Comptes Rendus

Mathématique

Jakub Kabat and Beata Strycharz-Szemberg

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Jakub Kabat[a] and Beata Strycharz-Szemberg[b]

[a] Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

[b] Department of Mathematics, Cracow University of Technology, Warszawska 24, PL-31-155 Kraków, Poland.

E-mails: jakub.kabat@up.krakow.pl, szemberg@pk.edu.pl.

Abstract. The purpose of this note is to present and study a new series of the so-called unexpected curves. They enjoy a surprising property to the effect that their degree grows to infinity, whereas the multiplicity at a general fat point remains constant, equal 3, which is the least possible number appearing as the multiplicity of an unexpected curve at its singular point. We show that additionally the BMSS dual curves inherits the same pattern of behaviour.

2020 Mathematics Subject Classification. 14C20, 14N10, 14N20.

1. Introduction

The notion of unexpected curves has been introduced by Cook II, Harbourne, Migliore and Nagel in [3]. Motivated by an example described first by Di Gennaro, Ilardi, and Vallés in [4], they observed that there exist configurations Z of points in $\mathbb{P}^2$ such that imposing an additional point P of multiplicity m on the linear system of curves of degree d vanishing at all points of Z imposes less conditions than expected. If this happens, we say that Z *admits an unexpected curve* of degree d with a point of multiplicity m. More generally, one studies unexpected hypersurfaces in projective spaces $\mathbb{P}^N$. Here we propose the following abbreviation: we say that a set $Z \subset \mathbb{P}^N$ has the $U(N; d, m)$ property, if the number of conditions imposed by a general point of multiplicity m on forms of degree d vanishing along Z is less than expected. In order to alleviate notation for $N = 2$, we drop the first index and speak simply of the $U(d, m)$ property.

After the appearance of [3] in 2017, the subject has attracted a lot of attention. The foundational article [3] studies sets Z with the $U(d, d-1)$ property. Bauer, Malara, Szemberg, and Szpond in [1] discovered the first example of an unexpected surface in $\mathbb{P}^3$, more precisely, they identified a set Z with the $U(3; 4, 3)$ property. In the same paper, they observed a phenomenon which is now

known as the BMSS duality. Harbourne, Migliore, Nagel and Teitler study in [7] more systematically higher dimensional unexpected hypersurfaces, in particular a series of such hypersurfaces attached to root systems. Unexpected cones are studied in [8]. Other constructions of unexpected hypersurfaces appear in [2, 5, 6]. In a series of papers [9–11], Szpond shows examples of unexpected hypersurfaces with multiple fat points and studies series of examples building upon Fermat-type arrangements of hyperplanes. Our study is related to Fermat-type arrangements as well.

2. Initial example

In this section, we analyse in detail an example underlying our construction. Let F be the Fermat cubic curve given by the defining equation

$$x^3 + y^3 + z^3 = 0.$$

It is well-known that F has 9 inflection points. The coordinates of these points can be computed explicitly intersecting F with its Hessian curve H, which in this case splits in the union of lines given by the equation

$$xyz = 0.$$

Thus the inflection points on F have coordinates

$$A_1 = (1 : -1 : 0), \quad A_2 = (1 : -\varepsilon : 0), \quad A_3 = (1 : -\varepsilon^2 : 0),$$
$$A_4 = (1 : 0 : -1), \quad A_5 = (1 : 0 : -\varepsilon), \quad A_6 = (1 : 0 : -\varepsilon^2),$$
$$A_7 = (0 : 1 : -1), \quad A_8 = (0 : 1 : -\varepsilon), \quad A_9 = (0 : 1 : -\varepsilon^2),$$

where ε is a primitive root of the unity of order 3. The lines tangent to F at points $A_1, \ldots, A_9$ are given by equations:

$$\ell_1 : x + y = 0, \quad \ell_2 : x + \varepsilon^2 y = 0, \quad \ell_3 : x + \varepsilon y = 0,$$
$$\ell_4 : x + z = 0, \quad \ell_5 : x + \varepsilon^2 z = 0, \quad \ell_6 : x + \varepsilon z = 0,$$
$$\ell_7 : y + z = 0, \quad \ell_8 : y + \varepsilon^2 z = 0, \quad \ell_9 : y + \varepsilon z = 0.$$

Note that the product of these lines is the polynomial

$$g_3 = (x^3 + y^3)(y^3 + z^3)(z^3 + x^3). \tag{1}$$

Consider the arrangement $\mathscr{L}_3 = \{\ell_1, \ldots, \ell_9\}$. It has 3 triple points in the coordinate points

$$X_1 = (1 : 0 : 0), \quad X_2 = (0 : 1 : 0), \quad X_3 = (0 : 0 : 1)$$

and 27 points where only 2 configuration lines meet. We denote the set of these 27 points by Y. The union of the coordinate points is denoted by X.

Lemma 1. *The points in Y form an almost complete intersection. The ideal $I(Y)$ of Y is generated in degree* 6 *by the following three polynomials*

$$f_1 = y^6 - z^6, \quad f_2 = x^6 - z^6, \quad f_3 = (x^3 + z^3)(y^3 + z^3).$$

Proof. The first two polynomials define a complete intersection W_6 of 36 points of the form

$$(1 : \tau^\alpha : \tau^\beta),$$

where τ is a primitive root of unity of order 6 and $\alpha, \beta = 1, \ldots, 6$. Our 27 points are the set difference of W_6 and the complete intersection W_3 defined by

$$g_1 = x^3 - y^3, \quad \text{and} \quad g_2 = x^3 - z^3.$$

Obviously, none of points in W_3 belongs to the set of zeroes of f_3, whereas f_3 vanishes at all points of $Y = W_6 \setminus W_3$.   □

We are interested in the set

$$Z = Y \cup X, \tag{2}$$

which associated ideal $I(Z)$ is

$$I(Z) = I(Y) \cap (xy, xz, yz).$$

Lemma 2. *The ideal $I(Z)$ is generated in degree 7 by the following polynomials*

$$h_1 = x(y^6 - z^6), \qquad h_2 = y(x^6 - z^6), \qquad h_3 = z(x^3 + y^3)(y^3 + z^3),$$
$$h_4 = z(x^3 + y^3)(x^3 + z^3), \quad h_5 = y(x^3 + z^3)(y^3 + z^3), \quad h_6 = x(x^3 + z^3)(y^3 + z^3).$$

Proof. Let J be the ideal generated by $h_1, \ldots, h_6$. It is easy to see that

$$J \subset I(Z),$$

i.e., the polynomials h_i for $i = 1, \ldots, 6$ vanish at all points of Z. For the reverse inclusion, let $F \in I(Z)$ be an arbitrary polynomial. Since, in particular, $F \in I(Y)$, there are polynomials $\alpha, \beta, \gamma \in \mathbb{C}[x, y, z]$ such that

$$F = \alpha f_1 + \beta f_2 + \gamma f_3. \tag{3}$$

From $F \in I(X)$ we obtain, evaluating at X_1, that $\beta(1 : 0 : 0) = 0$, hence there are polynomials $\beta_1, \beta_2 \in \mathbb{C}[x, y, z]$ such that

$$\beta = y\beta_1 + z\beta_2. \tag{4}$$

Analogously, evaluating at X_2 and X_3, we obtain

$$\alpha = x\alpha_1 + z\alpha_2 \text{ and } -\alpha + \gamma - \beta = x\gamma_1 + y\gamma_2, \tag{5}$$

for some polynomials $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{C}[x, y, z]$. Substituting (4) and (5) to (3), we obtain

$$\begin{aligned} F &= x\alpha_1(y^6 - z^6) + z\alpha_2(y^6 - z^6) + y\beta_1(x^6 - z^6) + z\beta_2(x^6 - z^6) \\ &\quad + (x\gamma_1 + y\gamma_2 + x\alpha_1 + z\alpha_2 + y\beta_1 + z\beta_2)(x^3 + z^3)(y^3 + z^3) \\ &= \alpha_1 h_1 + \beta_1 h_2 + \gamma_1 h_6 + \gamma_2 h_5 + \alpha_1 h_6 + \beta_1 h_5 \\ &\quad + z\alpha_2(y^3 + z^3)(y^3 - z^3 + x^3 + z^3) + z\beta_2(x^3 + z^3)(x^3 - z^3 + y^3 + z^3) \\ &= \alpha_1 h_1 + \beta_1 h_2 + \gamma_1 h_6 + \gamma_2 h_5 + \alpha_1 h_6 + \beta_1 h_5 + \alpha_2 h_3 + \beta_2 h_4. \end{aligned}$$

Thus $F \in J$ which completes the proof. $\qquad \square$

Lemma 2 yields the following, immediate, consequence.

Corollary 3. *The set Z defined in* (2) *imposes independent conditions of forms of degree 7.*

Proof. Indeed, the space of homogeneous polynomials in $\mathbb{C}[x, y, z]$ of degree 7 has dimension $\binom{9}{2} = 36$. There are 30 points in Z, so the expected dimension of the vector space of homogeneous polynomials of degree 7 vanishing along Z is $36 - 30 = 6$. This is equal to the actual dimension established in Lemma 2. $\qquad \square$

The main result in this part is the following statement.

Theorem 4. *The set Z defined in* (2) *has the $U(7, 3)$ property.*

Proof. Let $P = (a : b : c)$ be a general point in $\mathbb{P}^2$ and let Γ be the curve defined by

$$\gamma = \gamma_P(x : y : z) = a^2(5c^3 - a^3)h_1 + b^2(b^3 - 5c^3)h_2 + c^2(c^3 - 5a^3)h_3 + c^2(5b^3 - c^3)h_4$$
$$+ 5b^2(a^3 - c^3)h_5 + 5a^2(c^3 - b^3)h_6. \tag{6}$$

Obviously $\gamma \in I(Z)$. Moreover $\text{mult}_P \Gamma \geq 3$. This can be checked directly computing partial derivatives of γ of order 2 and checking that they all vanish at P. By the multiple use of the Euler formula this justifies the claim. We omit the simple calculations. $\qquad \square$

2.1. *The BMSS dual of* Γ

The idea of the BMSS duality is to consider γ as a polynomial in variables a, b, c with parameters x, y, z viewed as coordinates of a general point $Q = (x : y : z)$ in the projective plane with the a, b, c coordinates. Thus, reorganizing terms we have

$$\begin{aligned}
\gamma = \gamma_Q(a : b : c) = {} & x(z^6 - y^6)a^5 + y(x^6 - z^6)b^5 + z(y^6 - x^6)c^5 \\
& + 5y(x^3 + z^3)(y^3 + z^3)a^3b^2 - 5x(x^3 + z^3)(y^3 + z^3)a^2b^3 \\
& - 5z(x^3 + y^3)(z^3 + y^3)a^3c^2 + 5x(x^3 + y^3)(z^3 + y^3)a^2c^3 \\
& + 5z(x^3 + y^3)(x^3 + z^3)b^3c^2 - 5y(x^3 + y^3)(x^3 + z^3)b^2c^3.
\end{aligned}$$

We claim that $\mathrm{mult}_Q \Gamma \geq 3$. This is again easy to check verifying vanishing of all partial derivatives of order 2 of γ taken, this time, with respect to variables a, b, c.

Let Λ be the linear system of curves of degree 5 generated by

$$u_1 = a^2(5c^3 - a^3), \quad u_2 = b^2(b^3 - 5c^3), \quad u_3 = c^2(c^3 - 5a^3),$$
$$u_4 = c^2(5b^3 - c^3), \quad u_5 = 5b^2(a^3 - c^3), \quad u_6 = 5a^2(c^3 - b^3).$$

These are the coefficients in (6).

We establish the following easy fact.

Lemma 5. *The system Λ is base point free.*

Proof. The polynomial u_6 vanishes if $a = 0$ or $c = \varepsilon^\alpha b$ for some $\alpha \in \{0, 1, 2\}$.

Assume first that $a = 0$. Then $u_3 = c^5$, so it must be $c = 0$. But then $u_2 = b^5$ and it must be $b = 0$, which is not possible for coordinates of a point in the projective plane.

Now we turn to the second case $c = \varepsilon^\alpha b$. Then vanishing of u_2 implies that $b = 0$ and proceeding as in the previous case we conclude that all coordinates vanish. $\square$

Thus Λ is not defined by vanishing along a specific set of points in $\mathbb{P}^2$. Nevertheless it has the unexpected property of having a member with a point of multiplicity 3 in a general point, which does not follow by a naive dimension count. It would be very interesting to understand better how Λ appears and to investigate properties of the *companion surface*, i.e., the image of $\mathbb{P}^2$ in $\mathbb{P}^5$ under the morphism defined by Λ, see [11, Section 4.4] for this path of thought.

3. General case

In the present section we generalize the construction exhibited in Section 2. Recall, that the Fermat arrangement of lines is defined by linear factors of the polynomial

$$F_m = (x^m - y^m)(y^m - z^m)(z^m - x^m).$$

The singular points S_m of this arrangement (i.e., points where 2 or more lines intersect) are the union of a complete intersection grid W_m defined by the ideal $I(W_m) = \big((x^m - y^m), (x^m - z^m) \big)$ and the three coordinate points $X = \{X_1, X_2, X_3\}$. We consider sets of points Z_m defined as the set difference of S_{2m} and W_m (therefore the name diminished Fermat-type arrangements). It turns out that these sets behave surprisingly regularly.

Lemma 6. *The points in $Y_m = W_{2m} \setminus W_m$ form an almost complete intersection. The ideal $I(Y_m)$ of Y_m is generated in degree $2m$ by the following 3 polynomials*

$$f_1 = x^{2m} - y^{2m}, \ \ f_2 = x^{2m} - z^{2m}, \ \ f_3 = (x^m + z^m)(y^m + z^m).$$

Proof. The proof is the same (with obvious exponent changes) as of Lemma 1 and therefore omitted here. $\square$

Lemma 7. *The ideal $I(Z_m)$ is generated in degree $2m+1$ by the following polynomials*

$$h_1 = x(y^{2m} - z^{2m}), \ h_2 = y(x^{2m} - z^{2m}), \ h_3 = z(x^m + y^m)(y^m + z^m),$$

$$h_4 = z(x^m + y^m)(x^m + z^m), \ h_5 = y(x^m + z^m)(y^m + z^m), \ h_6 = x(x^m + z^m)(y^m + z^m).$$

Proof. The proof is almost verbatim as that of Lemma 2. We omit it here. □

The main result of this work is the following.

Theorem 8. *For $m \geq 3$ the sets Z_m have the $U(2m+1, 3)$ property.*

Proof. It is enough to produce the equation of the unexpected curve of degree $2m+1$ explicitly. To this end, let $P = (a : b : c)$ be a general point in $\mathbb{P}^2$ and let Γ_m be the curve defined by

$$\begin{aligned}
\gamma_m = \gamma_{m,P}(x : y : z) = {}& a^{m-1}((2m-1)c^m - a^m)h_1 + b^{m-1}(b^m - (2m-1)c^m)h_2 \\
& + c^{m-1}(c^m - (2m-1)a^m)h_3 + c^{m-1}((2m-1)b^m - c^m)h_4 \\
& + (2m-1)b^{m-1}(a^m - c^m)h_5 + (2m-1)a^{m-1}(c^m - b^m)h_6.
\end{aligned}$$

Obviously $\gamma_m \in I(Z_m)$. Moreover $\text{mult}_P \Gamma_m \geq 3$. This can be checked directly computing partial derivatives of γ_m up to order 2 and checking that they all vanish at P. By a multiple use of the Euler formula we can justify that claim – we omit here simple calculations. □

3.1. *The BMSS dual curve*

Of course, also in the general case, we can look for the equation of Γ_m from the perspective of coordinates $(a : b : c)$. We obtain a curve of degree $2m-1$

$$\begin{aligned}
\gamma_m = \gamma_Q(a : b : c) = {}& x(z^{2m} - y^{2m})a^{2m-1} + y(x^{2m} - z^{2m})b^{2m-1} + z(y^{2m} - x^{2m})c^{2m-1} \\
& + (2m-1)y(x^m + z^m)(y^m + z^m)a^m b^{m-1} \\
& - (2m-1)x(x^m + z^m)(y^m + z^m)a^{m-1}b^m \\
& - (2m-1)z(x^m + y^m)(z^m + y^m)a^m c^{m-1} \\
& + (2m-1)x(x^m + y^m)(z^m + y^m)a^{m-1}c^m \\
& + (2m-1)z(x^m + y^m)(x^m + z^m)b^m c^{m-1} \\
& - (2m-1)y(x^m + y^m)(x^m + z^m)b^{m-1}c^m
\end{aligned}$$

with a point of multiplicity 3 at $Q = (x : y : z)$.

As in the initial case, there is a full analogy with Lemma 5.

Lemma 9. *The linear system Λ_m generated by*

$$u_1 = a^{m-1}((2m-1)c^m - a^m), \ u_2 = b^{m-1}(b^m - (2m-1)c^m), \ u_3 = c^{m-1}(c^m - (2m-1)a^m),$$

$$u_4 = c^{m-1}((2m-1)b^m - c^m), \ u_5 = (2m-1)b^{m-1}(a^m - c^m), \ u_6 = (2m-1)a^{m-1}(c^m - b^m)$$

is base point free.

The existence of a member of Λ_m vanishing to order 3 at a general point is unexpected. It would be interesting to explore linear systems of this kind in a more systematic way. We formulate it as an open problem.

Problem 10. *Find more examples of base points free (even very ample) linear systems admitting members with exceptional high multiplicity at a general point.*

Acknowledgement

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