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Breaking down the reduced Kronecker coefficients

Analyse fine des coefficients de Kronecker réduits

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Abstract. We resolve three interrelated problems on *reduced Kronecker coefficients* $\overline{g}(\alpha,\beta,\gamma)$. First, we disprove the *saturation property* which states that $\overline{g}(N\alpha,N\beta,N\gamma)>0$ implies $\overline{g}(\alpha,\beta,\gamma)>0$ for all $N>1$. Second, we esimate the maximal $\overline{g}(\alpha,\beta,\gamma)$, over all $|\alpha|+|\beta|+|\gamma|=n$. Finally, we show that computing $\overline{g}(\lambda,\mu,\nu)$ is strongly #P-hard, i.e. #P-hard when the input (λ,μ,ν) is in unary.

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1. Introduction

The *reduced Kronecker coefficients* were introduced by Murnaghan in 1938 as the stable limit of *Kronecker coefficients*, when a long first row is added:

$$\overline{g}(\alpha,\beta,\gamma):=\lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big),\quad \text{where}\quad \alpha[n]:=(n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n\geq|\alpha|+\alpha_1, \tag{1}$$

see [15, 16]. They generalize the classical *Littlewood–Richardson* (LR–) *coefficients*:

$$\overline{g}(\alpha,\beta,\gamma)=c^{\alpha}_{\beta\gamma}\quad\text{for}\quad |\alpha|=|\beta|+|\gamma|,$$

see [12]. As such, they occupy the middle ground between the Kronecker and the LR–coefficients. While the latter are well understood and have a number of combinatorial interpretations, the former are notorious for their difficulty. It is generally believed that the reduced Kronecker coefficients are simpler and more accessible than the (usual) Kronecker coefficients, cf. [9, 18]. The results of this paper suggest otherwise, see Remark 12.

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1.1. *Saturation property*

The *Kronecker coefficients* $g(\lambda,\mu,\nu)$, are defined as

$$g(\lambda,\mu,\nu) := \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma),$$

where $\lambda,\mu,\nu \vdash n$, and χ^λ is the irreducible character of S_n corresponding to partition λ. Similarly, the *Littlewood–Richardson coefficients* are defined as

$$c^\lambda_{\mu\nu} := \langle \chi^\lambda, \chi^\mu \otimes \chi^\nu \uparrow^{S_n}_{S_k \times S_{n-k}} \rangle, \quad \text{where} \quad \lambda \vdash n, \mu \vdash k, \nu \vdash n-k.$$

It is easy to see that $c^{N\lambda}_{N\mu,N\nu} \ge c^\lambda_{\mu\nu}$ for all $N \ge 1$, where $N\lambda = (N\lambda_1, N\lambda_2, \ldots)$. The *saturation property* is the fundamental result by Knutson and Tao [11], giving a converse:

$$c^{N\lambda}_{N\mu,N\nu} > 0 \quad \text{for some } N \ge 1 \quad \Longrightarrow \quad c^\lambda_{\mu\nu} > 0.$$

For a partition $\alpha \vdash k$ and $n \ge k + \alpha_1$, we have $\alpha[n] = (n-k, \alpha_1, \alpha_2, \ldots) \vdash n$. It is known that $g(\alpha[n+1], \beta[n+1], \gamma[n+1]) \ge g(\alpha[n], \beta[n], \gamma[n])$ for all n, whenever the right hand side is defined. In this notation, Murnaghan's result (1) states that $\overline{g}(\alpha,\beta,\gamma) = g(\alpha[n],\beta[n],\gamma[n])$ for n large enough.

The saturation property fails for the Kronecker coefficients, i.e. $g(2^2,2^2,2^2) = 1$ but $g(1^2,1^2,1^2) = 0$. It is a long-standing open problem whether it holds for the reduced Kronecker coefficients. This was independently conjectured in 2004 by Kirillov [9, Conj. 2.33] and Klyachko [10, Conj. 6.2.4] :

Conjecture 1 (Kirillov, Klyachko). *The reduced Kronecker coefficients satisfy the saturation property:*

$$\overline{g}(N\alpha, N\beta, N\gamma) > 0 \quad \textit{for some } N \ge 1 \quad \Longrightarrow \quad \overline{g}(\alpha,\beta,\gamma) > 0.$$

This conjecture was motivated by the known converse:

$$\overline{g}(\alpha,\beta,\gamma) > 0 \quad \Longrightarrow \quad \overline{g}(N\alpha, N\beta, N\gamma) > 0 \qquad \text{for all } N \ge 1,$$

see below. Here is the first result of this paper.

Theorem 2. *For all $k \ge 3$, the triple of partitions $\left(1^{k^2-1}, 1^{k^2-1}, k^{k-1}\right)$ is a counterexample to Conjecture 1. Moreover, for every partition γ s.t. $\gamma_2 \ge 3$, there are infinitely many pairs $(a,b) \in \mathbb{N}^2$ for which the triple of partitions (a^b, a^b, γ) is a counterexample to Conjecture 1.*

These results both contrast and complement [5, Cor. 6], which confirms the saturation property for triples of the form (a^b, a^b, a).

1.2. *Maximal values*

Our second result is a variation on Stanley's recent bounds on the maximal Kronecker and LR–coefficients:

Theorem 3 (cf. [25,26], and also [21]). *We have:*

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} g(\lambda,\mu,\nu) = \sqrt{n!}\, e^{-O(\sqrt{n})}, \tag{2}$$

$$\max_{0 \le k \le n} \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} c^\lambda_{\mu,\nu} = 2^{n/2 - O(\sqrt{n})}. \tag{3}$$

In [21], we refine (3) and prove that the maximal Kronecker and LR–coefficients appear when all three partitions have near-maximal dimension, which in turn implies that they have a *Vershik–Kerov–Logan–Shepp shape*. See also [20] for refined upper bounds on (reduced) Kronecker coefficients with few rows. Here we obtain the following analogue of Stanley's Theorem 3.

Theorem 4. *We have:*

$$\max_{a+b+c\leq 3n}\max_{\alpha\vdash a}\max_{\beta\vdash b}\max_{\gamma\vdash c}\overline{g}(\alpha,\beta,\gamma)=\sqrt{n!}e^{O(n)}.$$

1.3. *Complexity*

Our final result is on complexity of computing the reduced Kronecker coefficients. Via reduction to LR–coefficients, computing the reduced Kronecker coefficients is classically #P-hard, see [17]. The following recent result by Ikenmeyer, Mulmuley and Walter is a far-reaching extension:

Theorem 5 (cf. [7] and Remark 14). *Computing the Kronecker coefficients $g(\lambda,\mu,\nu)$ is* strongly *#P-hard.*

Here by *strongly #P-hard* we mean #P-hard when the input (λ,μ,ν) is given in unary. In other words, the input size of the problem is the total number of squares in the three Young diagrams. The theorem is in sharp contrast with computing $\chi^{(n-k,k)}[\lambda]$ which is #P-complete but not strongly #P-complete, see [19, §7].

Theorem 6. *Computing the reduced Kronecker coefficients $\overline{g}(\alpha,\beta,\gamma)$ is* strongly *#P-hard.*

Let us mention that the problem of computing the (reduced) Kronecker coefficients is not known to be in #P, see [19]. In fact, finding a combinatorial interpretation for (reduced) Kronecker coefficients is a classical open problem [24, Prob. 10]. Note also that Theorem 6 is stronger than Theorem 5, since in the limit (1) it suffices to take $n\geq|\alpha|+|\beta|+|\gamma|$, see [3,27]. Indeed, this implies that the reduced Kronecker coefficient problem is a subset of instances of the usual Kronecker coefficient problem (cf., however, Remark 16).

2. Disproof of the saturation property

We assume the reader is familiar with basic results and standard notations in Algebraic Combinatorics, see [23]. We also need the following two results on Kronecker coefficients.

Lemma 7 (Symmetries). *For every $\lambda,\mu,\nu\vdash n$, we have:*

$$g(\lambda,\mu,\nu)=g(\lambda',\mu',\nu)=g(\mu,\lambda,\nu)=g(\lambda,\nu,\mu).$$

Lemma 8 (Semigroup property [4, 13]). *Suppose $\alpha,\beta,\gamma\vdash m$, such that $g(\alpha,\beta,\gamma)>0$. Then, for all partitions $\lambda,\mu,\nu\vdash n$, we have:*

$$g(\lambda+\alpha,\mu+\beta,\nu+\gamma)\geq g(\lambda,\mu,\nu).$$

This result is crucial for understanding of reduced Kronecker coefficients. First, since $g(1,1,1)=1$, we conclude that the sequence $\{g(\alpha[n],\beta[n],\gamma[n])\}$ is weakly increasing with n. Similarly, the sequence $\{g(N\lambda,N\mu,N\nu)\}$ is weakly increasing with N if $g(\lambda,\mu,\nu)>0$.

Let $\ell(\lambda)$ be the *number of parts* of the partition λ, and $d(\lambda):=\max\{k:\lambda_k\geq k\}$ be the *Durfee size*. For the proof of Theorem 2, we need several known technical results which we state below.

Lemma 9 (cf. [6]). *Let $\lambda,\mu,\nu\vdash n$ be such that $d(\lambda)>2d(\mu)d(\nu)$. Then $g(\lambda,\mu,\nu)=0$.*

Lemma 10 (cf. [1, Cor. 3.2]). *Let $\lambda=\lambda'$ be a self-conjugate partition. Then $g(\lambda,\lambda,\lambda)>0$.*

Lemma 11 (cf. [8, Thm. 1.10]). *Let $\mathscr{X}:=\{1,1^2,1^4,1^6,21,31\}$, and let partition $\nu\notin\mathscr{X}$. Denote $\ell:=\max\{\ell(\nu)+1,9\}$, and suppose $r>3\ell^{3/2}$, $s\geq 3\ell^2$, and $|\nu|\leq rs/6$. Then $g(s^r,s^r,\nu[rs])>0$.*

Proof of Theorem 2. We prove the first statement of the theorem. Let $k \geq 3$, and let $\alpha = \left(1^{k^2-1}\right)$, $\gamma = \left(k^{k-1}\right)$ be as in the theorem. Since $d(\alpha[n]) = 1$ and $d(\gamma[n]) = k$ for all $n \geq k^2$, we have $2d(\alpha[n])^2 = 2 < d(\gamma[n]) = k$. Thus, we have $\overline{g}(\alpha,\alpha,\gamma) = 0$ by Lemma 9.

By Lemma 10, the symmetry and semigroup properties (Lemma 7 and 8), we have:

$$\begin{aligned}\overline{g}(k\alpha,k\alpha,k\gamma) &= \overline{g}\left(k^{k^2-1},k^{k^2-1},(k^2)^{k-1}\right) \geq g\left(k^{k^2-1}[k^3],k^{k^2-1}[k^3],(k^2)^{k-1}[k^3]\right)\\ &\geq g\left(k^{k^2},k^{k^2},(k^2)^k\right) = g\left((k^2)^k,(k^2)^k,(k^2)^k\right) \geq g\left(k^k,k^k,k^k\right) > 0.\end{aligned}$$

This contradicts the saturation property in Conjecture 1 for $N = k$, and proves the first part of the theorem.

For the second part, we construct the counterexample based on Lemma 11. For a partition γ, let $\ell := \max\{\ell(\gamma)+1, 9\}$ as in the lemma. Let $b \geq \max\left\{3\ell^{3/2}, |\gamma|/(6\sqrt{d(\gamma[n])/2}-6)\right\}$. Since $\gamma_2 \geq 3$, we have $d(\gamma[n]) \geq 3$. Thus, there exists at least one $a \geq 1$, such that $|\gamma|/(6b) \leq a < \sqrt{d(\gamma[n])/2}$. Let us show now that (a,b) is a pair as in the theorem.

Take $\alpha := (a^b)$. Since $d(\alpha[n]) \leq a$, we have $2d(\alpha[n])^2 \leq 2a^2 < d(\gamma[n])$. Thus, we have $\overline{g}(\alpha,\alpha,\gamma) = 0$ by Lemma 9. On the other hand, let $N \geq 3\ell^2/a$, $\nu := N\gamma$, $r := b+1$, and $s := Na$. Then $|\nu| \leq Nab/6 < rs/6$, $r > 3\ell^{3/2}$, and $s = Na \geq 3\ell^2$, by construction. Since $\nu \notin \mathscr{X}$ for all $N > 1$, the conditions of Lemma 11 are satisfied. We conclude:

$$\overline{g}(N\alpha,N\alpha,N\gamma) = \overline{g}\left(Na^b,Na^b,N\gamma\right) \geq g\left(s^{b+1},s^{b+1},N\gamma[rs]\right) = g\left(s^r,s^r,\nu[rs]\right) > 0,$$

which implies that (α,α,γ) is a counterexample to the saturation property. Since the construction works for all b large enough as above, this proves the second part of the theorem. $\square$

3. Bounds and complexity via identities

Proof of Theorem 4. We follow [21] in our exposition. We start with the following identity [2, Cor. 4.5]:

$$\overline{g}(\alpha,\beta,\gamma) = \sum_{m=0}^{\lfloor k/2\rfloor} \sum_{\pi\vdash q+m-b} \sum_{\rho\vdash q+m-a} \sum_{\sigma\vdash m} \sum_{\lambda,\mu,\nu\vdash k-2m} c^{\alpha}_{\nu\pi\rho} c^{\beta}_{\mu\pi\sigma} c^{\gamma}_{\lambda\rho\sigma} g(\lambda,\mu,\nu), \tag{4}$$

where $a = |\alpha|$, $b = |\beta|$, $q = |\gamma|$, $k = a+b-q$, and

$$c^{\lambda}_{\alpha\beta\gamma} = \sum_{\tau} c^{\lambda}_{\alpha\tau} c^{\tau}_{\beta\gamma}.$$

For the upper bound, by [21, Thm. 1.5] which extends (3) in Theorem 3, we have:

$$c^{\lambda}_{\alpha\beta} \leq \binom{N}{a}^{1/2} \quad \text{for all } \lambda\vdash N, \alpha\vdash a, \beta\vdash N-a.$$

Using the *Vandermonde identity* for the sums of binomial coefficients, we have:

$$c^{\lambda}_{\alpha\beta\gamma} \leq \binom{N}{a,b,N-a-b}^{1/2} \leq 3^{N/2} \quad \text{for all } \lambda\vdash N, \alpha\vdash a, \beta\vdash b, \gamma\vdash N-a-b.$$

In this notation, the theorem is a maximum over $a+b+q \leq 3n$. Combining these with (2) in Theorem 3, we have:

$$\overline{g}(\alpha,\beta,\gamma) \leq (3n/2)\cdot p\,(3n)^6\cdot 3^{3n/2}\cdot\sqrt{n!} = \sqrt{n!}e^{O(n)}.$$

For the lower bound, let $\alpha,\beta,\gamma\vdash n$, and note that $\overline{g}(\alpha,\beta,\gamma) \geq g(\alpha,\beta,\gamma)$, which is achieved in (4) for $m = 0$. The result now follows from part (2) of Theorem 3. $\square$

Proof of Theorem 6. Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, which can be viewed as an infinite nonincreasing sequence by appending zeros at the end. Denote $\widetilde{\lambda} := (\lambda_2, \lambda_3, \ldots)$. For all $i \geq 1$, define

$$\lambda^{\langle i \rangle} := \big(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \lambda_{i+2}, \ldots\big),$$

so in particular $\lambda^{\langle 1 \rangle} = \widetilde{\lambda}$. The result is a direct consequence of the following identity:

$$g(\lambda, \mu, \nu) = \sum_{i=1}^{\ell(\mu)\ell(\nu)} (-1)^i \, \overline{g}\big(\lambda^{\langle i \rangle}, \widetilde{\mu}, \widetilde{\nu}\big), \tag{5}$$

see [3, Thm. 1.1]. From Theorem 5, computing $g(\lambda, \mu, \nu)$ is #P-hard in unary. The identity (5) has polynomially many terms, and thus gives a polynomial reduction. □

4. Final remarks and open problems

Remark 12. All three results in this paper are centered around the same (philosophical) claim, that the reduced Kronecker coefficients are closer in nature to the (usual) Kronecker coefficients than to the LR–coefficients. This is manifestly evident from both the statements and the proofs of the theorems. However, this should not be taken as a suggestion that the LR–coefficients are not strongly #P-hard. We do, in fact, conjecture that computing $c^{\lambda}_{\mu\nu}$ is strongly #P-hard [19, Conj. 8.1], but this remains beyond the reach of existing technology.

Remark 13. There is a general setting which extends the stability of Kronecker coefficients to other families of stable limits, see [22]. Manivel asks if the saturation property holds for all these families, but notes that "we actually have only very limited evidence for that" [13]. In view of our results, it would be interesting to see if the saturation property holds for *any* of these families of stable coefficients.

Remark 14. Theorem 5 is not stated in [7] in this form. It does however follow directly from the proof, which is essentially a parsimonious reduction from the 3-PARTITION problem classically known to be (strongly) NP-complete, and thus the counting is (strongly) #P-complete.

Remark 15. Among other consequences, the saturation property implies that the *vanishing problem* $c^{\lambda}_{\mu\nu} >^{?} 0$ is in P, see [14]. The main result of [7] proved that the vanishing problem $g(\lambda, \mu, \nu) >^{?} 0$ is NP-hard, refuting Mulmuley's conjecture (cf. e.g. [19, §2]). Following the pattern in Remark 12 above, we conjecture that the vanishing problem $\overline{g}(\alpha, \beta, \gamma) >^{?} 0$ for reduced Kronecker coefficients is also NP-hard.

Remark 16. There is a subtle but important technical differences between the way we state Theorem 5 and the way it is stated in [7]. While we use the (standard) *Turing reduction* to derive Theorem 6 from Theorem 5, the original proof in [7] uses a more restrictive *many-to-one reduction.* Such a reduction for Theorem 6 would also resolve our conjecture above on the vanishing problem.

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