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Descent for coherent sheaves along formal/open coverings

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Abstract. For a regular Noetherian scheme X with a divisor with strict normal crossings D we prove that coherent sheaves satisfy descent w.r.t. the "covering" consisting of the open parts in the various completions of X along the components of D and their intersections.

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Introduction

Let X be a regular Noetherian scheme and $D \subset X$ a divisor with strict normal crossings (cf. Definition 15). In applications it is useful to have a descent theory for coherent sheaves on X relative to a "covering" consisting of the open parts in the various completions of X along the components of D and their intersections. For example when X is the (integral model of a) toroidal compactification of a Shimura variety, these completions can be described as completions of relative torus embeddings on mixed Shimura varieties. Our motivation comes from this case, cf. [7]. In the case that D consists of one component and X is affine, these questions have been treated in the literature, see e.g. [1,3,4,10], [13, Tag 0BNI][1]. In this short article, we generalize these results to arbitrary divisors with normal crossings without any affineness assumption, sticking to the Noetherian case, however. Similar questions have also been investigated recently in [2,5,11]. Our result is as follows:

Let $\{Y\}$ be the coarsest stratification of X into locally closed subschemes such that every component of D is the closure $\overline{Y}$ of a stratum Y. For each stratum we define a sheaf of $\mathscr{O}_X$-algebras on X:

$$R_Y(U) := C_{\overline{Y}}\mathscr{O}_X(U') = \lim_n \mathscr{O}_X(U')/\mathscr{I}_{\overline{Y}}^n(U') \tag{1}$$

where U is an open subset of X, and $U' \subset U$ is such that $U' \cap \overline{Y} = U \cap Y$. The definition of R_Y does not depend on the choice of U' for R_Y may also be described as the push-forward $\iota_* \mathscr{O}_{C_{\overline{Y}}X|_Y}$,

[1] Note that there is a mistake in the treatment of the non-affine case in [1, p. 7–8].

where $C_{\overline{Y}}X$ is the formal completion of X along $\overline{Y}$, and $C_{\overline{Y}}X|_Y$ is the open formal subscheme with underlying topological space equal to Y and ι is the composed morphism of formal schemes $C_{\overline{Y}}X|_Y \to X$.

For any chain of disjoint strata $Y_1, Y_2, \ldots, Y_n$ such that $Y_i \subset \overline{Y_{i-1}}$ for $i = 2, \ldots, n$, we define inductively a sheaf $R_{Y_1,\ldots,Y_n}$ of $\mathscr{O}_X$-algebras on X which coincides with (1) for $n = 1$. For $n > 1$ we set

$$R'_{Y_1,\ldots,Y_n}(U) := C_{\overline{Y_1}}(R_{Y_2,\ldots,Y_n}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(U')). \tag{2}$$

Here $\otimes$ is the usual tensor product (not completed!) and $U' \subset U$ is again such that $U' \cap \overline{Y_1} = U \cap Y_1$. Again this definition does not depend on U'. $R_{Y_1,\ldots,Y_n}$ is defined to be the sheaf associated with the pre-sheaf $R'_{Y_1,\ldots,Y_n}$.

Let

$$[\, X\text{-}\mathbf{coh}\,]\left(\int S, R\right)^{\mathrm{cocart}}$$

be the category of the following descent data (it will be defined in a different way in the article, which will explain the notation): For each stratum Y a coherent sheaf M_Y of R_Y-modules together with isomorphisms

$$\rho_{Y,Z} : M_Y \otimes_{R_Y} R_{Y,Z} \longrightarrow M_Z \otimes_{R_Z} R_{Y,Z}$$

for any Y, Z with $Z \subset \overline{Y}$, which are compatible w.r.t. any triple Y, Z, W of strata with $Z \subset \overline{Y}$ and $W \subset \overline{Z}$ in the obvious way.

Let

$$[\, X\text{-}\mathbf{coh}\,](\mathscr{O}_X)$$

be the category of coherent sheaves on X.

Then we have

Main Theorem 19. *The natural functor*

$$[\, X\text{-}\mathbf{coh}\,](\mathscr{O}_X) \longrightarrow [\, X\text{-}\mathbf{coh}\,]\left(\int S, R\right)^{\mathrm{cocart}}$$

is an equivalence of categories.

1. Generalities

Let $\mathscr{D} \to \mathscr{S}$ be a bifibered category such that the fibers have all limits and all colimits. We will be interested mainly in the following two cases

$$[\,\mathbf{mod}\,] \longrightarrow [\,\mathbf{ring}\,]$$

where $[\,\mathbf{ring}\,]$ is the category of commutative rings with 1 and $[\,\mathbf{mod}\,]$ is the bifibered category of modules over such rings. Furthermore let X be a topological space. Then we consider

$$[\, X\text{-}\mathbf{mod}\,] \longrightarrow [\, X\text{-}\mathbf{ring}\,]$$

where $[\, X\text{-}\mathbf{ring}\,]$ is the category of ring sheaves on X *which are coherent over themselves in the sense of ringed spaces* [12, §2, Définition 2] and $[\, X\text{-}\mathbf{mod}\,]$ is the bifibered category of sheaves of modules over such ring sheaves. Rings and algebras will always be commutative unless otherwise specified.

Back in the general case, for a morphism $f \in \mathrm{Mor}(\mathscr{S})$ we denote by $f_\bullet$ and $f^\bullet$ the corresponding push-forward and pull-back functors. By definition of a bifibered category $f_\bullet$ is always left adjoint to $f^\bullet$. Those functors are only defined up to a (unique) natural isomorphism. Whenever we write $f_\bullet$ or $f^\bullet$ we assume that a choice has been made. For example, for a morphism $f : S \to T$ of rings, $f^\bullet$ is the forgetful map that considers a T-module as an S-module via f (hence there is a canonical choice in this case) and $f_\bullet$ is its left-adjoint, the functor $-\otimes_S T$ (which is also only defined up to a unique natural isomorphism).

Pairs (I,S) consisting of a diagram I (i.e. a small category) and a functor $S: I \to \mathscr{S}$ form a 2-category $\mathrm{Dia}^{\mathrm{op}}(\mathscr{S})$, called the category of diagrams in $\mathscr{S}$ (cf. also [8]). A morphism of $\mathscr{S}$-diagrams $(\alpha,\mu):(I,S)\to(J,T)$ is a functor $\alpha: I\to J$ together with a natural transformation $\mu: T\alpha \Rightarrow S$. A 2-morphism $(\alpha,\mu)\Rightarrow(\beta,\nu)$ is a natural transformation $\kappa:\alpha\Rightarrow\beta$ such that $\nu\circ(T*\kappa)=\mu$.

For each pair (I,S), we define the category $\mathscr{D}(I,S)$ of (I,S)-modules, whose objects are lifts of the functor S

$$\begin{tikzcd} & \mathscr{D} \arrow[d] \\ I \arrow[ur, "M"] \arrow[r, "S"'] & \mathscr{S} \end{tikzcd}$$

to the given bifibered category, and whose morphisms are natural transformations between those. An object is called *coCartesian* if all the morphisms $M(\rho)$ for $\rho: i\to j$ are coCartesian. This defines a full subcategory $\mathscr{D}(I,S)^{\mathrm{cocart}}$ of $\mathscr{D}(I,S)$. If $p: I\to E$ is a functor between small categories, we also define the subcategory of E*-coCartesian* objects as those for which the morphisms $M(\rho)$ are coCartesian for all ρ such that $p(\rho)$ is an identity.

We need a refinement of the categories defined above. Suppose we are given a full subcategory $\mathscr{D}^{\mathrm{f}}$ of $\mathscr{D}$ whose objects shall be called *finite*. We assume that $\mathscr{D}^{\mathrm{f}}\to\mathscr{S}$ is still opfibered (i.e. push-forward preserves finiteness) but not necessarily fibered. We define the full subcategories $\mathscr{D}(I,S)^{\mathrm{f}}$ and $\mathscr{D}(I,S)^{\mathrm{f,cocart}}$ requiring point-wise finiteness.

Definition 1. *In the example* [**mod**] $\to$ [**ring**] *an object* $M\in$ [**mod**] *over a ring* R *is finite, if it is a finitely generated* R*-module. We denote the corresponding full subcategory by* [**mod-f.g.**].

In the example [X-**mod**] $\to$ [X-**ring**] *an object* $M\in$ [X-**mod**] *lying over the sheaf of rings* R *is finite, if it is* coherent *in the sense of ringed spaces [12, §2, Définition 2]. In particular, locally on* X*, we have an exact sequence*

$$R|_U^n \longrightarrow R|_U^m \longrightarrow M|_U \longrightarrow 0$$

for some $n,m\in\mathbb{N}_0$*. We denote the corresponding full subcategory by* [**X-coh**].

For each morphism $(\alpha,\mu):(I,S)\to(J,T)$, we have a corresponding pull-back $(\alpha,\mu)^*$ given by

$$((\alpha,\mu)^*M)_i=\mu(i)_\bullet M_{\alpha(i)} \quad \forall\, i\in I.$$

$(\alpha,\mu)^*$ has a right adjoint $(\alpha,\mu)_*$ given by Kan's formula

$$((\alpha,\mu)_*M)_j=\lim_{j\times_{/J}I} T(\nu)^\bullet\mu(i)^\bullet M_i \quad \forall\, j\in J,$$

where an object in the slice category $j\times_{/J}I$ is denoted by a pair (i,ν), with $i\in I$ and $\nu: j\to\alpha(i)$. If $(\alpha,\mu):(I,S)\to(J,T)$ is purely of diagram type, i.e. if $\mu: T\circ\alpha\to S$ is the identity, then $(\alpha,\mu)^*$ does have also a left adjoint $(\alpha,\mu)_!$ given by

$$((\alpha,\mu)_!M)_j=\mathrm{colim}_{I\times_{/J}j}\, T(\nu)_\bullet M(i) \quad \forall\, j\in J,$$

where an object in the slice category $I\times_{/J}j$ is denoted by a pair (i,ν), with $i\in I$ and $\nu:\alpha(i)\to j$.

Example 2. For the fibered category [**mod**] $\to$ [**ring**] and for each morphism $(\alpha,\mu):(I,S)\to(J,T)$, the pull-back is given by

$$((\alpha,\mu)^*M)_i=M_{\alpha(i)}\otimes_{T_{\alpha(i)}}S_i \quad \forall\, i\in I,$$

and

$$((\alpha,\mu)_*M)_j=\lim_{j\times_{/J}I} M_i \quad \forall\, j\in J,$$

where each M_i is considered an T_j-module via the composition $T_j \to T_{\alpha(i)} \to S_i$. If $\mu : T \circ \alpha \to S$ is the identity, then

$$((\alpha,\mu)_! M)_j = \operatorname{colim}_{I \times_{/J} j} M(i) \otimes_{S_{\alpha(i)}} S_j \quad \forall\, j \in J.$$

2. Morphisms of (finite) descent

We keep the notations introduced in the previous section.

Definition 3. *We say that a morphism* (α,μ) *in* $\mathrm{Dia}^{\mathrm{op}}(\mathscr{S})$ *is* of descent *if*

$$(\alpha,\mu)^* : \mathscr{D}(J,T)^{\mathrm{cocart}} \longrightarrow \mathscr{D}(I,S)^{\mathrm{cocart}}$$

is an equivalence of categories.

We say that a morphism (α,μ) *in* $\mathrm{Dia}^{\mathrm{op}}(\mathscr{S})$ *is* of finite descent *if*

$$(\alpha,\mu)^* : \mathscr{D}(J,T)^{\mathrm{f,cocart}} \longrightarrow \mathscr{D}(I,S)^{\mathrm{f,cocart}}$$

is an equivalence of categories.

That (α,μ) is of descent does not imply that $(\alpha,\mu)_*$ is an inverse to the equivalence $(\alpha,\mu)^*$. This holds if and only if $(\alpha,\mu)_*$ preserves coCartesian objects. Obviously $(\alpha,\mu)_*$ always preserves coCartesian objects if $J = \{\cdot\}$.

The following proposition lists some of the basic properties of this formalism. Assertions (i)–(iv) are very general and hold also in the context of an arbitrary fibered derivator. Assertions (v)–(vi) are specific to the situation of an ordinary fibered category.

Proposition 4.

(i) *The morphisms of descent (resp. of finite descent) satisfy the 2-out-of-3 property.*

(ii) *For two morphisms of diagrams in* $\mathscr{S}$

$$(I,S) \underset{(\beta,\nu)}{\overset{(\alpha,\mu)}{\rightleftarrows}} (J,T)$$

such that sequences of 2-morphisms $(\alpha,\mu)\circ(\beta,\mu) \Rightarrow \cdots \Leftarrow \mathrm{id}$ *and* $(\beta,\mu)\circ(\alpha,\mu) \Rightarrow \cdots \Leftarrow \mathrm{id}$ *exist, we have that* (α,μ) *and* (β,μ) *are of descent (resp. of finite descent).*

(iii) *Let* $(\alpha,\mu) : (I,S) \to (J,T)$ *be a morphism of diagrams in* $\mathscr{S}$*. If for every* $j \in J$ *the morphism* $(\alpha_j,\mu_j) : (j \times_{/J} I, \mathrm{pr}_2^* S) \to (j, T_j)$ *is of descent (resp. of finite descent) then* (α,μ) *is of descent (resp. of finite descent).*

If $\alpha : I \to J$ *is a Grothendieck fibration then the statement holds with the slice category* $j \times_{/J} I$ *replaced by the fibered product* $j \times_J I$.

(iv) *Let* $\alpha : I \to J$ *be a morphism of diagrams and let* $(\alpha,\mathrm{id}) : (I,\alpha^* S) \to (J,S)$ *be a morphism of diagrams in* $\mathscr{S}$ *of pure diagram type. If* $(I \times_{/J} j, S_k) \to (\cdot, S_k)$ *is of descent (resp. of finite descent) for all* $j,k \in J$ *then* (α,id) *is of descent (resp. of finite descent).*

(v) *Let* $S \in \mathscr{S}$ *be an object and* I *a diagram. Denote by* (I,S) *the corresponding constant diagram. There is an equivalence of categories*

$$\mathscr{D}(I,S)^{\mathrm{cocart}} \longrightarrow \mathscr{D}(I[\mathrm{Mor}(I)^{-1}],S)^{\mathrm{cocart}}$$

(resp. decorated with f*) where* $I[\mathrm{Mor}(I)^{-1}]$ *is the universal groupoid to which* I *maps.* $I[\mathrm{Mor}(I)^{-1}]$ *is equivalent to the small category whose set of objects is* $\pi_0(I)$ *and whose morphism sets are*

$$\mathrm{Hom}(\xi,\xi') = \begin{cases} \pi_1(I,\xi) & \text{if } \xi = \xi', \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular $(I,S) \to (\cdot,S)$ *is of descent (resp. of finite descent) if* $\pi_0(I) = \pi_1(I) = \{\cdot\}$.

(vi) *Let Δ be the simplex category, and let Δ° be the injective simplex category. Let $(\Delta, S_\bullet)$, resp. $(\Delta^\circ, S_\bullet)$ be a cosimplicial, resp. a cosemisimplicial object in $\mathscr{S}$. Then the category $\mathscr{D}(\Delta, S_\bullet)^{\mathrm{cocart}}$, resp. $\mathscr{D}(\Delta^\circ, S_\bullet)^{\mathrm{cocart}}$, is equivalent to the category of classical descent data, whose objects are an object M in $\mathscr{D}(S_0)$ together with an isomorphism in $\mathscr{D}(S_1)$*

$$\rho : (\delta_1^0)^* M \longrightarrow (\delta_1^1)^* M$$

such that the following equality of morphisms in $\mathscr{D}(S_2)$ holds true:

$$(\delta_2^0)^* \rho \circ (\delta_2^2)^* \rho = (\delta_2^1)^* \rho.$$

Here δ_k^i is the strictly increasing map $\{0, \dots, k-1\} \to \{0, \dots, k\}$ omitting i.

Proof. **(i).** The statement is clear.

(ii). The statement follows from the fact that if $\rho : (\alpha, \mu) \to (\beta, \nu)$ is a 2-morphism between 1-morphisms $(I, S) \to (J, T)$ and $M \in \mathscr{D}(J, T)$ is a coCartesian object, then the morphism

$$\rho^* : (\alpha, \mu)^* M \longrightarrow (\beta, \nu)^* M$$

is an isomorphism.

(iii). We start by showing that both unit and counit

$$(\alpha, \mu)^* (\alpha, \mu)_* E \longrightarrow E \qquad F \longrightarrow (\alpha, \mu)_* (\alpha, \mu)^* F$$

are isomorphisms when restricted to the subcategories of coCartesian objects. Let $j : (\cdot, T_j) \to (J, T)$ be the embedding. We have to check that

$$j^* E \longrightarrow j^* (\alpha, \mu)_* (\alpha, \mu)^* E \tag{3}$$

is an isomorphism for all j.

Consider the 2-commutative diagram:

$$\begin{tikzcd}
(j \times_{/J} I, \iota_j^* S) \arrow[r, "\iota_j"] \arrow[d, "p_j"] & (I, S) \arrow[d, "(\alpha,\mu)"] \\
(j, T_j) \arrow[r] \arrow[ur, Rightarrow] & (J, T)
\end{tikzcd}$$

By the explicit point-wise formula for $(\alpha, \mu)_*$, the morphism (3) is the same as

$$j^* E \longrightarrow p_{j,*} \iota_j^* (\alpha, \mu)^* E_j.$$

The morphism induced by the 2-morphism in the diagram $\iota_j^* (\alpha, \mu)^* E \to p_j^* j^* E$ is an isomorphism on coCartesian objects by definition. Since the unit $\mathrm{id} \to p_{j,*} p_j^*$ is an isomorphism by assumption, we are done.

We now show that $(\alpha, \mu)_*$ preserves coCartesian objects. Let $\rho : j_1 \to j_2$ be a morphism in J. It induces a map of fibers (purely of diagram type) $\overline{\rho} : (j_2 \times_{/J} I, \mathrm{pr}_2^* S) \to (j_1 \times_{/J} I, \mathrm{pr}_2^* S)$. We have to show that

$$S(\rho)^* ((\alpha, \mu)_* M)_{j_1} \longrightarrow ((\alpha, \mu)_* M)_{j_2}$$

is an isomorphism. This can be checked after pull-back along $p_2 : (j_2 \times_{/J} I, \mathrm{pr}_2^* S) \to (j_2, T_{j_2})$ because this induces an equivalence of the categories of coCartesian objects by assumption. Since $S(\rho) p_2 = p_1 \overline{\rho}$ we get the morphism

$$\overline{\rho}^* p_1^* ((\alpha, \mu)_* M)_{j_1} \longrightarrow p_2^* ((\alpha, \mu)_* M)_{j_2}$$

which is the same as

$$\overline{\rho}^* \iota_1^* M \longrightarrow \iota_2^* M.$$

Since $\iota_2 = \iota_1 \circ \overline{\rho}$, this is an isomorphism.

To see that the counit is an isomorphism on coCartesian objects, we have to see that

$$\iota_j^*(\alpha,\mu)^*(\alpha,\mu)_* E \longrightarrow \iota_j^* E \tag{4}$$

is an isomorphism for all j. Since $(\alpha,\mu)_*$ preserves coCartesian objects, this is again the morphism

$$p_j^* j^*(\alpha,\mu)_* E \longrightarrow \iota^* E$$

and hence the morphism induced by the counit

$$p_j^* p_{j,*}\iota^* E \longrightarrow \iota^* E.$$

This is an isomorphism on coCartesian objects by assumption.

Proof of the additional statement: If α is a Grothendieck fibration, we have an adjunction

$$(j \times_J I, \iota_j^* S) \underset{\kappa_j}{\overset{\iota_j}{\rightleftarrows}} (j \times_{/J} I, \mathrm{pr}_2^* S)$$

with $\kappa_j\iota_j = \mathrm{id}$ and such that there is a 2-morphism $\iota_j\kappa_j \Rightarrow \mathrm{id}$. Hence by (ii), these morphisms are of descent (resp. of finite descent). Hence we may replace $j \times_{/J} I$ by $j \times_J I$ in the statement.

(iv). We will show again that both unit and counit

$$(\alpha,\mu)_!(\alpha,\mu)^* E \longrightarrow E \qquad F \longrightarrow (\alpha,\mu)^*(\alpha,\mu)_! F$$

are isomorphisms when restricted to the subcategories of coCartesian objects. Let $j : (\cdot, T_j) \to (J, T)$ be the embedding. We have to see that

$$j^*(\alpha,\mu)_!(\alpha,\mu)^* E \longrightarrow j^* E \tag{5}$$

is an isomorphism for all j.

Consider the 2-commutative diagram:

$$\begin{tikzcd}
(I \times_{/J} j, T_j) \arrow[r, "\iota"] \arrow[d, "p_j"'] & (I, S) \arrow[d, "{(\alpha,\mu)}"] \arrow[dl, Rightarrow] \\
(j, T_j) \arrow[r] & (J, T)
\end{tikzcd}$$

By the explicit point-wise formula for $(\alpha,\mu)_!$, the morphism (5) is the same as

$$p_{j,!}\iota^*(\alpha,\mu)^* E \longrightarrow j^* E_j.$$

The morphism induced by the 2-morphism in the diagram $\iota^*(\alpha,\mu)^* E \to p_j^* j^* E$ is an isomorphism on coCartesian objects by definition. Since the counit $p_{j,!}p_j^* \to \mathrm{id}$ is an isomorphism by assumption, we are done.

We now show that $(\alpha,\mu)_!$ preserves coCartesian objects. Let $\rho : j_1 \to j_2$ be a morphism in J. We have to show that

$$S(\rho)^*((\alpha,\mu)_! M)_{j_1} \longrightarrow ((\alpha,\mu)_! M)_{j_2}$$

is an isomorphism. After inserting the point-wise formula and denoting $p_2' : (I \times_{/J} j_1, T_{j_2}) \to (\cdot, T_{j_2})$, $\iota_2' : (I \times_{/J} j_1, T_{j_2}) \to (I, \alpha^* T)$ we get:

$$p'_{2,!}(\iota_2')^* M \longrightarrow p_{2,!}\iota_2^* M. \tag{6}$$

This is the morphism induced by the counit $\widetilde{\rho}_!\widetilde{\rho}^*$ for the morphism $\widetilde{\rho} : (I \times_{/J} j_1, T_{j_2}) \to (I \times_{/J} j_2, T_{j_2})$ (composition with ρ). Now observe that $p'_{2,!}$, resp. $p_{2,!}$, by assumption, can be computed on coCartesian elements just by evaluation at any element of $I \times_{/J} j_1$ resp. $I \times_{/J} j_2$. Therefore (6) is an isomorphism.

(v). The statement is obvious.

(vi). We have to show that the inclusion $\iota : (\Delta^\circ_{\le 3}, S_\bullet) \to (\Delta^\circ, S_\bullet)$ is of descent (resp. of finite descent), the category $\mathscr{D}(\Delta^\circ_{\le 3}, S_\bullet)^{\mathrm{cocart}}$ being clearly just the category of classical descent data. By (iv) and (v) this amounts to showing that $\Delta^\circ_{\le 3} \times_{/\Delta^\circ} \Delta_n$ is connected and $\pi_1(\Delta^\circ_{\le 3} \times_{/\Delta^\circ} \Delta_n)$ is trivial. This is well-known. Note that it is essential to take 3 terms of Δ° here. For example $\pi_1(\Delta^\circ_{\le 2} \times_{/\Delta^\circ} \Delta_3) \cong \mathbb{Z}$. The same holds with Δ° replaced by Δ because there is an adjunction

$$\Delta^\circ_{\le m} \times_{/\Delta^\circ} \Delta_n \rightleftarrows \Delta_{\le m} \times_{/\Delta} \Delta_n$$

and the morphism

$$(\Delta, S_\bullet) \longrightarrow (\Delta^\circ, S_\bullet)$$

is of descent (resp. of finite descent). To prove the latter assertion, by (iv) it suffices to show that $\Delta^\circ \times_{/\Delta} \Delta_n$ is contractible. To see this, consider the projection $p : \Delta^\circ \times_{/\Delta} \Delta_n \to \cdot$. It has a section s given by mapping $\cdot$ to id_{Δ_n}. We construct a morphism

$$\xi : \Delta^\circ \times_{/\Delta} \Delta_n \longrightarrow \Delta^\circ \times_{/\Delta} \Delta_n$$

mapping $\alpha : \Delta_k \to \Delta_n$ to $\alpha' : \Delta_{k+1} \to \Delta_n$ given by

$$\alpha'(i) = \begin{cases} \alpha(i) & i < k, \\ n-1 & i = k, \end{cases}$$

and mapping an injective morphism $\beta : \Delta_k \to \Delta_{k'}$ such that

$$\begin{tikzcd} \Delta_k \arrow[rr, "\beta"] \arrow[dr] & & \Delta_{k'} \arrow[dl] \\ & \Delta_n & \end{tikzcd}$$

commutes to the (still injective) morphism $\beta' : \Delta_{k+1} \to \Delta_{k'+1}$ given by

$$\beta'(i) = \begin{cases} \beta(i) & i < k, \\ k' & i = k. \end{cases}$$

We have $p \circ s = \mathrm{id}_{\{\cdot\}}$ and here are obvious 2-morphisms

$$s \circ p \Rightarrow \xi \qquad \mathrm{id}_{\Delta^\circ \times_{/\Delta} \Delta_n} \Rightarrow \xi$$

showing that $\Delta^\circ \times_{/\Delta} \Delta_n$ is contractible, or, what matters here, that $(\Delta^\circ \times_{/\Delta} \Delta_n, S) \to (\cdot, S)$ is of descent (resp. finite descent) for any $S \in \mathscr{S}$. □

We need a refinement of Proposition 4 (iii) / (iv) which also is specific to the situation of fibered categories and will not hold in any context of cohomological descent. Call an object $j \in J$ *initial* if no morphism $j' \to j$ exists with $j \ne j'$.

Lemma 5. *Let again $\mathscr{D} \to \mathscr{S}$ be a bifibered category with choice of a full subcategory of finite objects $\mathscr{D}^{\mathrm{f}}$ as above. Let $(\alpha, \mu) : (I, S) \to (J, T)$ be a morphism of diagrams in $\mathscr{S}$. Assume that for any object j there is a morphism $k \to j$ from an initial object k.*

Then (α, μ) is of descent (resp. of finite descent) if $p_j : (j \times_{/J} I, \mathrm{pr}_2^* S) \to (j, T_j)$ is such that p_j^* is fully-faithful for any $j \in J$ and such that p_j is of descent (resp. of finite descent) for any initial object j. If α is a Grothendieck fibration then the same holds with $j \times_{/J} I$ replaced by $j \times_J I$.

Proof. By the proof of Proposition 4 (iii), $(\alpha, \mu)^*$ is fully-faithful because all p_j^* are fully-faithful. We show by direct construction that $(\alpha, \mu)^*$ is essentially surjective. Let M be an (I, S)-module and $j \in J$ an object. Choose a morphism $\alpha_j : k \to j$ such that k is initial, which is the identity if j is already initial. Define

$$N(j) := (p_{k,*} \iota_k^* M) \otimes_{T(k)} T(j) = (\cdot, T(\alpha_j))^* (p_{k,*} \iota_k^* M).$$

Note that $(p_k)_*$ is an inverse to the equivalence $(p_k)^*$. For a morphism $\nu : j_1 \to j_2$ we must define a morphism $N(j_1) \otimes_{T(j_1)} T(j_2) \to N(j_2)$ or, in other words $(\cdot, T(\nu))^* N(j_1) \to N(j_2)$. We have the standard 2-commutative diagram

$$\begin{tikzcd} (j \times_{/J} I, \iota_j^* S) \arrow[r, "\iota_j"] \arrow[d, "p_j"] & (I,S) \arrow[d, "(\alpha,\mu)"] \\ (j, T_j) \arrow[r] \arrow[ur, Rightarrow] & (J,T) \end{tikzcd}$$

Denote μ_ν the morphism induced by ν: $(j_2 \times_{/J} I, \mathrm{pr}_2^* S) \to (j_1 \times_{/J} I, \mathrm{pr}_2^* S)$. We have

$$p_{j_1} \circ \mu_\nu = (\cdot, T(\nu)) \circ p_{j_2}$$

and

$$\iota_{j_1} \circ \mu_\nu = \iota_{j_2}.$$

We give the morphism

$$(\cdot, T(\nu))^* (\cdot, T(\alpha_{j_2}))^* (p_{k_1,*} \iota_{k_1}^* M) \longrightarrow (\cdot, T(\alpha_{j_2}))^* (p_{k_2,*} \iota_{k_2}^* M).$$

Because of fully-faithfulness we may do so after pulling back via $p_{j_2}^*$ and hence define $p_{j_2}^*$ applied to it as the following composition

$$\begin{tikzcd} p_{j_2}^*(\cdot, T(\nu))^* (\cdot, T(\alpha_{j_1}))^* (p_{k_1,*} \iota_{k_1}^* M) \arrow[d, "\sim"] & p_{j_2}^*(\cdot, T(\alpha_{j_2}))^* (p_{k_2,*} \iota_{k_2}^* M) \arrow[d, "\sim"] \\ \mu_{\nu\circ\alpha_{j_1}}^* p_{k_1}^* p_{k_1,*} \iota_{k_1}^* M \arrow[d, "\sim"] & \mu_{\alpha_{j_2}}^* p_{k_2}^* p_{k_2,*} \iota_{k_2}^* M \arrow[d, "\sim"] \\ \mu_{\nu\circ\alpha_{j_1}}^* \iota_{k_1}^* M \arrow[d, "\sim"] & \mu_{\alpha_{j_2}}^* \iota_{k_2}^* M \arrow[d, "\sim"] \\ \iota_{j_2}^* M \arrow[r, equal] & \iota_{j_2}^* M \end{tikzcd}$$

using that $p_{k_1}^*$ and $p_{k_1,*}$ define an equivalence. One checks that this association is functorial. □

3. Descent for modules

Lemma 6. *Let $R \to R'$ be a ring homomorphism. For a morphism $(I,S) \to (J,T)$ of diagrams of R-algebras the property of being of descent for arbitrary modules implies that $(I, S \otimes_R R') \to (J, T \otimes_R R')$ is of descent for arbitrary modules.*

If $R \to R'$ is finite then for a morphism $(I,F) \to (J,G)$ of diagrams of R-algebras the property of being of descent for finitely generated modules implies that $(I, S \otimes_R R') \to (J, T \otimes_R R')$ is of descent for finitely generated modules.

Proof. The category of $(I, S \otimes_R R')$-modules is equivalent to the category of (I,S)-modules with R'-action, i.e. to the category whose objects are pairs consisting of an object $X \in [\,\mathbf{mod}\,](I,S)$ and of a homomorphism of (non-commutative) R-algebras $\rho : R' \to \mathrm{End}_R(X)$. □

Lemma 7. *Let $(\alpha,\mu) : (I,S) \to (J,T)$ be a morphism of diagrams of rings such that $I \times_{/J} j$ is a finite diagram for all j. If $(\alpha,\mu)^*$ is faithful then $(\alpha,\mu)^* M$ finitely generated implies M finitely generated. In particular "of descent" implies "of descent for finitely generated modules".*

Proof. This is similar to the statement that a module is finitely generated if it becomes finitely generated after a faithfully flat ring extension. Let M be a coCartesian module over (J,T) such that $(\alpha,\mu)^*M$ is finitely generated. Let $j \in J$. For each $(i, \rho : \alpha(i) \to j) \in I \times_{/J} j$ we know that $M(\alpha(i)) \otimes_{T_{\alpha(i)}} S_i$ is a finitely generated S_i-module. Let $\{\xi_k^{i,\rho}\}_k$ be images in $M(j)$ of the (finitely many) $M(\alpha(i))$-components of those generators. We claim that the union over those finite sets for all objects in $I \times_{/J} j$ generates $M(j)$. For let $N(j)$ be the submodule generated by them, and assume that $N(j)$ is different from $M(j)$. The non-zero morphism $j^*M \to M(j)/N(j)$ induces a non-zero morphism $M \to j_*(M(j)/N(j))$ and therefore a non-zero morphism $(\alpha,\mu)^*M \to (\alpha,\mu)^* j_*(M(j)/N(j))$ because $(\alpha,\mu)^*$ is faithful. For any i consider the morphism $i^*(\alpha,\mu)^*M \to i^*(\alpha,\mu)^* j_* M(j)/N(j)$ which is

$$M(\alpha(i)) \otimes_{T(\alpha_i)} S(i) \longrightarrow (\alpha(i)^*(j_* M(j)/N(j))) \otimes_{T(\alpha_i)} S(i),$$

or also

$$M(\alpha(i)) \otimes_{T(\alpha_i)} S(i) \longrightarrow \left(\prod_{\alpha(i)\to j} M(j)/N(j) \right) \otimes_{T(\alpha_i)} S(i).$$

This is the tensor product with $S(i)$ of the map induced by the canonical ones $M(\alpha(i)) \to M(j)$. By construction of $N(j)$ this map is zero, a contradiction. $\square$

4. Descent for modules on ringed spaces

Lemma 8. *For the bifibered category*

$$[\, X\text{-}\mathbf{mod}\,] \longrightarrow [\, X\text{-}\mathbf{ring}\,]$$

we have that $(I,S) \to (J,T)$ is of descent (resp. of finite descent) if for any open set $U \subset X$ there is a cover $U = \bigcup_i U_i$ such that $(I, S|_{U_i}) \to (J, T|_{U_i})$ is of descent (resp. of finite descent) for the bifibered category

$$[\, U_i\text{-}\mathbf{mod}\,] \longrightarrow [\, U_i\text{-}\mathbf{ring}\,].$$

Proof. This is an obvious glueing argument. Alternatively one could construct a commutative square of diagram-morphisms

$$\begin{tikzcd}
(I \times \Delta^\circ, S_\bullet) \arrow[r] \arrow[d] & (J \times \Delta^\circ, T_\bullet) \arrow[d] \\
(I,S) \arrow[r] & (J,T)
\end{tikzcd}$$

where the vertical morphisms consist point-wise in I (resp. J) of the restrictions of S_i (resp. T_j) to a hypercovering of X such that the top horizontal morphism consists point-wise in Δ° of a morphism of descent (resp. of finite descent). By explicit construction one sees that the upper horizontal morphism is also of descent (resp. of finite descent). The vertical morphisms are then of descent by the definition of sheaf. This shows that also the lower horizontal morphism is of descent (resp. of finite descent). $\square$

5. Descent and projective systems

Let S be a Noetherian ring, $\mathfrak{a}$ an ideal of S and consider the diagram $(\mathbb{N}^{\mathrm{op}}, S_\bullet)$ where $S_n = S/\mathfrak{a}^n S$ for every $n \in \mathbb{N}$.

Lemma 9. *For an object $M_\bullet \in [\,\mathbf{mod}\,](\mathbb{N}^{\mathrm{op}}, S_\bullet)$ the following assertions are equivalent*

(i) $M_\bullet \in [\,\mathbf{mod\text{-}f.g.}\,](\mathbb{N}^{\mathrm{op}}, S_\bullet)^{\mathrm{cocart}}$.

(ii) *M_1 is finitely generated and for each for each $k \le l$, the sequence*

$$0 \longrightarrow \mathfrak{a}^k M_l \longrightarrow M_l \longrightarrow M_k \longrightarrow 0$$

is exact.

Proof. The exact sequence

$$0 \longrightarrow \mathfrak{a}^k S \longrightarrow S \longrightarrow S/\mathfrak{a}^k S \longrightarrow 0$$

tensored with M_l yields the sequence

$$0 \longrightarrow \mathfrak{a}^k M_l \longrightarrow M_l \longrightarrow (S/\mathfrak{a}^k S) \otimes_S M_l \longrightarrow 0$$

Hence coCartesianity of the diagram is equivalent to the exactness of the sequence above. It suffices to show that for a coCartesian diagram the condition that M_1 is finitely generated implies that M_k is finitely generated. Consider the sequence of $S/\mathfrak{a}^l S$-modules

$$0 \longrightarrow \mathfrak{a} M_l \longrightarrow M_l \longrightarrow M_1 \longrightarrow 0$$

Since $\mathfrak{a}$ is nilpotent in $S/\mathfrak{a}^l S$, this implies that M_l is finitely generated by Nakayama's lemma. □

Lemma 10. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal and consider a diagram (I, F) of $\mathfrak{a}$-adically complete and separated Noetherian R-algebras. Let $(I \times \mathbb{N}^{\mathrm{op}}, F_\bullet)$ be the diagram with value $F_n(i) := F(i) \otimes_R R/\mathfrak{a}^n R$. Let $p : (I \times \mathbb{N}^{\mathrm{op}}, F_\bullet) \to (I, F)$ be the obvious morphism.*

(i) *p^* and p_* induce an equivalence*

$$[\,\mathbf{mod\text{-}f.g.}\,](I \times \mathbb{N}^{\mathrm{op}}, F_\bullet)^{I-\mathrm{cocart}} \leftrightarrow [\,\mathbf{mod\text{-}f.g.}\,](I, F)$$

(ii) *This equivalence restricts to an equivalence*

$$[\,\mathbf{mod\text{-}f.g.}\,](I \times \mathbb{N}^{\mathrm{op}}, F_\bullet)^{\mathrm{cocart}} \leftrightarrow [\,\mathbf{mod\text{-}f.g.}\,](I, F)^{\mathrm{cocart}}$$

of the full subcategories of coCartesian modules. In other words the morphism $(I \times \mathbb{N}^{\mathrm{op}}, F_\bullet) \to (I, F)$ is of descent for finitely generated modules.

Proof. (i). By Proposition 4(iv) the statement can be checked point-wise. This reduces to the case $I = \cdot$. Then the equivalence results from [6, Chapitre 0, Proposition (7.2.9) and Corollaire (7.2.10)].

(ii). We will show that for a finitely generated coCartesian $(I \times \mathbb{N}, F_\bullet)$-module M, the module $p_* M$ is again coCartesian. Let $\nu : i \to j$ be a morphism in I. We have to show that the morphism

$$\left(\lim_n M_n(j)\right) \otimes_{F(j)} F(i) \longrightarrow \lim_n M_n(i) = \lim_n (M_n(j) \otimes_{F_n(j)} F_n(i))$$

is an isomorphism. Using what is already proven, we may write this denoting $M(j) := \lim_n M_n(j)$

$$M(j) \otimes_{F(j)} F(i) \longrightarrow \lim_n (M(j) \otimes_{F(j)} F_n(i)).$$

That this is an isomorphism follows from the following lemma. □

Lemma 11. *Let R be a ring with ideal $\mathfrak{a}$. Assume that R is Noetherian and $\mathfrak{a}$-adically complete and separated. Let M be a f.g. R-module, and let S_n be a projective system of $R/\mathfrak{a}^n R$-modules (or algebras). Then we have:*

$$M \otimes_R (\lim_n S_n) \cong \lim_n (M \otimes_R S_n).$$

Proof. Since R is Noetherian, we have an exact sequence of R-modules

$$R^k \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

hence the exact sequence

$$S_n^k \longrightarrow S_n^m \longrightarrow S_n \otimes_R M \longrightarrow 0$$

and, taking the limit, the exact sequence

$$(\lim_n S_n)^k \longrightarrow (\lim_n S_n)^m \longrightarrow \lim_n(S_n \otimes_R M) \longrightarrow 0.$$

On the other hand by tensoring the original sequence with $\lim_n S_n$, we get

$$(\lim_n S_n)^k \longrightarrow (\lim_n S_n)^m \longrightarrow M \otimes_R (\lim_n S_n) \longrightarrow 0.$$

Comparing the two exact sequences proves the assertion. □

Proposition 12. *Let R be a ring with ideal $\mathfrak{a}$.*

Let $(\alpha,\mu) : (I,F) \to (J,G)$ be a morphism of diagrams of Noetherian R-algebras such that $C_\mathfrak{a}F$ and $C_\mathfrak{a}G$ consist of separated and Noetherian R-algebras, where $C_\mathfrak{a}$ means completion w.r.t. to the $\mathfrak{a}$-adic topology.

If (α,μ) is of descent for finitely generated modules then $(I, C_\mathfrak{a}F) \to (J, C_\mathfrak{a}G)$ is of descent for finitely generated modules.

Proof. Define F_n as $F \otimes_R R/\mathfrak{a}^n R$. We have by definition $C_\mathfrak{a}F = \lim_n F_n$. Lemma 6 implies that $(I,F_n) \to (J,G_n)$ is of descent for finitely generated modules. Therefore by Proposition 4(iv) the morphism $(I \times \mathbb{N}^{\mathrm{op}}, F_\bullet) \to (J \times \mathbb{N}^{\mathrm{op}}, G_\bullet)$ is of descent for finitely generated modules. Now we have the commutative diagram of diagrams of rings

$$\begin{tikzcd} (I \times \mathbb{N}^{\mathrm{op}}, F_\bullet) \arrow[r] \arrow[d] & (J \times \mathbb{N}^{\mathrm{op}}, G_\bullet) \arrow[d] \\ (I, C_\mathfrak{a}F) \arrow[r] & (J, C_\mathfrak{a}G) \end{tikzcd}$$

in which the upper horizontal morphism is of descent for finitely generated modules and the vertical ones are of descent for finitely generated modules by Lemma 10. Hence so is the lower horizontal one. □

Lemma 13. *Let R be a ring and $\mathfrak{a}_1,\dots,\mathfrak{a}_n$ ideals of R. Let R' be a Noetherian R-algebra. Then we have an isomorphism*

$$C_{\mathfrak{a}_1}\dots C_{\mathfrak{a}_n}R' \cong C_{\mathfrak{a}_1+\dots+\mathfrak{a}_n}R'$$

In particular the left hand side does not depend on the ordering of the $\mathfrak{a}_i$.

Proof. First note that

$$C_\mathfrak{b}R'$$

for any ideal $\mathfrak{b} \subset R$ and R-algebra R' is the limit over

$$R'/\mathfrak{b}^j R' \cong R' \otimes_R (R/\mathfrak{b}^j).$$

Applying the Lemma to R' with ideals $\mathfrak{a}_i R'$ we may thus assume that $R = R'$ and R is Noetherian.

It obviously suffices to see this for two ideals $\mathfrak{a},\mathfrak{b}$.

Consider the exact sequence

$$R^n \xrightarrow{(b_1\ \dots\ b_n)} R \longrightarrow R/\mathfrak{b}^j \longrightarrow 0$$

in which $\mathfrak{b}^j = (b_1,\dots,b_n)$. Since $C_\mathfrak{a}$ is exact on finitely generated R-modules, we get an exact sequence

$$(C_\mathfrak{a}R)^n \xrightarrow{(b_1\ \dots\ b_n)} C_\mathfrak{a}R \longrightarrow C_\mathfrak{a}(R/\mathfrak{b}^j) \longrightarrow 0.$$

Hence

$$C_{\mathfrak{a}}R/\mathfrak{b}^j C_{\mathfrak{a}}R \cong \lim_i R/(\mathfrak{a}^i + \mathfrak{b}^j)$$

for all j. Taking the limit we get

$$C_{\mathfrak{b}} C_{\mathfrak{a}} R \cong \lim_{i,j} (R/(\mathfrak{a}^i + \mathfrak{b}^j)).$$

However, in this projective system of shape $(\mathbb{N}^{\mathrm{op}})^2$ the diagonal is initial, therefore this is the same as

$$\lim_i R/(\mathfrak{a}^i + \mathfrak{b}^i).$$

On the other hand, we also have that

$$(\mathfrak{a}+\mathfrak{b})^{2i} \subseteq \mathfrak{a}^i + \mathfrak{b}^i \subseteq (\mathfrak{a}+\mathfrak{b})^i$$

showing the statement. □

6. Descent along basic formal/open coverings

Proposition 14. *Let R be a Noetherian ring and f a non-zero divisor of R. Denote $\widehat{R}$ the completion of R w.r.t. (f)-adic topology and let R_f, and $\widehat{R}_f$, the rings $R[f^{-1}]$, and $\widehat{R}[f^{-1}]$, respectively. Then*

(i) *The morphism of diagrams*

$$p : D := \left(\begin{tikzcd} & \widehat{R} \arrow[d] \\ R_f \arrow[r] & \widehat{R}_f \end{tikzcd} \right) \longrightarrow R$$

is of descent for arbitrary modules (resp. for finitely generated modules).

(ii) *For any sequence of ideals $\mathfrak{a}_i$ and elements f_i such that $I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} R \hookrightarrow I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} R_f$ is injective, and the morphism*

$$\widetilde{p} : I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} D \longrightarrow I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} R,$$

the functor $\widetilde{p}^*$ is fully-faithful. Here I_f, for an element $f \in R$, denotes the functor $R' \mapsto R'[f^{-1}]$.

Proof. We will actually need only the following axioms on the diagram of rings D:

(a) $R \to R_f$ and $R \to \widehat{R}$ are *flat* R-algebras and $\widehat{R}_f \cong \widehat{R} \otimes_R R_f$.

(b) The sequence

$$0 \longrightarrow R \longrightarrow R_f \oplus \widehat{R} \longrightarrow \widehat{R}_f \longrightarrow 0 \tag{7}$$

is exact.

(c) For each object (i.e. descent datum) $(\widehat{M}, M_f, \widehat{M}_f) \in [\,\mathbf{mod}\,](D)^{\mathrm{cocart}}$ the map

$$\widehat{M} \oplus M_f \longrightarrow \widehat{M}_f$$

is surjective.

Let us verify that the axioms (a)–(c) hold in the situation of the lemma.

(a). Flatness of R_f is clear. $\widehat{R}$ is flat, because R is Noetherian. The tensor property holds by construction.

(b). That the last map is surjective is clear. Hence the statement boils down to the Cartesianity of the diagram of R-modules

$$\begin{tikzcd} R \arrow[r, "f^n"] \arrow[d] & R \arrow[d] \\ \widehat{R} \arrow[r, "f^n"] & \widehat{R} \end{tikzcd}$$

for any n. This diagram is Cartesian because f is not a zero divisor in R (and hence neither in $\widehat{R}$) and we have an isomorphism

$$R/f^nR \cong \widehat{R}/f^n\widehat{R}.$$

(c). Let ρ be the composition of the $\widehat{R}_f$-module isomorphism $M_f \otimes_{R_f} \widehat{R}_f \to \widehat{M}_f$ and the inverse of $\widehat{M} \otimes_{\widehat{R}} \widehat{R}_f \to \widehat{M}_f$. Let m_f be any element of M_f. We have for all $q \in \widehat{R}_f$:

$$\rho(m_f \otimes q) = \sum_j \widehat{m}_j \otimes f^{-n_j} p_j q,$$

where $p_j \in \widehat{R}$ are independent of q. Hence this element is of the form $\widehat{m} \otimes 1$ if q is sufficiently divisible by f. Therefore, writing any given $m_f \otimes p \in M_f \otimes_{R_f} \widehat{R}_f \cong \widehat{M}_f$ as $m'_f \otimes 1 + m''_f \otimes q$ where q is sufficiently divisible by f, we see that the map $\widehat{M} \oplus M_f \to \widehat{M}_f$ is surjective.

Now assume that the axioms (a)–(c) hold. First observe that also $\widehat{R}_f$ is flat over R (base change of flatness). Let M be an R-module. Tensoring the sequence (7) with M over R yields the exact sequence

$$0 \longrightarrow M \longrightarrow M_f \otimes \widehat{M} \longrightarrow \widehat{M}_f \longrightarrow 0$$

where $M_f := M \otimes_R R_f$, $\widehat{M} := M \otimes_R \widehat{R}$, and $\widehat{M} \otimes_R \widehat{R}_f$. This shows that M can be reconstructed as the limit over the diagram

$$\left(\begin{tikzcd} & \widehat{M} \arrow[d] \\ M_f \arrow[r] & \widehat{M}_f \end{tikzcd}\right)$$

Consequently the unit $\mathrm{id} \to p_* p^*$ of the adjunction is an isomorphism.

For the second assertion of the proposition observe that application of the functors $C_{\mathfrak{a}_i}$, and I_{f_i}, respectively, induce exact sequences

$$0 \longrightarrow I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} R \longrightarrow I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} R_f \oplus I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} \widehat{R} \longrightarrow I_{f_1} C_{\mathfrak{a}_1} \dots I_{f_n} C_{\mathfrak{a}_n} \widehat{R}_f \longrightarrow 0$$

Therefore also after applying those functors to D, and R, respectively, we have that the unit $\mathrm{id} \to \widetilde{p}_* \widetilde{p}^*$ of the new adjunction is an isomorphism.

Now let $(\widehat{M}, M_f, \widehat{M}_f)$ be a descent datum. We form the exact sequence (cf. axiom (c) for the surjectivety of the map to $\widehat{M}_f$):

$$0 \longrightarrow N \longrightarrow M_f \otimes \widehat{M} \longrightarrow \widehat{M}_f \longrightarrow 0$$

To see that the counit $p^* p_* \to \mathrm{id}$ is an isomorphism, we have to show that the natural maps $N \otimes_R R_f \to M_f$ and $N \otimes_R \widehat{R} \to \widehat{M}$ are isomorphisms. Exactness of the sequence (7) yields that the following diagram is Cartesian

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_f \arrow[d] \\ \widehat{R} \arrow[r] & \widehat{R}_f \end{tikzcd}$$

From the Cartesianity, we may conclude that in the following diagram the right vertical morphisms is an isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d] & R_f \arrow[r] \arrow[d] & R_f/R \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \widehat{R} \arrow[r] & \widehat{R}_f \arrow[r] & \widehat{R}_f/\widehat{R} \arrow[r] & 0
\end{tikzcd}$$

Now observe that after tensoring this diagram with $\widehat{R}$ over R the whole morphism of exact sequences splits (via the multiplication maps), hence we get a diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \ker \arrow[r, equal] \arrow[d] & \ker \arrow[d] & & \\
0 \arrow[r] & \widehat{R} \otimes_R \widehat{R} \arrow[r] \arrow[d] & \widehat{R}_f \otimes_R \widehat{R} \arrow[r] \arrow[d] & (\widehat{R}_f/\widehat{R}) \otimes_R \widehat{R} \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \widehat{R} \arrow[r] \arrow[d] & \widehat{R}_f \arrow[r] \arrow[d] & (R_f/R) \otimes \widehat{R} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Tensoring the diagram over $\widehat{R}$ with $\widehat{M}$ (letting $\widehat{R}$ act on the first factor in the first column) we get exact colums which we may insert in the following diagram which gets exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker \otimes_{\widehat{R}} \widehat{M} \arrow[r, equal] \arrow[d] & \ker \otimes_{\widehat{R}} \widehat{M} \arrow[d] & \\
0 \arrow[r] & N \otimes_R \widehat{R} \arrow[r] \arrow[d] & \widehat{M} \otimes_R \widehat{R} \oplus \widehat{M}_f \arrow[r] \arrow[d] & \widehat{M}_f \otimes_R \widehat{R} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{M} \arrow[r] & \widehat{M} \oplus \widehat{M}_f \arrow[r] \arrow[d] & \widehat{M}_f \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

This shows that the natural map $N \otimes_R \widehat{R} \to \widehat{M}$ is an isomorphism.

That also $N \otimes_R R_f \to M_f$ is an isomorphism follows because the axioms (a)–(c) are completely symmetric in $\widehat{R}$ and R_f. Actually, if we have, as in the formulation of the proposition, that $R \to R_f$ is an epimorphism of R-algebras, i.e. that $R_f = R_f \otimes_R R_f$, then this is even easier.

Finally the statement of descent for *finitely generated* modules follows from Lemma 7. □

7. Descent along completions w.r.t. a divisor with normal crossings

Definition 15. *A subscheme D of a regular scheme S is called a* divisor with strict normal crossings *if it is the zero-locus of a Cartier divisor which is Zariski locally around any $p \in D$ of the form $f_1 \cdots f_m$, where $f_1, \ldots, f_m$ are part of a sequence of minimal generators of the maximal ideal at p.*

This definition differs slightly from [9, Exposé I, 3.1.5, p. 24] to the extent that there the existence of $f_1, \ldots, f_m$ is assumed globally. Note that it follows from the definition that all $V(f_i)$ (defined locally) are regular themselves.

Let X be a regular Noetherian scheme and let $D \subset X$ be a divisor with strict normal crossings. Let $\{Y\}$ be the coarsest stratification of X into locally closed subvarieties such that every component of D is the closure of a stratum. For each stratum Y we define a sheaf of $\mathscr{O}_X$-algebras on X:

$$R_Y(U) := C_{\overline{Y}}\mathscr{O}_X(U') = \lim_n \mathscr{O}_X(U')/\mathscr{I}_{\overline{Y}}(U')^n$$

where $U' \subset U$ is such that $U' \cap \overline{Y} = U \cap Y$. The definition of R_Y does not depend on the choice of U' for R_Y may also be described as $\iota_* \mathscr{O}_{C_{\overline{Y}}X|_Y}$, where $C_{\overline{Y}}X$ is the formal completion of X along $\overline{Y}$, and $C_{\overline{Y}}X|_Y$ is the open formal subscheme with underlying topological space equal to Y and ι is the composed morphism of formal schemes $C_{\overline{Y}}X|_Y \to X$.

For any chain of disjoint strata $Y_1, Y_2, \ldots, Y_n$ such that $Y_i \subset \overline{Y_{i-1}}$, we define inductively a sheaf $R_{Y_1,\ldots,Y_n}$ of $\mathscr{O}_X$-algebras on X which coincides with the previous one for $n = 1$. For $n > 1$ we set

$$R'_{Y_1,\ldots,Y_n}(U) := C_{\overline{Y_1}}\left(R_{Y_2,\ldots,Y_n}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{O}_X(U')\right). \tag{8}$$

Again $U' \subset U$ is such that $U' \cap \overline{Y_1} = U \cap Y_1$, and this definition does not depend on the choice of U'. $R_{Y_1,\ldots,Y_n}$ is defined to be the sheaf associated with the pre-sheaf $R'_{Y_1,\ldots,Y_n}$. If X is affine, then $R'_{Y_1,\ldots,Y_n}$ is already a sheaf. Hence the sheafification may at most change the value at non-quasi-affine opens.

Example 16. Consider $X = \mathbb{A}^2_k$ with coordinates y, z and $D = V(yz)$. Then the strata are $X' = X \setminus V(yz)$, $Y = V(y) \setminus V(z)$, $Z = V(z) \setminus V(y)$ and $W = V(y,z)$. We have

$$R_{X'}(X) = k[y,z,y^{-1},z^{-1}] \quad R_Y(X) = k[z,z^{-1}][\![y]\!] \quad R_Z(X) = k[y,y^{-1}][\![z]\!] \quad R_W(X) = k[\![y,z]\!]$$

and (for example):

$$R_{X',W}(X) = k[\![y,z]\!][y^{-1},z^{-1}] \quad R_{X',Y}(X) = k[z,z^{-1}][\![y]\!][y^{-1}] \quad R_{X',Y,W}(X) = k[\![z]\!][z^{-1}][\![y]\!][y^{-1}]$$

Recall (cf. [12, §2, Proposition 7]) that for a sheaf of $\mathscr{O}_X$-algebras R as above (which is coherent over itself as an R-module), a *coherent sheaf M of R-modules* is a sheaf of R-modules such that on a covering $\{U_i\}$ of X, we have an exact sequence

$$R|_{U_i}^{n_i} \longrightarrow R|_{U_i}^{m_i} \longrightarrow M|_{U_i} \longrightarrow 0$$

for every i.

Lemma 17. *If $U \subset X$ is affine then for any of the sheaves $R = R_{Y_1,\ldots,Y_n}$ of $\mathscr{O}_X$-modules defined above, we have an equivalence of categories of coherent sheaves of $R|_U$-modules and finitely generated $R(U)$-modules.*

Proof. For a Noetherian topological space X consider the class of sheaves $\mathscr{O}$ of Noetherian rings (which are coherent over themselves) on X which have the property that for all $U \subset X$ in a fixed cofinal system of opens (here the affines) the functor

$$\begin{aligned} \Gamma_U : \{\text{ coherent sheaves of } \mathscr{O}|_U\text{-modules }\} &\longrightarrow \{\text{ f.g. } \mathscr{O}(U)\text{-modules }\} \\ \mathscr{E} &\longmapsto \mathscr{E}(U) \end{aligned}$$

is an equivalence of categories.

One shows that this class is closed under taking localizations and closed under taking completions w.r.t. a sheaf of ideals. This is shown as for coherent sheaves of $\mathscr{O}_{\mathscr{X}}$-modules on a Noetherian formal scheme $\mathscr{X}$. $\square$

Coming back to the stratification $\{Y\}$ of X, we define the following semi-simplicial set S. The set S_n consists of chains of *disjoint* strata $[Y_1,\ldots,Y_n]$ such that $Y_i \subset \overline{Y_{i-1}}$ (we write also $Y_i < Y_{i-1}$) for $i = 2\ldots n$ with the obvious face maps. Alternatively consider the stratification as a partially ordered set where $Z \le Y$ if $Z \subset \overline{Y}$. S is then the (semi-simplicial) nerve of this partially ordered set.

We define also the category $\int S$ whose objects are pairs (n,ξ), where $n \in \mathbb{N}$ and $\xi \in S_n$, and whose morphisms $\mu : (n,\xi) \to (m,\xi')$ are morphisms $\mu : \Delta_n \to \Delta_m$ in Δ° such that $S(\mu)(\xi') = \xi$. This is the fibered category associated with the functor $(\Delta^\circ)^{\mathrm{op}} \to [\,\mathbf{set}\,] \subset [\,\mathbf{cat}\,]$ via the Grothendieck construction.

The construction (8) actually defines a functor

$$\begin{aligned} R : \int S &\longrightarrow [\,\mathscr{O}_X\text{-}\mathbf{alg}\,] \\ (n, Y_1 > \cdots > Y_n) &\longmapsto R_{Y_1,\ldots,Y_n}. \end{aligned}$$

Lemma 18. *The category*

$$[\,X\text{-}\mathbf{coh}\,]\left(\int S, R\right)^{\mathrm{cocart}}$$

is equivalent to the category of the following descent data: For each stratum Y a coherent sheaf M_Y of R_Y-modules together with isomorphisms

$$\rho_{Y',Z} : M_Y \otimes_{R_Y} R_{Y,Z} \longrightarrow M_Z \otimes_{R_Z} R_{Y,Z}$$

for any Y, Z with $Z \subset \overline{Y}$, which are compatible w.r.t. any triple Y, Z, W of strata with $Z \subset \overline{Y}$ and $W \subset \overline{Z}$ in the obvious way.

Proof. This follows essentially from Proposition 4(vi). For this observe that the morphism $(\int S, R) \to (\Delta^\circ, R')$, where $R'(\Delta_m) := \prod_{\xi \in S_m} R(m,\xi)$ is of finite descent by Proposition 4(iii) and the fact that S_m is finite. $\square$

Main Theorem 19. *The morphism $(\int S, R) \to (\cdot, \mathscr{O}_X)$ is of descent for coherent sheaves. In other words the natural functor*

$$[\,X\text{-}\mathbf{coh}\,]\left(\int S, R\right)^{\mathrm{cocart}} \longrightarrow [\,X\text{-}\mathbf{coh}\,](\mathscr{O}_X)$$

is an equivalence of categories.

Proof. By Lemma 8 and Lemma 17 we are reduced to a local, affine situation of the following kind: We may assume that D is defined by an equation $f_1 \cdots f_n = 0$ such that $f_1, \ldots, f_n$ are part of a minimal sequence of generators of the maximal ideal of a point $p \in D$. We may also assume (by possibly shrinking the affine cover) that all $V(f_{i_1}, \ldots, f_{i_k})$ are irreducible for each subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$. The strata Y are then all of the form $V(f_{i_1}, \ldots, f_{i_k}) \setminus V(f_{i_{k+1}} \cdots f_{i_n})$ such that $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. Denote $R := \mathscr{O}_X(U)$ for such an open affine subset U. Then

$$R_Y = \lim_l R[f_{i_{k+1}}^{-1}, \ldots, f_{i_n}^{-1}]/(f_{i_1}^l, \ldots, f_{i_k}^l)$$

and inductively

$$R_{Y_1,Y_2,\ldots,Y_n} = \lim_l R_{Y_2,\ldots,Y_n}[f_{i_{k+1}}^{-1}, \ldots, f_{i_n}^{-1}]/(f_{i_1}^l, \ldots, f_{i_k}^l)$$

for $Y_1 = V(f_{i_1}, \ldots, f_{i_k}) \setminus V(f_{i_{k+1}} \cdots f_{i_n})$. We similarly get a diagram $(\int S, R)$ in $\mathrm{Dia}^{\mathrm{op}}([\,\mathbf{ring}\,])$ and have to show that

$$[\,\mathbf{mod\text{-}f.g.}\,]\left(\int S, R\right)^{\mathrm{cocart}} \cong [\,\mathbf{mod\text{-}f.g.}\,](R) \tag{9}$$

is an equivalence. Note that our semi-simplicial set S here is equal to the (semi-simplicial) nerve of the partially ordered set $\mathscr{P}(\{1,\dots,n\})$ (power set).

We show by induction on n that (9) is an equivalence as follows. Denote the diagram defined before by $(\int S^{(n)}, R^{(n)})$ where D has equation $f_1\cdots f_n$. Decreasing n by 1 means forgetting the last element f_n. For $n=1$, $S^{(1)}$ is the nerve of the partially ordered set $\{\} < \{1\}$ hence $\int S^{(1)}$ is the diagram $\ulcorner$ and we are precisely in the situation of Proposition 14 (i). Therefore $(\int S^{(1)}, R^{(1)}) \to (\cdot, R)$ is of descent. For $n>1$ consider the stratification $\{Y\}$ for $f_1,\dots,f_{n-1}$. Then cut each stratum Y in the two pieces $Y' = Y \cap V(f_n)$ and $Y'' = Y \setminus Y'$. This yields a chain of stratifications and refinements like considered in the following Lemma. □

Lemma 20. *We consider arbitrary stratifications of U such that each stratum is locally closed and a union of the ones considered before. They are all of the form $V(f_{i_1},\dots,f_{i_k}) \setminus V(f_{i_{k+1}}\cdots f_{i_l})$, where however not necessarily $\{i_1,\dots,i_l\} = \{1,\dots,n\}$. The members of such a stratification form a partially ordered set $\mathscr{P}$ as before and we denote by S the corresponding (semi-simplicial) nerve.*

Let $\nu : \mathscr{P}' \to \mathscr{P}$ be a morphism of such partially ordered sets which comes from a refinement of stratifications such that precisely two strata Y', Y'' get mapped to one stratum Y and we have $Y' = Y \cap V(f)$ and $Y'' = Y \setminus Y'$. It induces a map of semi-simplicial nerves, denoted ν by abuse of notation, as follows: $(\Delta_n, Y_1 > \cdots > Y_n)$ is mapped to $(\Delta_{n'}, \nu(Y_1) > \cdots > \nu(Y_n))$ where in the sequence $\nu(Y_1) > \cdots > \nu(Y_n)$ double entries have been deleted such that every entry appears only once.

Let R, resp. R' be the corresponding ring-valued functors.

Then the functor induced by ν

$$[\,\textbf{mod-f.g.}\,]\left(\int S', R'\right)^{\mathrm{cocart}} \cong [\,\textbf{mod-f.g.}\,]\left(\int S, R\right)^{\mathrm{cocart}}$$

is an equivalence.

Proof.

Claim. *$\nu : \int S' \to \int S$ is a Grothendieck fibration.*

Proof of the claim. For the fiber of ν over a $\xi = (\Delta_n, Y_1 > \cdots > Y_n)$ there are two possibilities:

(1) If Y does not occur in the list, it consists of ξ itself, considered as an element of $\int S'$.

(2) If Y occurs in the list, the fiber consists of the diagram

$$\begin{tikzcd} & (\Delta_n, Y_1 > \cdots > Y' > \cdots > Y_n) \arrow[d] \\ (\Delta_n, Y_1 > \cdots > Y'' > \cdots > Y_n) \arrow[r] & (\Delta_{n+1}, Y_1 > \cdots > Y' > Y'' > \cdots > Y_n) \end{tikzcd}$$

For a morphism in $\int S'$, say $(\Delta_{n'}, Y_{i_1} > \cdots > Y_{i_{n'}}) \to (\Delta_n, Y_1 > \cdots > Y_n)$, there is an obvious pull-back functor between these fibers which establishes ν as a Grothendieck fibration. □

Using the claim and Proposition 4 (iii) we would have to show that the fibers of $(S'_\xi, R'|_\xi) \to (\xi, R(\xi))$ are of descent for any $\xi = (\Delta_k, Y_1 > \cdots > Y_k) \in \int S'$. Actually we have the refinement Lemma 5 which reduces us to prove that for all Z the fiber above (Δ_0, Z) (these are the initial objects of $\int S$ in the sense of Lemma 5) is of descent, and that for all other $p_\xi : (S'_\xi, R'_\xi) \to (\xi, R(\xi))$ the pull-back p_ξ^* is fully faithful. In other words, we have to see that

(1): Only $Z = Y$ is non-trivial.

$$p_Y : \left(\begin{tikzcd} & R_{Y'} \arrow[d] \\ R_{Y''} \arrow[r] & R_{Y',Y''} \end{tikzcd}\right) \longrightarrow (R_Y) \tag{10}$$

is of descent for finitely generated modules. Observe that $Y' = V(f_{i_1},\dots,f_{i_k}) \setminus V(f\cdot f_{i_{k+1}}\cdots f_{i_l})$ and $Y'' = V(f_{i_1},\dots,f_{i_k},f) \setminus V(f_{i_{k+1}}\cdots f_{i_l})$ with f not among the other f_i. Therefore we have, denoting $R' := R[f_{i_{k+1}}^{-1},\dots,f_{i_l}^{-1}]$ and using Lemma 13:

$$\begin{aligned} R_Y &= C_{(f_{i_1},\dots,f_{i_k})}R' \\ R_{Y'} &= C_{(f_{i_1},\dots,f_{i_k})}(R'[f^{-1}]) \\ R_{Y''} &= C_{(f_{i_1},\dots,f_{i_k})}C_{(f)}R' \\ R_{Y',Y''} &= C_{(f_{i_1},\dots,f_{i_k})}((C_{(f)}R')[f^{-1}]) \end{aligned}$$

Proposition 14 (i) asserts that

$$p : \left(\begin{tikzcd} & R'[f^{-1}] \arrow[d] \\ C_{(f)}R' \arrow[r] & (C_{(f)}R')[f^{-1}] \end{tikzcd}\right) \longrightarrow (R')$$

is of descent for finitely generated modules. Therefore, using Proposition 12, the original morphism of diagrams (10) is of descent for finitely generated modules as well.

(2):

$$p_\xi : \left(\begin{tikzcd} & R_{Y_1,\dots,Y',\dots,Y_n} \arrow[d] \\ R_{Y_1,\dots,Y'',\dots,Y_n} \arrow[r] & R_{Y_1,\dots,Y',Y'',\dots,Y_n} \end{tikzcd}\right) \longrightarrow (R_{Y_1,\dots,Y,\dots,Y_n})$$

is such that p_ξ^* is fully-faithful (on finitely generated modules). This follows from Proposition 14 (ii). □

References

[1] A. Beauville, Y. Laszlo, "Un lemme de descente", *C. R. Math. Acad. Sci. Paris* **320** (1995), no. 3, p. 335-340.

[2] O. Ben-Bassat, M. Temkin, "Berkovich spaces and tubular descent", *Adv. Math.* **234** (2013), p. 217-238.

[3] B. Bhatt, "Algebraization and Tannaka duality", *Camb. J. Math.* **4** (2016), no. 4, p. 403-461.

[4] D. Ferrand, M. Raynaud, "Fibres formelles d'un anneau local noethérien", *Ann. Sci. Éc. Norm. Supér.* **3** (1970), p. 295-311.

[5] M. Groechenig, "Adelic descent theory", *Compos. Math.* **153** (2017), no. 8, p. 1706-1746.

[6] A. Grothendieck, J. A. Dieudonné, *Eléments de géométrie algébrique. I*, Grundlehren der Mathematischen Wissenschaften, vol. 166, Springer, 1971, ix+466 pages.

[7] F. Hörmann, "Generalized automorphic sheaves and the proportionality principle of Hirzebruch–Mumford", 2016, https://arxiv.org/abs/1603.04411.

[8] ———, "Fibered multiderivators and (co)homological descent", *Theory Appl. Categ.* **32** (2017), no. 38, p. 1258-1362.

[9] L. Illusie (ed.), *Cohomologie l-adique et fonctions L*, Lecture Notes in Mathematics, vol. 589, Springer, 1977, Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), xii+484 pages.

[10] L. Moret-Bailly, "Un problème de descente", *Bull. Soc. Math. Fr.* **124** (1996), no. 4, p. 559-585.

[11] D. Schäppi, "Descent via Tannaka duality", 2015, https://arxiv.org/abs/1505.05681.

[12] J.-P. Serre, "Faisceaux algébriques cohérents", *Ann. Math.* **61** (1955), p. 197-278.

[13] The Stacks Project Authors, "Stacks project", Available at http://stacks.math.columbia.edu, 2014.