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A note on flatness of non separable tangent cone at a barycenter

Une note sur la platitude du cône tangent à un barycentre

Thibaut Le Gouic

Abstract. Given a probability measure $P$ on an Alexandrov space $S$ with curvature bounded below, we prove that the support of the pushforward of $P$ on the tangent cone $T_{b^*}S$ at its (exponential) barycenter $b^*$ is a subset of a Hilbert space, without separability of the tangent cone.

Résumé. Étant donné une mesure de probabilité $P$ sur un espace d'Alexandrov $S$ avec courbure minorée, nous prouvons que le support de la mesure poussée de $P$ sur le cône tangent $T_{b^*}S$ à son barycentre (exponentiel) $b^*$ est un sous-ensemble d'un espace de Hilbert, sans condition de séparabilité du cône tangent.

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1. Introduction

Barycenter of a probability measure $P$ (a.k.a. Fréchet means) provides an extension of expectation on Euclidean space to arbitrary metric spaces. We present here a useful tool for the study of barycenters on Alexandrov spaces with curvature bounded below: the support of $\log_{b^*} P$ in the tangent cone at the barycenter is included in a Hilbert space. This rigidity result has been stated in [9] as Theorem 45, however the proof is not written. Moreover, there is an extra assumption of support of $\log_{b^*} P$ being separable, which does not even seem to be a consequence of the support of $P$ being separable. As pointed out by [7], it is not clear if even $S$ being proper ensures that the tangent cone is separable. This paper presents a proof of this rigidity result, without this extra separable assumption on the tangent cone. For measurability purposes (see Lemma 6), we suppose however that $S$ is separable. The proof is essentially the one of Theorem 45 of [9], with needed approximations dealt with a bit differently.
2. Setting and main result

We use a classical notion of curvature bounded below for geodesic spaces, referred to as Alexandrov curvature. We recall several notions whose formal definitions can be found for instance in [3] or in the work in progress [2].

For a metric space \((S, d)\), we denote by \(\mathcal{P}_1(S)\) the set of probability measures on \(S\) with finite moment of order 1 (i.e. such that there exists \(x \in S\) such that \(\int d(x, y) d\mathbf{P}(y) < \infty\)). The support of a measure \(\mathbf{P}\) will be denoted by \(\text{supp}\mathbf{P}\). We use both notation \(\int f d\mathbf{P}\) and \(\mathbf{P} f\) for the integral of \(f\) w.r.t. \(\mathbf{P}\).

A geodesic space is a metric space \((S, d)\) such that every two points \(x, y \in S\) at distance is connected by a curve of length \(d(x, y)\). Such shortest curves are called geodesics. For \(\kappa \in \mathbb{R}\), the model space \((M_\kappa, d_\kappa)\) denotes the 2-dimensional simply connected complete surface of constant Gauss curvature \(\kappa\). A geodesic space \((S, d)\) is an Alexandrov space with curvature bounded below by \(\kappa \in \mathbb{R}\) if for every triangle (3-uple) \((x_0, x_1, y) \in S\), and a constant speed geodesic \((x_t)_{t \in [0;1]}\) there exists an isometric triangle \((\tilde{x}_0, \tilde{x}_1, \tilde{y}) \in M_\kappa\), such that the geodesic \((\tilde{x}_t)_{t \in [0;1]}\) satisfies for all \(t \in [0;1]\),

\[
d(y, x_t) \geq d_{\kappa}(\tilde{y}, \tilde{x}_t).
\]

For such spaces, angles between two unit-speed geodesics \(\gamma_1, \gamma_2\) starting at the same point \(p \in S\) can be defined as follows:

\[
\cos \angle_p(\gamma_1, \gamma_2) = \lim_{t \to 0} \frac{d^2(\gamma_1(t), p) + d^2(\gamma_2(t), p) - d^2(\gamma_1(t), \gamma_2(t))}{2 d(p, \gamma_1(t)) d(p, \gamma_2(t))},
\]

where angle \(\angle_p(\gamma_1, \gamma_2) \in [0; \pi]\). Denote by \(\Gamma_p\) the set of all unit-speed geodesics emanating from \(p\). Using angles, we can define the tangent cone \(T_pS\) at \(p \in S\) as follows. First define \(T'_pS\) as the (quotient) set \(\Gamma_p \times \mathbb{R}^+\), equipped with the (pseudo-)metric defined by

\[
\| (y_1, t) - (y_2, s) \|^2 := s^2 + t^2 - 2s.t \cos \angle_p(y_1, y_2).
\]

Then, the tangent cone \(T_pS\) is defined as the completion of \(T'_pS\) equipped with the metric \(\| \cdot \|_p\). We will use the notation for \(u, v \in T_pS\),

\[
\langle u, v \rangle_p := \frac{1}{2} (\| u \|^2_p + \| v \|^2_p - \| u - v \|^2_p),
\]

We will often identify a point \(\gamma(t) \in S\) with \((\gamma, t) \in T_pS\). Although such \(\gamma\) might not be unique, we will assume a choice of a map \(\log_p : S \to T_pS\), called logarithmic map, such that for all \(x \in S\), there exists a unit-speed geodesic \(\gamma\) emanating from \(p\) such that, for some \(t > 0\), \(\gamma(t) = x\) and \(\log_p(x) = (\gamma, t)\).

This map can be chosen to be \(\mathcal{G}_B\)-measurable, where \(\mathcal{G}_B\) denotes the \(\sigma\)-algebra generated by open balls on the tangent cone \(T_pS\) (see Lemma 6) and this weak measurability is enough for our results to hold and will be assumed for the rest of the paper. Then the pushforward of \(\mathbf{P}\) by \(\log_p\) will be denoted by \(\log_p \# \mathbf{P}\).

The tangent cone is not necessarily a geodesic space (see [4]), however, it is included in a geodesic space, namely the ultratangent space (see for instance Theorem 14.4.2 and 14.4.1 in [2]) that is an Alexandrov space with curvature bounded below by 0.

The tangent cone \(T_pS\) contains the subspace \(\text{Lin}_p\) of all points with an opposite, formally defined as follows. A point \(u\) belongs to \(\text{Lin}_p \subset T_pS\) if and only if there exists \(v \in T_pS\) such that \(\| u \|^2_p = \| v \|^2_p\) and

\[
\langle u, v \rangle_p = -\| u \|^2_p.
\]

Our main result is based on the following Theorem.

Theorem (Theorem 14.5.4 in [2]). The set \(\text{Lin}_p\) equipped with the induced metric of \(T_pS\) is a Hilbert space.
A point $b^* \in S$ is a barycenter of the probability measure $P \in \mathcal{P}_{1}(S)$ if for all $b \in S$

$$0 \leq \int d^2(x, b) - d^2(x, b^*) \, dP(x).$$

Such barycenter might not be unique, neither exists. However, when they exist, they satisfy

$$\int \langle x, y \rangle_{b^*} \, dP \otimes P(x, y) = 0. \tag{1}$$

A point $b^* \in S$ satisfying (1) is called an exponential barycenter of $P$.

We can now state our main result.

**Theorem 1.** Let $(S, d)$ be an Alexandrov space with curvature bounded below by some $\kappa \in \mathbb{R}$ and $P \in \mathcal{P}_{1}(S)$. If $b^* \in S$ is an exponential barycenter of $P$, then $\text{supp}\log_{b^*} #P \subset \text{Lin}_{b^*} S$. In particular, $\text{supp}\log_{b^*} #P$ is included in a Hilbert space.

This result allows to prove the following Corollary, that has been implicitly used in [1].

**Corollary 2 (Linearity).** Let $b \in T_{b^*} S$. Then, the map $\langle \cdot, b \rangle_{b^*} : \text{Lin}_{b^*} \to \mathbb{R}$ is continuous and linear. In particular, if $b^*$ is an exponential barycenter of $P$, then

$$\int \langle x, b \rangle_{b^*} \, dP(x) = 0.$$

3. Proofs

Recall that we always identify a point in $S$ and its image in the tangent cone $T_p S$ by the log$_b$ map.

**Proof of Corollary 2.** Linearity is obvious from the definition of $\langle \cdot, b \rangle_{b^*}$. We check that $x \mapsto \langle x, b \rangle_{b^*}$ is a convex and concave function in $\text{Lin}_{b^*} S$. Let $t \in (0, 1)$, $x_0, x_1 \in \text{Lin}_{b^*} S$, and set $x_t = (1 - t)x_0 + tx_1$. Since the tangent cone is included in an Alexandrov space with curvature bounded below by 0 on one hand, and $\text{Lin}_{b^*}$ is a Hilbert space on the other hand,

$$\langle x_t, b \rangle_{b^*} = \frac{1}{2} \left( \|x_t\|_{b^*}^2 + \|b\|_{b^*}^2 - \|x_t - b\|_{b^*}^2 \right)$$

$$\leq \frac{1}{2} \left( (1 - t)(\|x_0\|_{b^*}^2 - \|x_0 - b\|_{b^*}^2) + t(\|x_1\|_{b^*}^2 - \|x_1 - b\|_{b^*}^2) + \|b\|_{b^*}^2 \right)$$

$$= (1 - t)\langle x_0, b \rangle_{b^*} + t\langle x_1, b \rangle_{b^*}.$$

The same lines applied to $-x_0$ and $-x_1$ gives the converse inequality

$$\langle -x_t, b \rangle_{b^*} \leq (1 - t)\langle -x_0, b \rangle_{b^*} + t\langle -x_1, b \rangle_{b^*}.$$

The second statement follows from the fact that $b^*$ is a Pettis integral of the pushforward of $P$ onto $\text{Lin}_{b^*} \subset T_{b^*} S$, as a direct consequence of Theorem 1.

**Proof of Theorem 1.** Let $L \subset \{x \in S | \int \langle x, \cdot \rangle_{b^*} \, dP = 0\}$ be a measurable set such that $P(L) = 1$ given by Lemma 3. Let $x \in L$. For $U = \{x\}$, use Lemma 5 with $Q = P$ and $B_\delta$ a ball of radius $\delta$ around $x$ in $T_{b^*} S$, to get a sequence $(y^n_0)_n \subset T_{b^*} S$ such that,

$$\limsup_n \cos \angle (1_{b^*}, y^n_0) = \limsup_n \frac{\langle x, y^n_0 \rangle_{b^*}}{d(b^*, x) d(b^*, y^n_0)} = \limsup_n \frac{\langle x, y^n_0 \rangle_{b^*}}{d(b^*, x) d(b^*, y^n_0)} \frac{1}{\lim_n d(b^*, y^n_0)} \leq \frac{1}{d(b^*, x)} \frac{\int_{B_\delta} \langle x, y \rangle_{b^*} \, dP(x)}{P(B_\delta)} \left( \int_{B_\delta} \langle x, y \rangle_{b^*} \, dP \otimes P(x, y) \right)^{1/2}.$$
Then, since \( \int \langle x, y \rangle_{b^*} \, d\mathbb{P}(y) = 0 \), letting \( \delta \to 0 \), one gets
\[
\frac{1}{\mathbb{P}(B_\delta)} \int_{B_\delta} \langle x, y \rangle_{b^*} \, d\mathbb{P}(y) = -\frac{1}{\mathbb{P}(B_\delta)} \int_{B_\delta} \langle x, y \rangle_{b^*} \, d\mathbb{P}(y) \to -d^2(b^*, x).
\]
and
\[
\left( \int_{B_\delta} \int_{B_\delta} \langle x, y \rangle_{b^*} \, d\mathbb{P} \otimes \mathbb{P}(x, y) \right)^{1/2} \to d(b^*, x)
\]
Thus,
\[
\lim_{\delta \to 0^+} \limsup_n \cos \angle(\triangle x, \triangle y) = -1
\]
One can choose \((\hat{y}^n)_n\) a sequence in \((y^n)_n, \delta \) such that \(\cos \angle(\triangle x, \triangle y) \to -1\). Since \(T_{b^*} S\) is a subspace of an Alexandrov space of curvature bounded below by 0, we also have
\[
\angle(\triangle x, \triangle y) \leq 2\pi - \angle(\triangle x, \triangle y) - \angle(\triangle x, \triangle y) \to 0,
\]
as \(n, k \to \infty\). Thus \((\hat{y}^n)_n\) corresponds to a Cauchy sequence in the space of direction, and thus admits a limit in \(T_{b^*} S\), since its “norm” also admits a limit \(d(b^*, x)\). Its limit \(\hat{y}\) satisfies \(\cos \angle(\triangle x, \triangle y) = -1\), and therefore, it is the opposite \(\hat{y} = -x\).

Finally, by definition of the support, for \(x \in supp(\log_{b^*} \mathbb{P})\), every ball centered at \(x\) have a positive probability, and thus there exists a sequence \((x_n)_{n \geq 1} \subset L\) such that \(x_n \to x\). We conclude with the completeness of \(Lin_{b^*}\).

Lemma 3 (Proposition 1.7 of [8] for non separable metric space). Suppose \((S, d)\) is an Alexandrov space with curvature bounded below. Then, for any probability measure \(Q \in \mathcal{P}_1(S)\), and any \(b^* \in S\),
\[
\int \langle x, y \rangle_{b^*} \, dQ \otimes Q(x, y) \geq 0.
\]
Moreover, if \(b^*\) is an exponential barycenter of \(Q\), then for \(Q\)-almost all \(x \in S\),
\[
\int \langle x, y \rangle_{b^*} \, dQ(y) = 0.
\]

Proof. For brevity, we will adopt the notation \(Q g \) for \(\int g \, d\mathbb{Q}\).

The result for \(Q\) finitely supported is the Lang–Schroeder inequality (Proposition 3.2 in [5]). Thus, we just need to approximate \(Q \otimes Q(\cdot, \cdot)_{b^*}\) by some \(Q_n \otimes Q_n(\cdot, \cdot)_{b^*}\) for some finitely supported \(Q_n\).

To approximate \(Q \otimes Q(\cdot, \cdot)_{b^*}\), draw two independent sequences of i.i.d. random variables \((X^1_i)_i\) and \((X^2_i)_i\) of common law \(Q\), and denote \(Q_n^1\) and \(Q_n^2\) the corresponding empirical measures. In particular, \(Q_n^1 \otimes Q_n^2\) and \(Q_n^2 \otimes Q_n^1\) are both empirical measures of \(Q \otimes Q\). Since \(S\) is not separable, we can not apply the fundamental theorem of statistics that ensures almost sure weak convergence of \(Q_n^1 \otimes Q_n^1\) to \(Q \otimes Q\). However, for a measurable function \(f : S \times S \to \mathbb{R}\), such that \(Q \otimes \mathbb{Q} f < \infty\), the law of large number ensures that almost surely
\[
Q_n^1 \otimes Q_n^2 f \to Q \otimes \mathbb{Q} f
\]
and
\[
Q_n^2 \otimes Q_n^1 f \to Q \otimes \mathbb{Q} f.
\]
Since the sequence \((X^1_1, X^1_2, X^1_3, X^2_1, X^2_2, X^2_3, \ldots)\) is also an i.i.d. sequence of random variables of common law \(Q\), the subsequence of the associated empirical measures \((Q_n^3)_{n}\) defined by
\[
Q_n^3 := \frac{1}{2}(Q_n^1 + Q_n^2)
\]
also satisfies the almost sure convergence
\[
Q_n^3 \otimes Q_n^3 f \to Q \otimes \mathbb{Q} f.
\]
Now, since
\[ Q_n^1 \otimes Q_n^2 = \frac{1}{4} (Q_n^1 \otimes Q_n^2 + Q_n^1 \otimes Q_n^1 + Q_n^2 \otimes Q_n^1 + Q_n^2 \otimes Q_n^2), \]
we proved that almost surely
\[ Q_n^1 \otimes Q_n^1 f + Q_n^2 \otimes Q_n^2 f \to 2Q \otimes Q f. \]
And since \((Q_n^1)_n\) and \((Q_n^2)_n\) are independent and with same law, it implies that both \(Q_n^1 \otimes Q_n^1 f\) and \(Q_n^2 \otimes Q_n^2 f\) converge to \(Q \otimes Q f\) almost surely. In particular, since \(Q_n^1\) is supported on \(n\) points, there exists a sequence of finitely supported measures \((\text{we rename } (Q_n)_n)\) such that \(Q_n \otimes Q_n f \to Q \otimes Q f\). We thus proved the first result applying \(f = \langle \cdot, \cdot \rangle_{b^*}\).

Now, for any \(x \in S\), applying this first result to the measure \(Q_e := \frac{1}{1+ \epsilon} Q + \frac{\epsilon}{1+ \epsilon} \delta_x\), we get
\[ 0 \leq (1+ \epsilon)Q_e \otimes Q_e \langle \cdot, \cdot \rangle_{b^*} = Q \otimes Q \langle \cdot, \cdot \rangle_{b^*} + 2\epsilon Q(x, \cdot)_{b^*} + \epsilon^2 \|x\|_{b^*}^2. \]
Letting \(\epsilon \to 0^+\), we get
\[ Q \langle x, \cdot \rangle_{b^*} \geq 0. \]
Then equality follows from the hypothesis \(Q \otimes Q \langle \cdot, \cdot \rangle_{b^*} = 0\) meaning that \(b^*\) is an exponential barycenter. \(\square\)

**Lemma 4 (Subadditivity, Lemma A.6 of [5]).** Let \((S, d)\) be an Alexandrov space with curvature bounded below. Take \(b^* \in S\). Let \(x_1, \ldots, x_n \in T_{b^*} S\) and \(U \subset T_{b^*} S\) finite. Then, for all \(\epsilon > 0\), there exists \(y \in T_{b^*} S\) such that for all \(u \in U\),
\[ \langle y, u \rangle_{b^*} \leq \sum_{i=1}^n \langle x_i, u \rangle_{b^*} + \epsilon, \]
and
\[ \|y\|^2 \leq \sum_{i,j=1}^n \langle x_i, x_j \rangle_{b^*} + \epsilon. \]

**Lemma 5 (Approximation).** Let \(U \subset T_{b^*} S\) finite. Take \(B \subset S\) measurable and a probability measure \(P \in \mathcal{P}_1(S)\) such that \(P \otimes P \langle \cdot, \cdot \rangle_{b^*} = 0\) and \(P(B) > 0\). Then, there exists a sequence \((y_n)_n\) such that for all \(u \in U\)
\[ \frac{1}{P(B)} \int_B \langle u, x \rangle_{b^*} dP(x) \geq \limsup_n \langle u, y_n \rangle_{b^*} \quad (2) \]
and
\[ \frac{1}{P(B)^2} \int_B \int_B \langle x, y \rangle_{b^*} dP \otimes P(x, y) = \lim_n d^2(b^*, y_n). \quad (3) \]

**Proof.** Using the same arguments as in Lemma 3, we see that the empirical measures \((P_n)_n\) satisfy
\[ P_n \otimes P_n f \to P \otimes P f, \]
almost surely, for any \(f : S \times S \to \mathbb{R} \in L^1(S \times S,\, P \otimes P)\). In particular, taking \(f(x, y) = \langle x, y \rangle_{b^*} 1_{B \times B}(x, y)\), the following convergence holds in \(L^2(P^{\otimes \infty})\),
\[ \int_B \int_B \langle \cdot, \cdot \rangle_{b^*} dP_n \otimes P_n \to \int_B \int_B \langle \cdot, \cdot \rangle_{b^*} dP \otimes P, \quad (4) \]
and similarly for \(B^c\). Also, the law of large number ensures that almost surely, for all \(u \in U\),
\[ \int_B \langle \cdot, u \rangle_{b^*} dP_n \to \int_B \langle \cdot, u \rangle_{b^*} dP, \quad (5) \]
and again, the same for \(B^c\). Thus, there exists a subsequence (of a deterministic realization of) \(P_n\) (that we rename \(P_n\)) such that \((4)\) and \((5)\) both hold for all \(u \in U\).
Then, applying Lemma 4 to finite sum
\[
\frac{1}{\mathbb{P}(B)} \int_{B^c} \langle \cdot, t \rangle_{b^*} \, d\mathbb{P}_n,
\]
shows that there exists a sequence \((y^n)_n \in T_{b^*}S\) such that (2) holds and for a sequence \((\varepsilon_n)_n\) s.t. \(\varepsilon_n \to 0\),
\[
\|y^n\|_{b^*}^2 \leq \frac{1}{\mathbb{P}(B)^2} \int_{B^c} \int_{B^c} \int_{B^c} \int_{B^c} \langle x, y \rangle_{b^*} \, d\mathbb{P}_n \otimes \mathbb{P}_n + \varepsilon_n.
\]
Then, applying the same Lemma 4 again shows that there exists a sequence \((z^n)_n \subset T_{b^*}S\), such that
\[
0 \leftarrow \frac{1}{\mathbb{P}(B)^2} \int_{B^c} \int_{B^c} \langle x, y \rangle_{b^*} \, d\mathbb{P}_n \otimes \mathbb{P}_n(x, y)
= \frac{1}{\mathbb{P}(B)^2} \left( \int_{B^c} \int_{B^c} \int_{B^c} \int_{B^c} \langle x, y \rangle_{b^*} \, d\mathbb{P}_n \otimes \mathbb{P}_n(x, y) + 2 \int_{B^c} \langle x, y \rangle_{b^*} \, d\mathbb{P}_n \otimes \mathbb{P}_n(x, y) \right)
\geq \|z^n\|_{b^*}^2 + \|y^n\|_{b^*}^2 + 2\|y^n, z^n\|_{b^*} - \varepsilon_n.
\]
(6)

Letting \(n \to \infty\), one obtains
\[
0 \geq \lim_{n} \|z^n\|_{b^*}^2 + 2\|y^n, z^n\|_{b^*} + \|y^n\|_{b^*}^2
\geq \lim_{n} \|z^n\|_{b^*}^2 - 2\|y^n\|_{b^*}^2 \|z^n\|_{b^*} + \|y^n\|_{b^*}^2
= \lim_{n} (\|z^n\|_{b^*}^2 - \|y^n\|_{b^*}^2) \geq 0.
\]
and which shows \(\lim_n \|y^n\| = \lim_n \|z^n\|\) and also that (6) becomes an equality at the limit and therefore
\[
\lim_n \|z^n\|_{b^*} = \frac{1}{\mathbb{P}(B)^2} \int_{B^c} \int_{B^c} \langle x, y \rangle_{b^*} \, d\mathbb{P} \otimes \mathbb{P}(x, y) \quad \square
\]

This Lemma appears in a remark of [6].

**Lemma 6 (Measurability of the log map).** Let \((S, d)\) be a separable Alexandrov space. Let \(p \in S\). Then \(\log_p : S \to T_pS\) can be chosen to be \(\mathcal{G}_B\)-measurable.

**Proof.** Denote \(G_p\) the space of all constant speed geodesics emanating from \(p\) equipped with the sup distance \(\|\cdot\|_{\infty}\). Then \((G_p, \|\cdot\|_{\infty})\) is separable and complete too. Using Kuratowski and Ryll-Nardzewski measurable selection theorem, one can choose a Borel map \(g : S \to G_p\) such that \(g\) maps \(x\) to a geodesic from \(p\) to \(x\). Then, using the (proof of) Lemma 4.2 of [7], the map \(l : G_p \to T_pS\) is measurable \(T_pS\) is equipped with the \(\sigma\)-algebra \(\mathcal{G}\) generated by open balls. Therefore, \(\log_p := l \circ g\) is \(\mathcal{G}_B\)-measurable. \(\square\)

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**References**


