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On pairs of equations involving unlike powers of primes and powers of 2

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Abstract. In this paper, it is proved that every pair of sufficiently large even integers can be represented by a pair of equations, each containing one prime, one prime square, two prime cubes and 302 powers of 2. This result constitutes a refinement upon that of L. Q. Hu and L. Yang.

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1. Introduction

In the 1950s, Linnik [2, 3] proved that each large even integer N is a sum of two primes and a bounded number of powers of 2,

$$N = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{k_1}}, \tag{1}$$

where here and hereafter, the letters p and ν, with or without subscripts, denote a prime number and a positive integer respectively. The famous Goldbach conjecture implies that $k_1 = 0$. The explicit value for the number k_1 was improved by many authors.

In 1999, Liu, Liu and Zhan [5] proved that every sufficiently large even integer N can be represented in the form

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{k_2}}, \tag{2}$$

and they also showed that there is a representation of the form (2) for some finite value of ν_{k_2}. The best result now is Zhao [11], who obtained $k_2 = 39$.

In 2001, Liu and Liu [4] proved that every large even integer N can be written as a sum of eight cubes of primes and k_3 powers of 2,

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{k_3}}. \tag{3}$$

The acceptable value for the number k_3 was determined by D. Platt and T. Trudgian [11], where $k_3 = 341$.

In 2011, Liu and Lü [7] considered the hybrid problem of (1), (2) and (3), i.e.

$$N = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \tag{4}$$

and proved that every sufficiently large even integer can be written as one prime, one square of primes, two cubes of primes and 161 powers of 2. In 2015, D. Platt and T. Trudgian [10] improved the value of k to 156, then to 16 by Zhao [12] and finally 15 by Lü [9].

It is of interest to investigate the simultaneous representation of pairs of positive even integers satisfying $N_2 \gg N_1 > N_2$, in the form

$$\begin{cases} N_1 = p_1 + p_2^2 + p_3^3 + p_4^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \\ N_2 = p_5 + p_6^2 + p_7^3 + p_8^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_k}, \end{cases} \tag{5}$$

where k is a positive integer. In 2017, Hu and Yang [1] proved that for $k = 455$, the equations (5) are solvable. In this paper, we obtain a further improvement of the value of k by giving the following theorem.

Theorem. *For $k = 302$, the equations* (5) *are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

2. Notation and Some Preliminary Lemmas

For the proof of the Theorem, in this section we introduce the necessary notation and Lemmas.

Throughout this paper, by N_i we denote a sufficiently large even integer. In addition, let $\eta < 10^{-10}$ be a fixed positive constant, and let $\varepsilon < 10^{-10}$ be an arbitrarily small positive constant not necessarily the same in different formulae. The letter p, with or without subscripts, is reserved for a prime number. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. By $A \sim B$ we mean that $B < A \leqslant 2B$. We denote by (m,n) the greatest common divisor of m and n. As usual, $\varphi(n)$ and $\mu(n)$ denote Euler's function and the Möbius function respectively. For $i = 1,2$, let

$$P_i = \frac{1}{2}\sqrt{(1-\eta)N_i}, \quad U_i = \frac{1}{2}\left(\frac{\eta N_i}{2}\right)^{\frac{1}{3}}, \quad V_i = \frac{1}{2}\left(\frac{\eta N_i}{2}\right)^{\frac{5}{18}}, \quad L = \frac{\log\left(\frac{N_1}{\log N_1}\right)}{\log 2},$$

$$F_i = F_i(\alpha_i, N_i) = \sum_{p \le N_i} (\log p) e(\alpha_i p), \quad G_i = G_i(\alpha_i, P_i) = \sum_{p \sim P_i} (\log p) e(\alpha_i p^2),$$

$$S_i = S_i(\alpha_i, U_i) = \sum_{p \sim U_i} (\log p) e(\alpha_i p^3), \quad T_i = T_i(\alpha_i, V_i) = \sum_{p \sim V_i} (\log p) e(\alpha_i p^3),$$

$$H(\alpha_i) = \sum_{\nu \le L} e\left(\alpha_i 2^{\nu}\right), \quad \mathscr{E}(\lambda) = \{\alpha_i \in (0,1] : |H(\alpha_i)| \ge \lambda L\}.$$

For the application of the Hardy–Littlewood method, we need to define the Farey dissection. For this purpose, we set

$$Q_{1i} = N_i^{\frac{1}{9}-2\varepsilon}, \ Q_{2i} = N_i^{\frac{8}{9}+\varepsilon},$$

and for $(a_i, q_i) = 1$, $1 \le a_i \le q_i$, put

$$\mathfrak{M}_i(q_i, a_i) = \left(\frac{a_i}{q_i} - \frac{1}{q_i Q_{2i}}, \frac{a_i}{q_i} + \frac{1}{q_i Q_{2i}}\right], \qquad \mathfrak{M}_i = \bigcup_{1 \le q_i \le Q_{1i}} \bigcup_{\substack{a_i=1 \\ (a_i,q_i)=1}}^{q_i} \mathfrak{M}_i(q_i, a_i),$$

$$\mathfrak{J}_0 = \left(\frac{1}{Q_{2i}}, 1 + \frac{1}{Q_{2i}}\right], \quad \mathfrak{m}_i = \mathfrak{J}_0 \setminus \mathfrak{M}_i.$$

Then it follows from orthogonality that

$$\begin{aligned}R(N_1,N_2) &= \sum_{\substack{N_1=p_1+p_2^2+p_3^3+p_4^3+2^{v_1}+2^{v_2}+\cdots+2^{v_k}\\ N_2=p_5+p_6^2+p_7^3+p_8^3+2^{v_1}+2^{v_2}+\cdots+2^{v_k}\\ p_1\le N_1,\, p_2\sim P_1,\, p_3\sim U_1,\, p_4\sim V_1,\, p_5\le N_2,\\ p_6\sim P_2,\, p_7\sim U_2,\, p_8\sim V_2,\, v_j\le L\,(j=1,2,\dots,k)}} (\log p_1)(\log p_2)\cdots(\log p_8)\\ &= \int_0^1\int_0^1 F_1G_1S_1T_1F_2G_2S_2T_2H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2. \end{aligned} \tag{6}$$

Now we state the lemmas required in this paper.

Lemma 1. *For $\frac{N_i}{2}<n\leqslant N_i$ $(i=1,2)$, we have*

$$\int_{\mathfrak{M}_i} F_iG_iS_iT_ie(-\alpha n)\mathrm{d}\alpha_i = \frac{1}{2\cdot 3^2}\mathfrak{S}(n)J(n)+O\left(\frac{N_i^{\frac{10}{9}}}{L}\right).$$

Here $\mathfrak{S}(n)$ is defined as

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{S_2^*(q,a){S_3^*}^2(q,a)e_q(-an)\mu(q)}{\varphi^4(q)},$$

$$S_k^*(q,a)=\sum_{\substack{r=1\\(r,q)=1}}^{q} e_q(ar^k),$$

and satisfies $\mathfrak{S}(n)>0.24485083$ for $n\equiv 0\,(\bmod 2)$. $J(n)$ is defined as

$$J(n)=\sum_{\substack{n=m_1+m_2+m_3+m_4\\ m_1\le N_i,\, {P_i}^2<m_2\le 4{P_i}^2,\\ {U_i}^3<m_3\le 8{U_i}^3,\, {V_i}^3<m_4\le 8{V_i}^3}} (m_2)^{-\frac{1}{2}}(m_3m_4)^{-\frac{2}{3}},$$

and satisfies

$$N_i^{\frac{10}{9}} \ll J(n) \ll N_i^{\frac{10}{9}}.$$

Proof. This can be found in Lemma 2.1 and Section 3 in Liu and Lü [7]. □

Lemma 2. *For $\alpha\in\mathfrak{m}_i$ $(i=1,2)$, we have*

(i) $\max_{\alpha\in\mathfrak{m}_i}|F_i|\ll N_i^{\frac{17}{18}+\varepsilon}$,

(ii) $\max_{\alpha\in\mathfrak{m}_i}|G_i|\ll N_i^{\frac{4}{9}+\varepsilon}$,

(iii) $\max_{\alpha\in\mathfrak{m}_i}|S_i|\ll N_i^{\frac{13}{42}+\varepsilon}$,

(iv) $\max_{\alpha\in\mathfrak{m}_i}|T_i|\ll N_i^{\frac{5}{18}+\varepsilon}$.

Proof. See Lemma 2.5 in Hu and Yang [1]. □

Lemma 3. *For $i=1,2$, we have*

(i) $\int_0^1 |G_iS_iT_i|^2\mathrm{d}\alpha_i \le 6.4894513{U_i}^2{V_i}^2$,

(ii) $\int_0^1 |F_iH^6(2\alpha_i)|^2\mathrm{d}\alpha_i \le 2.009\dfrac{N_iL^{12}}{\log^2 N_i}$.

Proof. For (i), we follow the notation as (3.3) in Zhao [12] and define $\mathfrak{J}_i(3)$ as

$$\mathfrak{J}_i(3)=\frac{1}{2^2\cdot 3^4}\sum_{\substack{m_1+m_2+m_3=n_1+n_2+n_3\\ P_i^2<m_1,n_1^2\le 4P_i^2\\ {U_i}^3<m_2,n_2\le 8{U_i}^3\\ {V_i}^3<m_3,n_3\le 8{V_i}^3}}(m_1n_1)^{-\frac{1}{2}}(m_2n_2m_3n_3)^{-\frac{2}{3}}.$$

Moreover, noting from the fact that

$$\begin{aligned}m_1&=n_1+n_2+n_3-m_2-m_3\\&\ge n_1+U^3+V^3-8U^3-8V^3\\&\ge(1-4\eta)n_1,\end{aligned}\tag{7}$$

and

$$\sum_{{U_i}^3<m\le 8{U_i}^3}m^{-\frac{2}{3}}\sim 3U_i,\quad \sum_{{P_i}^2<m\le 4{P_i}^2}m^{-1}\sim 2\log 2,\tag{8}$$

we obtain

$$\begin{aligned}\mathfrak{J}_i(3)&\le\frac{1}{2^2\cdot 3^4}\sum_{\substack{m_1+m_2+m_3=n_1+n_2+n_3\\ {P_i}^2<m_1,n_1\le 4{P_i}^2\\ {U_i}^3<m_2,n_2\le 8{U_i}^3\\ {V_i}^3<m_3,n_3\le 8{V_i}^3}}(1-4\eta)^{-\frac{1}{2}}{n_1}^{-1}(m_2n_2m_3n_3)^{-\frac{2}{3}}\\&\le\frac{1+o(1)}{2^2\cdot 3^4}(1+4\eta)\cdot 2\log 2\cdot(3U_i)^2(3V_i)^2\\&\le\left(\frac{\log 2}{2}+o(1)\right){U_i}^2{V_i}^2.\end{aligned}$$

Thus we deduce from Lemma 3.1 and Lemma 4.1 in Zhao [12] that

$$\int_0^1|G_iS_iT_i|^2\mathrm{d}\alpha\le 6.4894513{U_i}^2{V_i}^2.$$

We then complete the proof of Lemma 3 (i). For (ii), it is Lemma 2.4 in Lü [8]. □

Lemma 4. *Let $\Xi(N_i,k)=\{(1-\eta)N_i\le n_i\le N_i: n_i=N_i-2^{\nu_1}-2^{\nu_2}-\cdots-2^{\nu_k}\}$ $(i=1,2)$. For $k\ge 2$ and $N_1\equiv N_2\equiv 0\ (\mathrm{mod}\ 2)$, we have*

$$\sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)\\ n_1\equiv n_2\equiv 0\ (\bmod\ 2)}}J(n_1)J(n_2)\ge 43.407769N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}U_1U_2V_1V_2L^k.$$

Proof. The domain of $J(n_i)$ can be written as

$$\mathfrak{D}=\left\{(m_1,m_2,m_3,m_4): \begin{array}{l}m_1\le N_i,\ P_i^2<m_2\le(2P_i)^2,\ U_i^3<m_3\le(2U_i)^3,\\ V_i^3<m_3\le(2V_i)^3,\ m_1=n-m_2-m_3-m_4\end{array}\right\}.$$

Define

$$\mathfrak{D}^*=\left\{(m_2,m_3,m_4): P_i^2<m_2\le 3P_i^2,\ U_i^3<m_3\le(2U_i)^3,\ V_i^3<m_3\le(2V_i)^3\right\}.$$

For $(m_1,\cdots,m_6)\in\mathfrak{D}^*$, we can deduce from $(1-\eta)N_i\leqslant n_i\leqslant N_i$ that

$$m_1=n_i-m_2-m_3-m_4\le N_i.$$

Thus $\mathfrak{D}^*$ is a subset of $\mathfrak{D}$. Then it follows from (8) and Euler–Maclaurin summation that

$$\begin{aligned}\mathfrak{J}(n_i) &\geq \sum_{(m_2,m_3,m_4)\in\mathfrak{D}^*} m_2^{-\frac{1}{2}} m_3^{-\frac{2}{3}} m_4^{-\frac{2}{3}} \\ &\geq \sum_{P_i^2<m_2\leq 3P_i^2} m_2^{-\frac{1}{2}} \sum_{U_i^3<m_3\leq(2U_i)^3} m_3^{-\frac{2}{3}} \sum_{V_i^3<m_3\leq(2V_i)^3} m_4^{-\frac{2}{3}} \\ &\geq 2\cdot 3\cdot 3\cdot\Big(\frac{\sqrt{3}}{2}-\frac{1}{2}\Big)(1-\eta)^{\frac{1}{2}} N_i^{\frac{1}{2}} U_i V_i \\ &\geq (9\sqrt{3}-9)(1-\eta)^{\frac{1}{2}} N_i^{\frac{1}{2}} U_i V_i. \end{aligned} \tag{9}$$

It follows from (9) and Lemma 4.1 in Liu [6] that

$$\begin{aligned}\sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)\\ n_1\equiv n_2\equiv 0\ (\bmod 2)}} J(n_1)J(n_2) &\geq (9\sqrt{3}-9)^2(1-\eta) N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1U_2V_1V_2 \sum_{\substack{n_1\in\Xi(N_1,k)\\ n_2\in\Xi(N_2,k)\\ n_1\equiv n_2\equiv 0\ (\bmod 2)}} 1 \\ &\geq (9\sqrt{3}-9)^2(1-\eta)(1-\varepsilon) N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1U_2V_1V_2L^k \\ &\geq 43.407769 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1U_2V_1V_2L^k. \end{aligned}$$

□

Lemma 5. *We have*

$$\operatorname{meas}(\mathscr{E}(\lambda)) \ll N_i^{-E(\lambda)}, \quad \textit{with } E(0.9457435) > \frac{109}{126} + 10^{-10}.$$

Proof. See Table 1 in Platt and Trudgian [10]. □

3. Auxiliary Estimates

We initiate our proof by recalling the Farey dissections (6) that

$$\begin{aligned} R(N_1,N_2) &= \int_0^1\int_0^1 F_1G_1S_1T_1F_2G_2S_2T_2H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\ &= \left(\int_{\mathfrak{M}_1} + \int_{\mathfrak{m}_1\bigcap\mathscr{E}(\lambda)} + \int_{\mathfrak{m}_1\setminus\mathscr{E}(\lambda)}\right)\left(\int_{\mathfrak{M}_2} + \int_{\mathfrak{m}_2\bigcap\mathscr{E}(\lambda)} + \int_{\mathfrak{m}_2\setminus\mathscr{E}(\lambda)}\right) \\ &\qquad F_1G_1S_1T_1F_2G_2S_2T_2H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\ &= \sum_{s=1}^{3}\sum_{t=1}^{3} R_{st}, \end{aligned}$$

where R_{st} denotes the combination of s-th term in the first bracket and the t-th term in the second bracket.

Proposition 6. *We have*

$$R_{11} \geq 0.008032 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1U_2V_1V_2L^k.$$

Proof. By the definition of $\Xi(N_i, k)$, Lemma 1 and Lemma 4, we get

$$\begin{aligned}
R_{11} &= \int_{\mathfrak{M}_1} F_1 G_1 S_1 T_1 H^k(\alpha_1) e(-\alpha_1 N_1) \mathrm{d}\alpha_1 \int_{\mathfrak{M}_2} F_2 G_2 S_2 T_2 H^k(\alpha_2) e(-\alpha_2 N_2) \mathrm{d}\alpha_2 \\
&= \sum_{\substack{n_1 \in \Xi(N_1,k) \\ n_2 \in \Xi(N_2,k)}} \int_{\mathfrak{M}_1} F_1 G_1 S_1 T_1 e(-\alpha_1 n_1) \mathrm{d}\alpha_1 \int_{\mathfrak{M}_2} F_2 G_2 S_2 T_2 e(-\alpha_2 n_2) \mathrm{d}\alpha_2 \\
&\geq \frac{1}{2^2 \cdot 3^4} \sum_{\substack{n_1 \in \Xi(N_1,k) \\ n_2 \in \Xi(N_2,k)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) J(n_1) J(n_2) + O\left(N_1^{\frac{10}{9}} N_2^{\frac{10}{9}} L^{k-1}\right) \\
&\geq \frac{1}{2^2 \cdot 3^4} \times 43.407769 \times (0.24485083)^2 N_1 N_2 U_1 U_2 V_1 V_2 L^k \\
&\geq 0.008032 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^k.
\end{aligned}$$

This completes the proof of Proposition 6. □

We now introduce three lemmas essential in our proof of the following propositions and the theorem.

Lemma 7. *We have*

$$I_{1i} = \int_{\mathfrak{M}_i} \left| F_i G_i S_i T_i H^{\frac{k}{2}}(2\alpha_i) \right| \mathrm{d}\alpha_i \leq 3.610722 N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}}.$$

Proof. It follows from Lemma 3 and Cauchy's inequality that

$$\begin{aligned}
I_{1i} &\leq \left(\max_{\alpha \in \mathfrak{M}_i} |H(2\alpha_i)| \right)^{\frac{k}{2}-6} \int_{\mathfrak{M}_i} \left| F_i G_i S_i T_i H^6(2\alpha_i) \right| \mathrm{d}\alpha_i \\
&\leq L^{\frac{k}{2}-6} \left(\int_0^1 |G_i S_i T_i|^2 \mathrm{d}\alpha_i \right)^{\frac{1}{2}} \left(\int_0^1 |F_i H^6(2\alpha_i)|^2 \mathrm{d}\alpha_i \right)^{\frac{1}{2}} \\
&\leq 3.610722 N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}}.
\end{aligned}$$

□

Lemma 8. *We have*

$$I_{2i} = \int_{\mathfrak{m}_i \cap \mathscr{E}(\lambda)} \left| F_i G_i S_i T_i H^{\frac{k}{2}}(2\alpha_i) \right| \mathrm{d}\alpha_i \ll N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}-1}.$$

Proof. Using the definition of $\mathscr{E}(\lambda)$, the trivial bound of $H(2\alpha_i)$, Lemma 2 and Lemma 5, we get

$$\begin{aligned}
I_{2i} &\ll \max_{\alpha \in \mathfrak{m}_i} |F_i G_i S_i T_i| L^{\frac{k}{2}} \left(\int_{\mathscr{E}(\lambda)} 1 \mathrm{d}\alpha_i \right) \\
&\ll N_i^{\frac{1}{2}} U_i V_i L^{\frac{k}{2}-1}.
\end{aligned}$$

□

Lemma 9. *We have*

$$I_{3i} = \int_{\mathfrak{m}_i \setminus \mathscr{E}(\lambda)} \left| F_i G_i S_i T_i H^{\frac{k}{2}}(2\alpha_i) \right| \mathrm{d}\alpha_i \leq 3.610722 N_i^{\frac{1}{2}} U_i V_i \lambda^{\frac{k}{2}-6} L^{\frac{k}{2}}.$$

Proof. We deduce from the trivial bound $|H(2\alpha_i)| \leq |H(\alpha_i)| + 2 \leq (1+o(1))\lambda L$, Lemma 3 and Cauchy's inequality that

$$\begin{aligned}
I_{3i} &\leq (\lambda L)^{\frac{k}{2}-6} \left(\int_0^1 |G_i S_i T_i|^2 \mathrm{d}\alpha_i \right)^{\frac{1}{2}} \left(\int_0^1 |F_i H^6(2\alpha_i)|^2 \mathrm{d}\alpha_i \right)^{\frac{1}{2}} \\
&\leq 3.610722 N_i^{\frac{1}{2}} U_i V_i \lambda^{\frac{k}{2}-6} L^{\frac{k}{2}}.
\end{aligned}$$

This completes the proof of Lemmas 7–9. □

According to Cauchy's inequality, we have

$$|H(\alpha_1+\alpha_2)| \le \sqrt{|H(2\alpha_1)H(2\alpha_2)|}.$$

Now we turn to give an upper bound of $R_{12}, R_{21}, R_{22}, R_{23}$ and R_{32}.

Proposition 10. *We have*

$$R_{12}, R_{21}, R_{22}, R_{23}, R_{32} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}.$$

Proof.

$$\begin{aligned} R_{12} &= \int_{\mathfrak{M}_1} \int_{\mathfrak{m}_2 \bigcap \mathscr{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1+\alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ &\ll I_{11} I_{22} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}. \end{aligned} \tag{10}$$

Similiarily to (10), we obtain

$$R_{21} \ll I_{12} I_{21} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}.$$

Moreover,

$$\begin{aligned} R_{22} &= \int_{\mathfrak{m}_1 \bigcap \mathscr{E}(\lambda)} \int_{\mathfrak{m}_2 \bigcap \mathscr{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1+\alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ &\ll I_{21} I_{22} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}. \end{aligned} \tag{11}$$

Besides, we have

$$\begin{aligned} R_{23} &= \int_{\mathfrak{m}_1 \bigcap \mathscr{E}(\lambda)} \int_{\mathfrak{m}_2 \backslash \mathscr{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1+\alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ &\ll I_{21} I_{32} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}. \end{aligned} \tag{12}$$

Analogously to (12), we get

$$R_{32} \ll I_{22} I_{31} \ll N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 L^{k-1}.$$

Now the proof of Proposition 10 is complete. □

Next we give the estimation for R_{13} and R_{31}.

Proposition 11. *We have*

$$R_{13}, R_{31} \ll (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k.$$

Proof.

$$\begin{aligned} R_{13} &= \int_{\mathfrak{M}_1} \int_{\mathfrak{m}_2 \backslash \mathscr{E}(\lambda)} F_1 G_1 S_1 T_1 F_2 G_2 S_2 T_2 H^k(\alpha_1+\alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ &\le I_{11} I_{32} \le (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k. \end{aligned}$$

In a similar manner, we get

$$R_{31} \le I_{12} I_{31} \le (3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 V_1 U_2 V_2 \lambda^{\frac{k}{2}-6} L^k.$$

This completes the proof of Proposition 11. □

It remains to estimate R_{33}.

Proposition 12. *We have*

$$R_{33} \le (0.3486527(3.610722)^2 N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} U_1 U_2 V_1 V_2 \lambda^{k-12} L^k.$$

Proof.

$$\begin{aligned} R_{33} &= \int_{\mathfrak{m}_1\setminus\mathscr{E}(\lambda)}\int_{\mathfrak{m}_2\setminus\mathscr{E}(\lambda)} F_1G_1S_1T_1F_2G_2S_2T_2H^k(\alpha_1+\alpha_2)e(-\alpha_1N_1-\alpha_2N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\ &\le I_{31}I_{32} \le (3.610722)^2N_1^{\frac12}N_2^{\frac12}U_1U_2V_1V_2\lambda^{k-12}L^k. \end{aligned}$$ □

4. Proof of Theorem

On combining recalling Propositions 6–12, we arrive at the conclusion that

$$\begin{aligned} R(N_1,N_2) &\ge 0.008032N_1^{\frac12}N_2^{\frac12}U_1U_2V_1V_2L^k - 2\times(3.610722)^2N_1^{\frac12}N_2^{\frac12}U_1V_1U_2V_2\lambda^{\frac{k}{2}-6}L^k \\ &\quad - (3.610722)^2N_1^{\frac12}N_2^{\frac12}U_1U_2V_1V_2\lambda^{k-12}L^k \\ &\ge N_1^{\frac12}N_2^{\frac12}U_1U_2V_1V_2L^k(0.008032-2\times(3.610722)^2\lambda^{\frac{k}{2}-6}-(3.610722)^2\lambda^{k-12}) \end{aligned} \tag{13}$$

When $k\ge 302$ and $\lambda=0.9457435$, we get

$$R(N)>0 \tag{14}$$

for all sufficiently large even integers N. Now by (14), the proof of the Theorem is completed.

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