poisson groups are the orbits of the *infinitesimal dressing actions* [19]; for Poisson 2-groups these dressing actions are multiplicative actions, meaning that the action maps are Lie groupoid morphisms. The simplest example of a Poisson 2-group is a Lie 2-group endowed with the zero bracket. In this case, the dressing action coincides with the coadjoint action on the dual of its Lie algebra so we get, in particular, that the coadjoint orbits of Lie 2-groups which contain a unit, endowed with their canonical Kostant–Kirillov–Souriau symplectic forms, are symplectic groupoids.

2. Preliminaries

2.1. Lie groupoids and Lie algebroids

A *smooth groupoid* is a groupoid object in the category of not necessarily Hausdorff smooth manifolds such that its source map is a submersion. A *Lie groupoid* is a smooth groupoid such that its base and source-fibers are Hausdorff manifolds; see e.g. [17] for an introduction to these concepts. We denote the source, target, multiplication, unit and inversion maps of a Lie groupoid M_1 over M_0 , $M_1 \rightrightarrows M_0$, as s , t and m , u , i respectively. In order to avoid ambiguity when dealing with several groupoids we use a subindex $s = s_{M_1}$, t_{M_1} , m_{M_1} to specify the groupoid under consideration. A *Lie algebroid* consists of a vector bundle A over a manifold M_0 endowed with a vector bundle map $a : A \rightarrow TM_0$, called the *anchor*, and a Lie algebra bracket on $\Gamma(A)$ that satisfies the Leibniz rule: $[X, fY] = f[X, Y] + (\mathcal{L}_{a(X)} f)Y$ for all $X, Y \in \Gamma(A)$ and all $f \in C^\infty(M)$. Given a Lie groupoid $M_1 \rightrightarrows M_0$, there is a canonical way to endow $\ker Ts|_{M_0}$ with a Lie algebroid structure over M_0 , see [17]; we call it the *Lie algebroid of $M_1 \rightrightarrows M_0$* and we denote it by A_{M_1} .

2.2. LA-groupoids and double Lie groupoids

Definition 1 ([15]). *An LA-groupoid is a Lie groupoid in the category of Lie algebroids, that is, a Lie groupoid $A_1 \rightrightarrows A_0$ where A_1 and A_0 are Lie algebroids, the structure maps are Lie algebroid morphisms over the structure maps of a base groupoid $M_1 \rightrightarrows M_0$, and the map $A_1 \rightarrow s_{M_1}^* A_0$ induced by s_{A_1} is surjective.*

Example 2. Let $M_1 \rightrightarrows M_0$ be a Lie groupoid. In general, $TM_1 \rightrightarrows TM_0$ (with the structure maps given by applying the tangent functor) and the usual Lie bracket is an LA-groupoid.

Let $M_1 \rightrightarrows M_0$ be a Lie groupoid. Then $T^*M_1 \rightrightarrows A_{M_1}^*$ (where $A_{M_1} \rightarrow M_0$ is the Lie algebroid of $M_1 \rightrightarrows M_0$) is also a Lie groupoid, called the *cotangent groupoid* of $M_1 \rightrightarrows M_0$. The multiplication \widehat{m} in T^*M_1 is characterized by the following property:

$$\langle \widehat{m}(\xi, \eta), Tm(u, v) \rangle = \langle \xi, u \rangle + \langle \eta, v \rangle, \tag{1}$$

for all composable $u \in T_x M_1, v \in T_y M_1$ and $\xi \in T_x^* M_1, \eta \in T_y^* M_1$ [7].

LA-groupoids are the infinitesimal counterparts to the following higher categorical structures.

Definition 3 ([4, 15]). *A double topological groupoid is a groupoid object in the category of topological groupoids. A double topological groupoid is represented as a diagram of the form*

$$\begin{array}{ccc} G_1 & \rightrightarrows & M_1 \\ \Downarrow & & \Downarrow \\ G_0 & \rightrightarrows & M_0. \end{array}$$

A double Lie groupoid is a double topological groupoid as in the previous diagram such that: (1) each of the side groupoids is a smooth groupoid, (2) M_1 and G_0 are Lie groupoids over M_0 , and (3) the double source map $(s^{M_1}, s^{G_0}) : G_1 \rightarrow M_1 \times_{M_0} G_0$ is a submersion (the superindices $^{M_1}, ^{G_0}$ denote the groupoid structures $G_1 \rightrightarrows M_1, G_1 \rightrightarrows G_0$ respectively).

It is known that double Lie groupoids differentiate to LA-groupoids [15], but the integration problem is not fully understood, see [20]. The next simple example shows that the naïve attempt to integrate an LA-groupoid by applying Lie’s second theorem to all its structure maps fails in general. Other integrations of the Lie algebroids involved can still carry a double Lie groupoid structure, the best candidate being an integration whose vertical source-fibers are 1-connected in the stacky sense, see [18].

Example 4. Consider a free action of \mathbb{R}^2 on a manifold M with at least three different orbits O_i . Consider $x_i \in O_i$ for $i = 1, 2, 3$ and $v \in \mathbb{R}^2$ with $|v| > 1$ and take

$$N = M - (\{v \cdot x_1\} \cup \{z \cdot x_3 \mid z \in \mathbb{S}^1\}).$$

On N there is an infinitesimal action of \mathbb{R}^2 which induces a diagonal action on N^2 ; the image of this action is a distribution D which is a subgroupoid of the tangent groupoid $TN^2 \rightrightarrows TN$ with respect to the pair groupoid structure on $N^2 \rightrightarrows N$ and hence it is an LA-groupoid.

The leaf of D through (x_1, x_2) is diffeomorphic to \mathbb{R}^2 with a point removed and the leaves through (x_1, x_3) and (x_2, x_3) are diffeomorphic to a disk. Call \mathcal{O}_{ij} the leaf through (x_i, x_j) . We have that $\pi_1(\mathcal{O}_{12}, (x_1, x_2)) \cong \mathbb{Z}$. Take a non trivial homotopy class a in that group. Then $s(a) \in \pi_1(O_2) = 1$ is the trivial class so a and the constant path based on (x_2, x_3) should be composable. However, there is no element in $\pi_1(\mathcal{O}_{13}, (x_1, x_3)) = 1$ which projects to $t(a) \in \pi_1(O_1 - \{v \cdot x_1\})$ which is non trivial. So the monodromy groupoid of D carries no compatible groupoid structure over the monodromy groupoid of $D|_N$.

3. The main result

Definition 5. Let $A_1 \rightrightarrows A_0$ be an LA-groupoid over $M_1 \rightrightarrows M_0$. Let $\mathcal{O} \subset M_0$ be an A_0 -orbit and take \mathcal{O}' the A_1 -orbit which contains \mathcal{O} . Let us denote by $\widehat{\mathcal{O}}$ the union of the connected components of $\mathcal{O}' \cap s_{M_1}^{-1}(x)$ which contain a unit for all $x \in \mathcal{O}$.

Theorem 6. Let $A_1 \rightrightarrows A_0$ be an LA-groupoid over $M_1 \rightrightarrows M_0$ and let $\mathcal{O} \subset M_0$ be an A_0 -orbit. Then $\widehat{\mathcal{O}}$ is an immersed Lie subgroupoid of M_1 over \mathcal{O} .

What motivates our main result is the following algebraic observation. If an LA-groupoid is integrable by a double Lie groupoid, then we can use this integration to show that the orbits through the units inherit a smooth groupoid structure.

Proposition 7. *Let G_1 be a double Lie groupoid with sides M_1 and G_0 over M_0 . Then the G_1 -orbits of the units in M_1 are immersed smooth subgroupoids over the corresponding G_0 -orbits in M_0 .*

Proof. Let $g, h \in G_1$ be such that $s^{M_1}(g) = s^{M_1}(h) = u \in M_0$ and $t^{G_0}(g) = x, t^{G_0}(h) = y$ are composable, i.e. $s_{M_1}(x) = t_{M_1}(y)$. Then $k = m^{M_1}(i^{M_1}(s^{G_0}(g)), t^{G_0}(h))$ is defined and so is $g' = m^{M_1}(g, k)$. Since $s^{G_0}(g') = t^{G_0}(h)$ we can take $l = m^{G_0}(g', h)$ and then $s^{M_1}(l) = u, t^{M_1}(l) = m^{G_0}(x, y)$. So these orbits are subgroupoids. In general, the orbits of a smooth groupoid are immersed submanifolds even when the base is non Hausdorff. \square

In general, we have to use an infinitesimal argument in order to prove the result.

Proof of Theorem 6. Let \mathcal{O} be an A_0 -orbit and let \mathcal{O}' be the A_1 -orbit which contains \mathcal{O} . Let us denote by $s_{M_1} = s$ and $t_{M_1} = t$ for simplicity in what follows. Recall that $A_{M_1} = \ker T s|_{M_0}$ is the Lie algebroid of $M_1 \rightrightarrows M_0$ and $a = T t|_{A_{M_1}}$ is its anchor. This proof consists of the following steps:

- (1) we construct a Lie subalgebroid B of the tangent Lie algebroid of $M_1 \rightrightarrows M_0$ over \mathcal{O} such that the span of the right-invariant vector field in M_1 induced by its inclusion in A_{M_1} coincides with $(T\mathcal{O}' \cap \ker T s)|_{\mathcal{O}}$;
- (2) using Lie's second theorem we show that $\widehat{\mathcal{O}}$ is the image of a Lie groupoid morphism $\Psi : H \rightarrow M_1$, where H is the source-simply-connected integration of B . Since Ψ is the identity on \mathcal{O} , we see that $\widehat{\mathcal{O}} = \Psi(H)$ inherits a groupoid structure over \mathcal{O} . Let us stress that our reliance on Lie's second theorem implies the existence of a groupoid structure only on $\widehat{\mathcal{O}}$, which can be seen as the "source-connected component" of \mathcal{O}' .

Step 1. Since s_{A_1} is fiberwise surjective, the pullback of A_1 to M_0 splits as the sum $A_1|_{M_0} = \ker s_{A_1}|_{M_0} \oplus A_0$.

By definition, we have that $T_x \mathcal{O}' = a_1(A_1|_x)$ for all $x \in \mathcal{O}$. But the anchor map a_1 of A_1 , being a groupoid morphism, induces a square of morphisms of vector bundles over G_0 :

$$\begin{array}{ccc} \ker s_{A_1}|_{M_0} & \xrightarrow{t_{A_1}} & A_0 \\ a_1| \downarrow & & \downarrow a_0 \\ A_{M_1} & \xrightarrow{a} & TM_0, \end{array}$$

where a_0 is the anchor of A_0 . So we have that $T_x \mathcal{O}' = a_1(\ker s_{A_1}|_x) \oplus a_0(A_0|_x)$ if $x \in \mathcal{O}$.

It is known that $C := \ker s_{A_1}|_{M_0}$ is a Lie algebroid over M_0 called the *core Lie algebroid* of $A_1 \rightrightarrows A_0$ [15]. Let us recall its structure maps: $\Gamma(C)$ is identified with the space of right invariant sections of A_1 as follows: to $X \in \Gamma(C)$ we associate the section $p \mapsto X^r(p) = m_{A_1}(X(t(p)), 0_p)$ for all $p \in M_1$. By the compatibility of the bracket with the multiplication on A_1 , the space of right invariant sections is closed under the bracket of A_1 . Since a_1 preserves the bracket and it takes right invariant sections of A_1 to right invariant vector fields on M_1 , the map

$$\phi := a_1| : C \rightarrow A_{M_1}$$

is a Lie algebroid morphism. As a consequence, C is a Lie algebroid over M_0 with anchor $a \circ a_1|_C$.

Let X be a section of C , define $\rho(X) \in \mathfrak{X}(\mathcal{O})$ as the vector field $(a \circ \phi)(X)|_{\mathcal{O}} = (a_0 \circ t_{A_1})(X)|_{\mathcal{O}}$. We have that ρ defines an action Lie algebroid $i^* C$ over \mathcal{O} , where $i : \mathcal{O} \hookrightarrow M_0$ is the inclusion. In fact, take $X, Y \in \Gamma(C)$, then $t_{A_1}|_p(X^r|_p) = t_{A_1}|_{t(p)}(X)$ for all $p \in G_1$. But t_{A_1} is a Lie algebroid morphism so we get that

$$t_{A_1}|_p([X^r, Y^r]|_p) = [t_{A_1}(X^r), t_{A_1}(Y^r)]|_{t(p)}$$

for all $p \in M_1$. Since $[X, Y] = [X^r, Y^r]|_{M_0}$ by definition, we get that

$$\mathfrak{t}_{A_1}|_x([X, Y]|_x) = [\mathfrak{t}_{A_1}(X), \mathfrak{t}_{A_1}(Y)]|_x$$

for all $x \in M_0$. Therefore, $\rho([X, Y]) = \mathfrak{a}_0([\mathfrak{t}_{A_1}(X), \mathfrak{t}_{A_1}(Y)])|_{\mathcal{O}} = [\rho(X), \rho(Y)]$ since \mathfrak{a}_0 commutes with the Lie brackets.

By construction, ϕ induces a Lie algebroid morphism $\phi' : i^*C \rightarrow A_{M_1}$ over i . Indeed, let $\sum_a f^a \otimes X_a, \sum_b g^b \otimes Y_b$ be elements of $C^\infty(\mathcal{O}) \otimes_{C^\infty(M_0)} \Gamma(C) \cong \Gamma(i^*C)$. Then

$$\phi' \left(\sum_a f^a \otimes X_a \right) = \sum_a f^a \otimes \phi(X_a), \quad \phi' \left(\sum_b g^b \otimes Y_b \right) = \sum_b g^b \otimes \phi(Y_b)$$

are elements of $C^\infty(\mathcal{O}) \otimes_{C^\infty(M_0)} \Gamma(A_{M_1}) \cong \Gamma(i^*A_{M_1})$. Hence we have that

$$\begin{aligned} \phi' \left(\left[\sum_a f^a \otimes X_a, \sum_b g^b \otimes Y_b \right] \right) &= \sum_{a,b} f^a g^b \otimes [\phi(X_a), \phi(Y_b)] \\ &\quad + \sum_{a,b} f^a \mathcal{L}_{\rho(X_a)} g^b \otimes \phi(Y_b) - \sum_{a,b} g^b \mathcal{L}_{\rho(Y_b)} f^a \otimes \phi(X_a); \end{aligned}$$

and so ϕ' is a Lie algebroid morphism, see [17] for the definition of Lie algebroid morphism that we used.

The image $\phi'(i^*C)$ is a Lie subalgebroid of A_{M_1} as long as it is a smooth subbundle. This follows from the following dimension counting:

$$\dim \mathcal{O}'_x = \text{rank } \mathfrak{a}_1|_{A_1|_x} = \text{rank } \mathfrak{a}_1|_{C_x} + \text{rank } \mathfrak{a}_0|_x$$

for all $x \in \mathcal{O}$. The map \mathfrak{a}_1 has constant rank over \mathcal{O} and the term $\text{rank } \mathfrak{a}_0|_x$ is constant over \mathcal{O} so ϕ'_x is of rank equal to $\text{rank } \mathfrak{a}_1|_{C_x}$ which is constant for $x \in \mathcal{O}$. Since the anchor of i^*C restricted to $\ker \phi'$ vanishes, we have that the quotient $B := i^*C / \ker \phi'$ inherits a Lie algebroid structure over \mathcal{O} . Then ϕ' induces an injective Lie algebroid morphism $\psi : B \hookrightarrow A_{M_1}$ over i .

Step 2. Since $\mathcal{O} \hookrightarrow M_0$ is an immersed submanifold and $B \hookrightarrow A_{M_1}$ is a Lie subalgebroid, Lie's second theorem implies that B is integrable and the morphism $\psi : B \hookrightarrow A_{M_1}$ is integrable by a Lie groupoid morphism $\Psi : H \rightarrow M_1$ which is an immersion, see [17]. We claim that $\widehat{\mathcal{O}} = \Psi(H)$. In fact, $T\mathcal{O}'$ fits into the fiberwise exact sequence

$$0 \longrightarrow T\mathcal{O}' \cap \ker Ts \longrightarrow T\mathcal{O}' \xrightarrow{Ts} T\mathcal{O} \longrightarrow 0$$

and $T\Psi(H)$ spans both $(T\mathcal{O}' \cap \ker Ts)|_{\widehat{\mathcal{O}}}$ and $T\mathcal{O}$ by construction. Since \mathcal{O}' is weakly embedded in M_1 [12], Ψ is also smooth with respect to the immersed submanifold structure on \mathcal{O} . Then $\widehat{\mathcal{O}}$ is an open subset of \mathcal{O} thanks to the fact that Ψ is an immersion and $\dim H = \dim \mathcal{O}$. Therefore, $\widehat{\mathcal{O}}$ is also an immersed weakly embedded submanifold. Finally, Ψ is a groupoid morphism which is injective on its base so its image is a subgroupoid of M_1 . Therefore, the result holds. \square

Remark 8. It is natural to ask whether the groupoid structure that we find on $\widehat{\mathcal{O}}$ extends to the whole A_1 -orbit \mathcal{O}' . We do not know if that is the case. Let us note that, thanks to Proposition 7, a counterexample would have to come from an LA-groupoid which does not admit any source-connected double groupoid integrating it, even in a weak sense as the Weinstein groupoids integrate general Lie algebroids [8].

4. Applications

4.1. Symplectic leaves of Poisson groupoids

Definition 9 ([24]). A Poisson groupoid is a Lie groupoid $M_1 \rightrightarrows M_0$ with a Poisson structure on M_1 such that the graph of the multiplication map is a coisotropic submanifold¹ of $M_1 \times M_1 \times \overline{M_1}$, where $\overline{M_1}$ denotes M_1 with the opposite Poisson structure.

A Poisson groupoid over a point is a Poisson group [10]; a Poisson groupoid whose Poisson bracket is nondegenerate, and hence induced by a symplectic form, is a symplectic groupoid [11, 23]. The infinitesimal counterpart of a Poisson groupoid is provided by the following objects.

Definition 10 ([16]). Consider a Lie algebroid A . If A^* possesses also a Lie algebroid structure such that its differential d_* satisfies

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y],$$

for all $X, Y \in \Gamma(A)$, then we call (A, A^*) a Lie bialgebroid.

The cotangent bundle of a Poisson manifold is canonically endowed with a Lie algebroid structure. In the case of a Poisson groupoid $M_1 \rightrightarrows M_0$ such a Lie algebroid structure is compatible with the cotangent groupoid structure thus producing an LA-groupoid. In fact, it follows from equation (1) that the canonical symplectic form on T^*M_1 makes $T^*M_1 \rightrightarrows A^*_{M_1}$ into a symplectic groupoid; from this fact it follows that the conormal bundle of the graph Γ of m in G_1^3 coincides with the bundle

$$E = \{(\xi, \eta, -\widehat{m}(\xi, \eta)) : \xi, \eta \in T^*M_1 \text{ composable}\} \subset (T^*M_1)^3$$

since $E \subset (TT)^\circ$ and both vector bundles have the same rank. Using these observations it is not difficult to prove the following.

Theorem 11 ([16]). Let Π be a bivector field on a Lie groupoid $M_1 \rightrightarrows M_0$. The map $\Pi^\sharp : T^*M_1 \rightarrow TM_1$ is a groupoid morphism if and only if the graph of m in $(M_1^3, \Pi \oplus \Pi \oplus \Pi)$ is coisotropic.

As a consequence of Theorem 11, if (M_1, Π) is a Poisson groupoid, then its cotangent groupoid with the Lie bracket induced by Π is an LA-groupoid.

Let A be the Lie algebroid of a Poisson groupoid $M_1 \rightrightarrows M_0$. We have that M_0 inherits a unique Poisson structure such that the target map $\tau : M_1 \rightarrow M_0$ is a Poisson morphism [24]. The symplectic leaves of this Poisson structure are contained in the connected components of the intersection of the A and A^* -orbits [16]. Let \mathcal{O} be an A^* -orbit, we have that \mathcal{O} is a Poisson submanifold of M_0 , being a union of symplectic leaves. Theorem 6 applied to $T^*M_1 \rightrightarrows A^*$ gives us a symplectic groupoid structure on $\widehat{\mathcal{O}} \rightrightarrows \mathcal{O}$ which integrates this Poisson structure.

Proposition 12. Let $M_1 \rightrightarrows M_0$ be a Poisson groupoid with tangent Lie bialgebroid (A, A^*) and let $\mathcal{O} \subset M_0$ be an A^* -orbit. Then $\widehat{\mathcal{O}}$ is an immersed subgroupoid of M_1 and it inherits a symplectic groupoid structure which integrates the Poisson structure on \mathcal{O} .

Proof. The symplectic structure on $\widehat{\mathcal{O}}$ is automatically multiplicative: if we see it as a nondegenerate Poisson bracket, then we have that $\text{graph}(m_G) \cap (\widehat{\mathcal{O}} \times \widehat{\mathcal{O}} \times \widehat{\mathcal{O}}) = \text{graph}(m_{\widehat{\mathcal{O}}})$ is coisotropic in $\widehat{\mathcal{O}} \times \widehat{\mathcal{O}} \times \widehat{\mathcal{O}}$.

Now we just have to compare the Poisson structure on \mathcal{O} induced by the target map of $\widehat{\mathcal{O}}$ with the Poisson structure on \mathcal{O} induced as a submanifold of M_0 . We have that $i : \widehat{\mathcal{O}} \hookrightarrow M_1$ is a Poisson morphism and hence so it is the composition $\tau \circ i : \widehat{\mathcal{O}} \rightarrow M_0$. But $\tau \circ i$ is the target map of $\widehat{\mathcal{O}} \rightrightarrows \mathcal{O}$ so both Poisson structures on \mathcal{O} coincide. □

¹Let M be a Poisson manifold with Poisson tensor π . A submanifold C of M is coisotropic if $\pi^\sharp(T^\circ C) \subset TC$, where $T^\circ C$ is the annihilator of TC .

4.2. Orbits of Lie 2-algebra actions on Lie groupoids

The symmetries of Lie groupoids are described by the following objects.

Definition 13 ([2,5]). A Lie 2-group is a groupoid object in the category of Lie groups.

A groupoid in the category of (Lie) groups is the same thing as a group in the category of (Lie) groupoids. This means that, given a Lie 2-group, $\mathcal{G} : G_1 \rightrightarrows G_0$, the multiplication maps $G_i \times G_i \rightarrow G_i$ and the group inversion map $G_i \rightarrow G_i$, $i = 0, 1$, determine Lie groupoid morphisms, in the first case from $\mathcal{G} \times \mathcal{G}$ to \mathcal{G} and in the second from \mathcal{G} to itself.

Definition 14 ([1]). A groupoid in the category of Lie algebras is called a strict Lie 2-algebra. An abelian Lie 2-algebra is called a 2-vector space.

It is straightforward to see that the Lie functor establishes an equivalence of categories between the categories of 1-connected Lie 2-groups and Lie 2-algebras.

Lie 2-algebras are equivalent to crossed modules of Lie algebras.

Definition 15 ([1]). Let $\phi : \mathfrak{h} \rightarrow \mathfrak{g}$ be a Lie algebra morphism and let \mathfrak{g} act on \mathfrak{h} by derivations in such a way that for all $x \in \mathfrak{g}$ and $a, b \in \mathfrak{h}$

$$\begin{aligned} \phi(x \cdot a) &= [x, \phi(a)] \\ \phi(a) \cdot b &= [a, b]; \end{aligned}$$

then this structure is called a differential crossed module or a crossed module of Lie algebras. The Lie 2-algebra associated to ϕ is the action groupoid $\mathfrak{G} : \mathfrak{h} \rtimes \mathfrak{g} \rightrightarrows \mathfrak{g}$ with action map $\tau(a, x) = \phi(a) + x$, for $a \in \mathfrak{h}$, $x \in \mathfrak{g}$. Here $\mathfrak{h} \rtimes \mathfrak{g}$ is the semidirect product Lie algebra given by the \mathfrak{g} -action.

Let $M_1 \rightrightarrows M_0$ be a Lie groupoid and let $\mathfrak{G} = \mathfrak{h} \rtimes \mathfrak{g} \rightrightarrows \mathfrak{g}$ be a Lie 2-algebra. Suppose that we have an infinitesimal action $\mathfrak{G} \rightarrow \mathfrak{X}(M_1)$ such that the induced vector bundle map $a : \mathfrak{G} \times M_1 \rightarrow TM_1$ is a groupoid morphism with the product groupoid structure on $\mathfrak{G} \times M_1$. Then the action Lie algebroid $\mathfrak{G} \times M_1$ becomes an LA-groupoid over M_1 . As another corollary of Theorem 6 we get the following result.

Proposition 16. Let $\mathcal{O} \subset M_0$ be a \mathfrak{g} -orbit, then $\widehat{\mathcal{O}}$ is a Lie subgroupoid of M_1 over \mathcal{O} . □

4.3. Dressing orbits of Poisson 2-groups

Now we shall see a family of examples that illustrate both Proposition 12 and Proposition 16.

Definition 17. Consider a Lie 2-algebra $\mathfrak{G} : \mathfrak{h} \rtimes \mathfrak{g} \rightrightarrows \mathfrak{g}$ associated to the differential crossed module $\phi : \mathfrak{h} \rightarrow \mathfrak{g}$. The map $-\phi^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ produces an action groupoid (given by the induced action by translations) we shall denote \mathfrak{G}^* ; the source and target maps of \mathfrak{G}^* have to be reversed in order to have the pairing $\mathfrak{G}^* \times \mathfrak{G} \rightarrow \mathbb{R}$ as a groupoid morphism, that is, for all $(\alpha, \theta) \in (\mathfrak{h} \rtimes \mathfrak{g})^*$:

$$s(\alpha, \theta) = \alpha - \phi^*(\theta), \quad \tau(\alpha, \theta) = \alpha.$$

Definition 18. The coadjoint action of a Lie 2-group $\mathcal{G} : H \rtimes G \rightrightarrows G$ on \mathfrak{G}^* is the usual (right) coadjoint action of $H \rtimes G$ on $(\mathfrak{h} \rtimes \mathfrak{g})^*$ at the level of arrows and, at the level of objects, it is the dual of the G -action on \mathfrak{h} .

Remark 19. The coadjoint action above is multiplicative in the sense that the action map $\mathfrak{G}^* \times \mathcal{G} \rightarrow \mathfrak{G}^*$ is a groupoid morphism.

Definition 20 ([6]). A Poisson 2-group is a Lie 2-group which is a Poisson groupoid and whose group of arrows is a Poisson group with the same Poisson bivector.

Example 21. The pair groupoid of a Poisson group is a Poisson 2-group.

Example 22. Let \mathcal{G} be a Lie 2-group over G . Then the dual of its Lie 2-algebra \mathfrak{G}^* is a Poisson 2-group with respect to the vector space addition as group structure and with the classical Lie-Poisson bracket as its Poisson structure. This follows from Theorem 11 by observing that the coadjoint action $\mathfrak{G}^* \times \mathcal{G} \rightarrow \mathfrak{G}^*$ of Definition 18 is multiplicative and hence the contraction map $\Pi^\sharp : T^*\mathfrak{G}^* \cong \mathfrak{G}^* \times \mathfrak{G} \rightarrow T\mathfrak{G}$ is a groupoid morphism since it is just the associated infinitesimal action.

The Lie algebroid structure induced by the Poisson bracket on the cotangent bundle of a Poisson group is isomorphic to an infinitesimal Lie algebra action called *dressng action* [19]. If \mathcal{G} is a Poisson group, we have that the cotangent groupoid $T^*\mathcal{G}$ is isomorphic to the product groupoid $\mathcal{G} \times \mathfrak{G}^*$ and, as a consequence of the previous observation and Theorem 11, there is a Lie 2-algebra structure on \mathfrak{G}^* in such a way that the dressing action $\Pi^\sharp : T^*\mathcal{G} \cong \mathcal{G} \times \mathfrak{G}^* \rightarrow T\mathcal{G}$ is a groupoid morphism. Proposition 12 implies the following:

Corollary 23. *Let \mathcal{G} be a Poisson 2-group over G with tangent Lie bialgebroid (A, A^*) and let \mathcal{O} be an A^* -orbit on G . Then $\widehat{\mathcal{O}}$ is a symplectic groupoid.* □

Now we shall see a situation in which the cotangent groupoid of a Poisson 2-group is integrable by a double Lie groupoid. Suppose that a Lie 2-algebra \mathfrak{G} is such that there is also a Lie 2-algebra structure on the 2-vector space \mathfrak{G}^* with the property that $(\mathfrak{G}, \mathfrak{G}^*)$ is a usual Lie bialgebra. Then we say that $(\mathfrak{G}, \mathfrak{G}^*)$ is a *Lie 2-bialgebra* [6]. The classical correspondence between Lie bialgebras and 1-connected Poisson groups implies, in particular, that 1-connected Poisson 2-groups are classified by Lie 2-bialgebras [6]. If $(\mathfrak{G}, \mathfrak{G}^*)$ is a Lie 2-bialgebra, it is immediate to check that the double $\mathfrak{G} \oplus \mathfrak{G}^*$ with its classical Lie bracket [10, 13] is also a Lie 2-algebra.

Suppose that \mathcal{G} and \mathcal{G}^* are 1-connected Poisson 2-groups which correspond to the Lie 2-bialgebra $(\mathfrak{G}, \mathfrak{G}^*)$. Then there are Lie 2-group morphisms $i : \mathcal{G} \rightarrow \mathcal{D}$ and $j : \mathcal{G}^* \rightarrow \mathcal{D}$, where \mathcal{D} is the 1-connected integration of the double $\mathfrak{G} \oplus \mathfrak{G}^*$. Let us recall the integration of the Poisson structures on Poisson groups given in [14]. We take

$$\mathcal{S} = \{(g, u, v, h) \in \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^* \times \mathcal{G} : i(g)j(u) = i(v)j(h)\}.$$

The source and target of this groupoid are the projections to \mathcal{G} and the multiplication is given by

$$m((a, u, v, b), (b, u', v', c)) = (a, uu', vv', c);$$

see [13] for a description of the symplectic form on \mathcal{S} . Since i and j are Lie 2-group morphisms, we have that \mathcal{S} is a double Lie groupoid with sides \mathcal{G} and $S_0 = \{(g, u, v, h) \in \mathcal{G}_0 \times \mathcal{G}_0^* \times \mathcal{G}_0^* \times \mathcal{G}_0 : i(g)j(u) = i(v)j(h)\}$ over \mathcal{G}_0 , where \mathcal{G}_0 and \mathcal{G}_0^* denote the groups of units of the corresponding Lie 2-groups. So in this situation we can apply Proposition 7 instead of Theorem 6 to deduce that the \mathcal{S} -orbits that pass through the units in \mathcal{G} are symplectic groupoids, which is a stronger statement than the one provided by Corollary 23.

Remark 24. The double Lie groupoid \mathcal{S} is actually a *double symplectic groupoid* [14], i.e. its symplectic form is multiplicative with respect to both groupoid structures on \mathcal{S} . This means that also $S_0 \rightrightarrows \mathcal{G}_0$ is a Poisson groupoid. Therefore, Corollary 12 gives us another family of symplectic groupoid as subgroupoids of $S_0 \rightrightarrows \mathcal{G}_0$. It would be interesting to know whether Poisson 2-groups are integrable by double symplectic groupoids in general.

Example 25. In the case of Example 22, Proposition 7 implies that the whole coadjoint orbit of a unit is a subgroupoid of \mathfrak{G}^* and hence it is a symplectic groupoid. In fact, as we have seen, the infinitesimal coadjoint action on \mathfrak{G}^* is integrable by the double Lie groupoid

$$\begin{array}{ccc} \mathfrak{G}^* \times \mathcal{G} & \rightrightarrows & \mathfrak{G}^* \\ \Downarrow & & \Downarrow \\ \mathfrak{h}^* \times G & \rightrightarrows & \mathfrak{h}^* \end{array}$$

given by the coadjoint action. The Poisson structures that these symplectic groupoids integrate are the G -orbits in \mathfrak{h}^* and these can be quite varied, for instance:

- if \mathfrak{G}^* is isomorphic to the pair groupoid on the dual of a Lie algebra $\mathfrak{G}^* \cong (\mathfrak{g}^* \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*)$, then the symplectic groupoids that we get in this manner are the pair groupoids over the coadjoint orbits in \mathfrak{g}^* endowed with their canonical symplectic structures,
- if \mathfrak{G}^* is given by a differential crossed module defined by the zero map $0: \mathfrak{h} \rightarrow \mathfrak{g}$, then the coadjoint orbits of the units in \mathfrak{G}^* integrate the null Poisson structures on the G -orbits in \mathfrak{h}^* .
- Take V a symplectic vector space and let $\phi: \mathfrak{h} \rightarrow \mathfrak{g}$ be the differential crossed module associated to the Heisenberg Lie algebra $\mathfrak{h} = V \times \mathbb{R}$, where \mathfrak{g} is the Lie algebra of derivations of \mathfrak{h} and ϕ is defined by $u \mapsto \text{ad}_u$. In this case we can integrate \mathfrak{h} with the Heisenberg group H and we can integrate \mathfrak{g} with $G = \text{Aut}(\mathfrak{h})$. So we can see that there are three kinds of G -orbits in \mathfrak{h}^* : two open orbits $\{(\alpha, \mu) \in \mathfrak{h}^* \mid \mu \neq 0\}$ isomorphic to $V \times (0, \infty)$ with its canonical Lie–Poisson structure and the orbits of the form $\mathcal{O} \times \{0\} \subset \mathfrak{h}^*$, where \mathcal{O} is an orbit of the linear symplectic group of V , which are endowed with the zero Poisson bracket.

Acknowledgements

The author thanks CNPq for the financial support and D. Martínez Torres for fruitful conversations. This work is part of the author's PhD thesis written under the helpful guidance of H. Bursztyn. The author also thanks the referee whose comments and suggestions significantly improved this work.

References

- [1] J. C. Baez, A. S. Crans, “Higher Dimensional Algebra VI: Lie 2-algebras”, *Theory Appl. Categ.* **12** (2004), no. 15, p. 492–528.
- [2] J. C. Baez, A. D. Lauda, “Higher Dimensional Algebra V: 2-Groups”, *Theory Appl. Categ.* (2004), no. 12, p. 423–491.
- [3] K. Behrend, P. Xu, “Differentiable stacks and gerbes”, *J. Symplectic Geom.* **9** (2011), no. 3, p. 285–341.
- [4] R. Brown, K. C. H. Mackenzie, “Determination of a double Lie groupoid by its core diagram”, *J. Pure Appl. Algebra* **80** (1992), no. 3, p. 237–272.
- [5] R. Brown, C. B. Spencer, “G-groupoids, crossed modules and the fundamental groupoid of a topological group”, *Nederl. Akad. Wet., Proc., Ser. A* **79** (1976), no. 4, p. 296–302.
- [6] Z. Chen, M. Stiénon, P. Xu, “Poisson 2-groups”, *J. Differ. Geom.* **94** (2013), no. 2, p. 209–240.
- [7] A. Coste, P. Dazord, A. Weinstein, *Groupoïdes symplectiques*, Université Claude Bernard, Dép. de Mathématiques, 1987.
- [8] M. Crainic, R. L. Fernandes, “Integrability of Lie brackets”, *Ann. Math.* **157** (2003), no. 2, p. 575–620.
- [9] ———, “Integrability of Poisson brackets”, *J. Differ. Geom.* **66** (2004), no. 1, p. 71–137.
- [10] V. G. Drinfeld, “Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations”, *Sov. Math., Dokl.* (1983), no. 27, p. 68–71.
- [11] M. V. Karasev, “Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets”, *Izv. Akad. Nauk SSSR, Ser. Mat.* **50** (1986), no. 3, p. 508–538.
- [12] J. M. Lee, *Manifolds and Differential Geometry*, Graduate Studies in Mathematics, vol. 107, American Mathematical Society, 2009.

- [13] J.-H. Lu, “Multiplicative and affine Poisson structures on Lie groups”, PhD Thesis, University of California, Berkeley, 1990.
- [14] J.-H. Lu, A. Weinstein, “Groupoïdes symplectiques doubles des groupes de Lie–Poisson”, *C. R. Math. Acad. Sci. Paris* **309** (1989), no. 18, p. 951-954.
- [15] K. C. H. Mackenzie, “Double Lie algebroids and second-order geometry. I”, *Adv. Math.* **94** (1992), no. 2, p. 180-239.
- [16] K. C. H. Mackenzie, P. Xu, “Lie bialgebroids and Poisson groupoids”, *Duke Math. J.* **73** (1994), no. 2, p. 415-452.
- [17] I. Moerdijk, J. Mrčun, *Introduction to foliations and Lie groupoids*, vol. 91, Cambridge University Press, 2003.
- [18] B. Noohi, “Fundamental groups of topological stacks with slice property”, *Algebr. Geom. Topol.* **8** (2008), no. 3, p. 1333-1370.
- [19] M. A. Semenov-Tian-Shansky, “Dressing transformations and Poisson group actions”, *Publ. Res. Inst. Math. Sci.* **21** (1985), no. 6, p. 1237-1260.
- [20] L. Stefanini, “On morphic actions and integrability of LA-groupoids”, PhD Thesis, ETH Zürich (Switzerland), 2008.
- [21] J. Waldron, “Lie Algebroids over Differentiable Stacks”, PhD Thesis, University of York (UK), 2014.
- [22] A. Weinstein, “The local structure of Poisson manifolds”, *J. Differ. Geom.* **18** (1983), no. 3, p. 523-557.
- [23] ———, “Symplectic groupoids and Poisson manifolds”, *Bull. Am. Math. Soc.* **16** (1987), no. 1, p. 101-104.
- [24] ———, “Coisotropic calculus and Poisson groupoids”, *J. Math. Soc. Japan* **40** (1988), no. 4, p. 705-727.