Zdzisław Brzeźniak and Nimit Rana

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Low regularity solutions to the stochastic geometric wave equation driven by a fractional Brownian sheet

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Abstract. We announce a result on the existence of a unique local solution to a stochastic geometric wave equation on the one-dimensional Minkowski space $\mathbb{R}^{1+1}$ with values in an arbitrary compact Riemannian manifold. We consider a rough initial data in the sense that its regularity is lower than the energy critical.

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1. Introduction

Recently, the existence and the uniqueness of a global solution, in the strong and weak sense, for the stochastic geometric wave equations (SGWEs) on the Minkowski space $\mathbb{R}^{1+m}$, $m \geq 1$, with the target manifold $(N, g)$ being a suitable $n$-dimensional Riemannian manifold, e.g. a sphere, has been established under various sets of assumptions by the first named author and M. Ondreját, see [1–3] for details. To the best of our knowledge, the most general result in the case $m = 1$, is a construction of a global $H^1_{loc}(N) \times L^2_{loc}(TN)$-valued weakly continuous solution of SGWE, where $TN$ denotes the tangent bundle of $N$, see [2].

The purpose of this note is to present a method by which we can prove the existence of a unique local solution to SGWE with $m = 1$ in the case of the initial data belonging to $H^s_{loc}(N) \times H^{s-1}_{loc}(TN)$ for $s \in (\frac{3}{2}, 1)$. In particular, we generalize the corresponding deterministic theory result of [8] to the stochastic setting, as well as the results of [1–3] to the wave maps equation with low regularity initial data (i.e. $s < 1$) and fractional (both in time and space) Gaussian noise.

A more detailed account of this work and the global theory, with complete proofs, will be presented in forthcoming papers.

* Corresponding author.
2. Problem formulation

We are interested in solutions having continuous paths and hence, motivated by [8] and [11], we find that it is suitable to formulate the Cauchy problem for the SGWE using local coordinates on the target manifold $N$. To be precise, given a sufficiently smooth function $\sigma$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, for the wave map $z : \mathbb{R}^{1+1} \to N$ composed with a given local chart $\phi$ of $N$ we consider the following Cauchy problem

\[ \square u = N_0(u) + \sigma(u) \xi, \quad u(0,x) = u_0(x), \quad \text{and} \quad \partial_t u(0,x) = u_1(x), \]  

(1)

where $\phi \circ z := u : \mathbb{R}^{1+1} \to \mathbb{R}^n$, $\square := \partial^2_t - \Delta_x$;

\[ \partial_0 = \partial_t, \quad \vartheta_0 = -\partial_t, \quad \partial_1 = \partial^1 = \partial_x; \]

\[ N_0(u) := -\sum_{a,b=1}^n \sum_{\mu=0}^1 \Gamma^k_{ab}(u) (\partial_\mu u^a \partial^\mu u^b) \]  

with $\Gamma^k_{ab}$ denoting the Christoffel symbols on $N$ in the chosen local coordinate system and $\xi$ is a suitable random field. The necessary assumptions will be given later in a precise manner.

An efficient way to simplify the computations of the required a’priori estimates for (1) is to switch the coordinate-axis of $(t,x)$-variables to the null coordinates, see for instance [8, 9], and respectively [13], for the deterministic and the stochastic problem. Our approach is in line with these references. By performing the following transformation, which can be made rigorous for sufficiently regular case,

\[ u^*(\alpha, \beta) := u \left( \frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2} \right) = u(t,x) \text{ and } u(t,x) = u^*(t + x, t - x), \]

(2)

the problem (1) can be re-written as

\[ \diamond u^* = \mathcal{N}(u^*) + \sigma(u^*) \Xi_{\alpha \beta}, \]

(3)

where $\Xi_{\alpha \beta} := \frac{\partial^2 \Xi}{\partial \alpha \partial \beta}$, subject to the following boundary conditions

\[ u^*(\alpha, -\alpha) = u_0(\alpha), \quad \partial_\alpha u^*(\alpha, -\alpha) + \partial_\beta u^*(\alpha, -\alpha) = u_1(\alpha). \]

(4)

Here $\Xi$ is a fractional Brownian sheet (fBs) on $\mathbb{R}^2$ with Hurst indices $H_1, H_2 \in (0,1)$, i.e. $\Xi$ is a centered Gaussian process such that

\[ \mathbb{E}[\Xi(\alpha_1, \beta_1) \Xi(\alpha_2, \beta_2)] = R_{H_1}(|\alpha_1|, |\alpha_2|) R_{H_2}(|\beta_1|, |\beta_2|), \quad (\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{R}^2, \]

where $R_{H}(a, b) = \frac{1}{2} \left( a^{2H} + b^{2H} - |a - b|^{2H} \right)$, $a, b \in \mathbb{R}$ and

\[ \diamond u^* := 4 \frac{\partial^2 u^*}{\partial \alpha \partial \beta}, \quad \mathcal{N}(u^*) := 4 \sum_{a,b=1}^n \Gamma_{ab}(u^*) \frac{\partial u^a}{\partial \alpha} \frac{\partial u^b}{\partial \beta}. \]

From now on we will only work in the $(\alpha, \beta)$-coordinates and hence, we will write $u$ instead of $u^*$ in the sequel. As usual in the SPDE theory, we understand the SGWE (3) in the following integral/mild form

\[ u = S(u_0, u_1) + \diamond^{-1} \mathcal{N}(u) + \diamond^{-1}[\sigma(u) \Xi_{\alpha \beta}], \]

(5)

where, for $(\alpha, \beta) \in \mathbb{R}^2$,

\[ [S(u_0, u_1)](\alpha, \beta) := \frac{1}{2} \left( u_0(\alpha) + u_0(-\beta) \right) + \frac{1}{2} \int_{-\beta}^\alpha u_1(r) \, dr; \]

(6)

\[ [\diamond^{-1} \mathcal{N}(u)](\alpha, \beta) := \frac{1}{4} \int_{-\beta}^\alpha \int_{-\alpha}^\beta \mathcal{N}(u(a,b)) \, db \, da; \]

(7)

and

\[ [\diamond^{-1}[\sigma(u) \Xi_{\alpha \beta}]](\alpha, \beta) := \frac{1}{4} \int_{-\beta}^\alpha \int_{-\alpha}^\beta \sigma(u(a,b)) \Xi_{\alpha \beta}(da, db). \]

(8)

The integral on the right hand side of (8) is well-defined pathwise, see Proposition 4.
3. Relevant notation and function spaces

If \( x \) and \( y \) are two quantities (typically non-negative), we will write \( x \lesssim y \) or \( y \gtrsim x \) to denote the statement that \( x \leq C y \) for some positive constant \( C > 0 \).

By \( L^p(\mathbb{R}^d) \), for \( p \in [1, \infty) \), we denote the classical real Banach space of all (equivalence classes of) \( \mathbb{R} \)-valued \( p \)-integrable functions on \( \mathbb{R}^d \). For \( s \in \mathbb{R} \), we set

\[
H^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \| f \|_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi < \infty \},
\]

where \( \mathcal{S}'(\mathbb{R}^d) \) is the set of all tempered distributions on \( \mathbb{R}^d \), i.e. the dual of the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) of all rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^d \), and \( \langle \xi \rangle := (1 + |\xi|^2)^{1/2} \), \( \xi \in \mathbb{R}^d \) and \( \hat{f} \) is the \( d \)-dimensional Fourier transform of \( f \).

**Definition 1.** Let \( s, \delta \in \mathbb{R} \). The hyperbolic \( H^{s,\delta} \) and the product \( H^s H^\delta_x \) Sobolev spaces are the sets of all \( u \in \mathcal{S}'(\mathbb{R}^2) \) for which, the appropriate norm is finite, where, with \( \mathcal{F}(u) \) being the space-time Fourier transform of \( u \in \mathcal{S}'(\mathbb{R}^2) \),

\[
\| u \|_{H^{s,\delta}} := \left( \int_{\mathbb{R}^2} |\tau| + \langle \xi \rangle |\tau| \right)^{2s} |\tau|^{-2\delta} |\mathcal{F}(u)(\tau, \xi)|^2 \, d\tau \, d\xi \right)^{1/2},
\]

\[
\| u \|_{H^s H^\delta_x} := \left( \int_{\mathbb{R}^2} |\tau|^{2s} |\xi|^{2\delta} |\mathcal{F}(u)(\tau, \xi)|^2 \, d\tau \, d\xi \right)^{1/2}.
\]

Let \( \Phi(\mathbb{R}^d) \) be the set of all systems \( \varphi = (\varphi_j)_{j=0}^\infty \in \mathcal{S}(\mathbb{R}^d) \) such that

1. \( \text{supp} \varphi_0 \subset \{ x : |x| \leq 2 \}, \) \( \text{supp} \varphi_j \subset \{ x : 2^j \leq |x| \leq 2^{j+1} \}, \) if \( j \in \mathbb{N} \setminus \{0\} \).
2. For every multi-index \( \alpha \), there exists a positive number \( C_\alpha \) such that

\[
2^{j|\alpha|} D^\alpha \varphi_j(x) \leq C_\alpha \quad \text{for all } j \in \mathbb{N} \text{ and all } x \in \mathbb{R}^d.
\]
3. \( \sum_{j=0}^\infty \varphi_j(x) = 1 \) for every \( x \in \mathbb{R}^d \).

It is known, see [12, Remark 2.3.1/1], that the system \( \Phi(\mathbb{R}^d) \) is not empty. Given a dyadic partition of unity \( \varphi := (\varphi_j)_{j=0}^\infty \in \Phi(\mathbb{R}) \) and a tempered distribution \( f \in \mathcal{S}'(\mathbb{R}^2) \), the Littlewood–Paley blocks of \( f \) are defined as \( \Delta_{j,k} f := 0, j, k \leq -1 \), and

\[
\Delta_{j,k} f := \mathcal{F}^{-1} (\varphi_j(\tau) \varphi_k(\xi) \mathcal{F}(f)(\tau, \xi)), \quad j, k \geq 0,
\]

where \( \mathcal{F}^{-1} \) stands for the inverse Fourier transform on \( \mathcal{S}'(\mathbb{R}^2) \). Next, for \( (s_1, s_2) \in \mathbb{R}^2, p, q \in (1, \infty) \), we define the following Banach space

\[
S_{p,q}^{s_1,s_2} B(\mathbb{R}^2) = \{ f \in \mathcal{S}'(\mathbb{R}^2) : \| f \|_{S_{p,q}^{s_1,s_2} B(\mathbb{R}^2)} < \infty \},
\]

where

\[
\| f \|_{S_{p,q}^{s_1,s_2} B(\mathbb{R}^2)} := \left( \sum_{k=0}^\infty \sum_{j=0}^\infty 2^{q(s_1 j + s_2 k)} \| \Delta_{j,k} f \|_{L^p(\mathbb{R}^2)}^q \right)^{1/q}.
\]

One can prove that the space \( S_{p,q}^{s_1,s_2} B(\mathbb{R}^2) \) does not depend on the chosen system \( \varphi \in \Phi(\mathbb{R}) \), see [12, Proposition 2.3.2/1], and the norms are pairwise equivalent.

It is known that \( S_{2,2}^{s,\delta} B(\mathbb{R}^2) = H^s_x H^\delta_t (\mathbb{R}^2) \), for \( s, \delta \in \mathbb{R} \), with equivalent norms.

The next proposition justifies the coordinate transformation (2) from the computation perspective, since in \((\alpha, \beta)\)-coordinate the knowledge of product Sobolev spaces is enough to have the local theory.
Proposition 2. If \( s \geq \delta \in \mathbb{R} \), then the map
\[
H^{s,\delta} \ni u(t,x) \mapsto u^* (\alpha, \beta) \in H_{\alpha}^{s,\delta} H_{\beta}^{s,\delta} =: H^{s,\delta},
\]
is an isomorphism, where as usual the space \( H^{s,\delta} \) is equipped with the norm
\[
\| u^* \|_{H^{s,\delta}} := \sqrt{\| u^* \|_{H_{\alpha}^{s,\delta}}^2 + \| u^* \|_{H_{\beta}^{s,\delta}}^2}.
\]
In particular, we have
\[
\| u^* \|_{H^{s,\delta}} \lesssim \| u^* \|_{H^{s,\delta}} \lesssim \| u^* \|_{H^{s,\delta}}.
\]

With the spaces \( H^{s,\delta} \) defined above introduced in the previous Proposition 2, we have the following result.

Proposition 3. Assume that \( H_1, H_2 \in \{0, 1\} \) and \( H'_i \in (0, H_1 \wedge H_2), i = 1, 2 \). Then there exists a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a map,
\[
\Xi : \mathbb{R}^2 \times \Omega \to \mathbb{R},
\]
such that \( \mathbb{P} \)-a.s. \( \Xi(\cdot, \cdot, \omega) \in H^{s,\delta} \) locally, i.e. for every bump function \( \eta \),
\[
\eta(\alpha) \eta(\beta) \Xi(\alpha, \beta, \omega) \in H^{s,\delta}.
\]
Moreover, for \((\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{R}^2\),
\[
\mathbb{E} \left[ \Xi(\alpha_1, \beta_1) \Xi(\alpha_2, \beta_2) \right] = R_{H_1}(|\alpha_1|, |\alpha_2|) R_{H_2}(|\beta_1|, |\beta_2|).
\]
Here \( \mathbb{E} \) is the Expectation operator w.r.t. \( \mathbb{P} \).

The above result, somehow related to a result proved in [11], can be proved by using Proposition 2 and a combination of results from [4] and [5].

4. The main result: the local well-posedness

Let us fix \( s \geq \delta \in \left(\frac{3}{4}, 1\right) \) for the whole present section. To solve the SGWE problem (5) locally, which is sufficient to prove the local-wellposedness result we are aiming, let \( \eta, \chi \in C_c^\infty(\mathbb{R}; [0, 1]) \) be even cut-off functions such that \( \text{supp} \eta = \text{supp} \chi \subseteq [-4,4] \) and \( [-2,2] \subseteq \eta^{-1}(\{1\}) = \chi^{-1}(\{1\}) \). Let us put \( \eta_T(x) := \eta(x/T), x \in \mathbb{R} \), for any \( T > 0 \). Similarly, we define \( \chi_T \).

To simplify the exposition, without loss of generality, we restrict ourselves to the target manifold of dimension 2 and which can be covered by a family of charts such that the Christoffel symbols \( \Gamma_{ab}^k \) depend polynomially on \( u \), that is, for every \( k = 1, 2 \), one can find \( r \in \mathbb{N} \) and \( A_{ab}^k \in \mathbb{R}^2 \) such that \( \Gamma_{ab}^k(u) = \sum_{i,j \leq r} A_{ab}^{kj} u^i u^j \), \( u = (u^1, u^2) \in \mathbb{R}^2 \), where, for \( l = (l_1, l_2) \in \mathbb{N}^2 \), \( u^l = [u^1]^{l_1} [u^2]^{l_2} \).

Our first result in this section is a generalization of [10, Lemma 2.2].

Proposition 4. Assume that \( s, \delta \in \left(\frac{3}{4}, 1\right), \) such that \( s \geq \delta, \) and \( f \in H^{s-1,\delta-1} \) are given. Then \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( F := H + I + J + G \) where, for \( \alpha, \beta \in \mathbb{R} \),
\[
H(\alpha, \beta) := \int_{-\beta}^{\alpha} \int_{-\gamma}^{\gamma} (\Delta_{0,0} f)(\gamma, \tau) \, d\tau \, d\gamma,
\]
\[
I(\alpha, \beta) := \int_{-\beta}^{\alpha} \sum_{n=1}^{\infty} \mathcal{F}^{-1} \left[ \frac{1}{i} 2 \phi_0(\tau) \phi_n(\xi)(\mathcal{F} f)(\tau, \xi) \right] (\gamma, \beta) \, d\gamma,
\]
\[
J(\alpha, \beta) := \int_{-\alpha}^{\beta} \sum_{m=1}^{\infty} \mathcal{F}^{-1} \left[ \frac{1}{i} 2 \phi_0(\xi) \phi_m(\tau)(\mathcal{F} f)(\tau, \xi) \right] (\alpha, \gamma) \, d\gamma
\]
\[
- \int_{-\alpha}^{\beta} \sum_{m=1}^{\infty} \mathcal{F}^{-1} \left[ \frac{1}{i} 2 \phi_0(\xi) \phi_m(\tau)(\mathcal{F} f)(\tau, \xi) \right] (\alpha, \gamma) \, d\gamma.
\]
and

\[ G(\alpha, \beta) := \sum_{j,k=1}^{\infty} \left[ \mathcal{F}^{-1} \left( \frac{1}{(i\tau)(i\xi)} \varphi_j(\tau)\varphi_k(\xi)(\mathcal{F} \phi)(\tau, \xi) \right) \right](\alpha, \beta) \]

\[ - \frac{1}{2} \sum_{j,k=1}^{\infty} \left[ \mathcal{F}^{-1} \left( \frac{1}{(i\tau)(i\xi)} \varphi_j(\tau)\varphi_k(\xi)(\mathcal{F} \phi)(\tau, \xi) \right) \right](\alpha, -\alpha) \]

\[ - \frac{1}{2} \sum_{j,k=1}^{\infty} \left[ \mathcal{F}^{-1} \left( \frac{1}{(i\tau)(i\xi)} \varphi_j(\tau)\varphi_k(\xi)(\mathcal{F} \phi)(\tau, \xi) \right) \right](-\beta, \beta) \]

\[ - \frac{1}{2} \int_{-\beta}^{\alpha} \sum_{j,k=1}^{\infty} \left[ \mathcal{F}^{-1} \left( \frac{1}{(i\tau)(i\xi)} \varphi_j(\tau)\varphi_k(\xi)(\mathcal{F} \phi)(\tau, \xi) \right) \right](\gamma, -\gamma) d\gamma \]

\[ - \frac{1}{2} \int_{-\beta}^{\alpha} \sum_{j,k=1}^{\infty} \left[ \mathcal{F}^{-1} \left( \frac{1}{(i\tau)(i\xi)} \varphi_j(\tau)\varphi_k(\xi)(\mathcal{F} \phi)(\tau, \xi) \right) \right](\gamma, -\gamma) d\gamma, \]

is the unique tempered distribution such that \( \frac{\partial^2 F}{\partial \alpha \partial \beta} = f \), and satisfy the following homogeneous boundary conditions

\[ F(\alpha, -\alpha) = 0 \quad \text{and} \quad \frac{\partial F}{\partial \alpha}(\alpha, -\alpha) + \frac{\partial F}{\partial \beta}(\alpha, -\alpha) = 0. \]

Moreover, for every \( \eta, \chi \) and \( T > 0 \), there exists \( C(\eta, \chi, T) > 0 \) such that

\[ \| \eta T(\alpha)\chi T(\beta)F(\alpha, \beta) \|_{H_\alpha, \delta} \leq C(\eta, \chi, T) \| f \|_{H_{\alpha-1, \delta-1}}. \]

We will use the following notation

\[ F(\alpha, \beta) := \int_{-\beta}^{\alpha} f(da, db), \quad (\alpha, \beta) \in \mathbb{R}^2. \]

**Proof.** Using the properties of \( S_{2,2}^{\gamma/\delta} B(\mathbb{R}^2) \) spaces, we need to show that \( G, H, I, J \) are well-defined elements of \( H^{s, \delta} \).

By following the approach of [7] we get the next required result.

**Proposition 5.** Assume that \( \sigma \in C^3_b(\mathbb{R}^2) \). Then \( \sigma \circ u \in H^{s, \delta} \) for every \( u \in H^{s, \delta} \) and there exist constants \( C_i(\sigma) := C_i(\| \sigma \|_{C^1_b}), \quad i = 1, 2 \) such that for \( u, u_1, u_2 \in H^{s, \delta} \),

\[ \| \sigma \circ u \|_{H^{s, \delta}}^2 \leq C_1(\sigma) \| u \|_{H^{s, \delta}}^2 \left[ 1 + \| u \|_{H^{s, \delta}}^2 \right], \]

\[ \| \sigma \circ u_1 - \sigma \circ u_2 \|_{H^{s, \delta}}^2 \leq C_2(\sigma) \| u_2 - u_1 \|_{H^{s, \delta}}^2 \left[ 1 + \sum_{i,k=1}^{2} \| u_i \|_{H^{s, \delta}}^{2k} \right]. \]

We now state and provide a sketch of proof of the main result of this note. Below we fix a realisation of the random field belonging to the space \( H^{s, \delta} \), see Proposition 3.

**Theorem 6.** Assume \( s, \delta \in \left( \frac{3}{4}, 1 \right) \) such that \( \delta \leq s \) and \( (u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \). Let \( \Xi \) be a fractional Brownian sheet with Hurst indices \( H_1, H_2 \in (s, 1) \). There exist a \( R_0 \in (0, 1) \) and a \( \lambda_0 := \lambda_0(\| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}}, R_0) \gg 1 \) such that for every \( \lambda \geq \lambda_0 \) there exists a unique \( u := u(\lambda, R_0) \in \mathbb{B}_R \), where \( \mathbb{B}_R := \{ u \in H^{s, \delta} : \| u \|_{H^{s, \delta}} \leq R \} \), which satisfies the following integral equation

\[ u(\alpha, \beta) = \eta(\lambda a)\eta(\lambda \beta) \left( |S(\chi(\lambda)u_0, \chi(\lambda)u_1)|(\alpha, \beta) + |\chi^{-1}(\mathcal{N}(u))(\alpha, \beta) + |\chi^{-1}(\sigma(u)\Xi_{\alpha \beta})|(\alpha, \beta) \right), \]

\( (\alpha, \beta) \in \mathbb{R}^2. \)

Here the right hand side terms are, respectively, defined in (6), (7) and (8).
**Sketch of proof of Theorem 6.** Our proof is based on the Banach Fixed Point Theorem in the space $\mathcal{H}^{s,\delta}$. Note that all the constants below are positive and depend on $\eta$ unless mentioned otherwise.

**Step 1.** Using the following well-known result, see e.g. [6],
\[ \left\| \chi(x) \int_0^x f(y) \, dy \right\|_{H^s} \lesssim \|f\|_{H^{s-1}}, \]
we can estimate the localized homogeneous part of the solution as
\[ \|\eta(\alpha)\chi(\beta)\phi(\alpha)\|_{H^s} \leq C_S \left( \|\phi\|_{H^s} + \|\phi\|_{H^{s-1}} \right). \]

**Step 2.** In view of the polynomiality of the Christoffel symbols, by using Proposition 4, we deduce the existence of a unique $u$ contraction as a map from $\mathcal{H}^{s,\delta}$ into itself, where
\[ \|\eta(\alpha)\chi(\beta)\phi(\alpha)\|_{H^s} \leq C_N \|\phi\|_{H^s} + \|\phi\|_{H^{s-1}} \]
for some positive constants $C_N$ and $C_N(\|\phi\|_{H^s})$.

**Step 3.** By Propositions 4 and 5 followed by the continuity of the multiplication map
\[ \mathcal{H}^{s,\delta} \times \mathcal{H}^{s-1,\delta-1} \rightarrow \mathcal{H}^{s-1,\delta-1}, \]
we get
\[ \|\eta(\alpha)\chi(\beta)\phi(S(u_0, u_1))\|_{H^{s-1,\delta-1}} \leq C_{N} C_{N}(\|\phi\|_{H^s} + \|\phi\|_{H^{s-1}}) \]
for some positive constants $C_{N}$ and $C_{N}(\|\phi\|_{H^s})$.

**Step 4.** We consider a map $\Theta^A : \mathcal{H}^{s,\delta} \ni u \rightarrow u_{\Theta^A} \in \mathcal{H}^{s,\delta}$ defined by
\[ u_{\Theta^A} := \eta(\alpha)\eta(\beta) \left( S(u_0^A, u_1^A) + \phi^{-1}(\mathcal{N}(\phi)) + \phi^{-1}(\mathcal{N}(\phi)) \right), \]
where
\[ u_0^A := \chi(\alpha) \left[ u_0 \left( \frac{\alpha}{A} \right) - \tilde{u}_0^A \right], \quad u_1^A := \chi(\alpha) \left[ u_1 \left( \frac{\alpha}{A} \right) \right], \quad \text{and} \quad \phi_A := \lambda^{-1} \Pi_A \phi_B, \]
with
\[ \tilde{u}_0^A := \int_R u_0 \left( \frac{Y}{A} \right) \psi(y) \, dy. \]

**Step 5.** By working with another suitable translated coordinate chart on $\mathcal{N}$ (which will remove the dependence on $\tilde{n}_0^A$) and by defining the inverse scaling $u(\alpha, \beta) := u^A(\lambda \alpha, \lambda \beta)$ for the fixed point $u^A$ from Step 4, we deduce that
\[ u(\alpha, \beta) = \Theta^A \left( u^A \right) (\lambda \alpha, \lambda \beta) \]
and
\[ u(\alpha, \beta) = \eta(\lambda \alpha)\eta(\lambda \beta) \left[ S(\chi(\lambda) u_0, \chi(\lambda) u_1) \right] + \left[ \phi^{-1}(\mathcal{N}(u)) \right] \]
for $\phi^{-1}(\mathcal{N}(u))$. Hence we conclude the proof of Theorem 6.
Theorem 7. Under the above mentioned assumptions, there exist an open set $\mathcal{O}$, containing the diagonal $\mathcal{D} := \{(\alpha, -\alpha) : \alpha \in \mathbb{R}\}$, and a function $u : \mathcal{O} \rightarrow \mathbb{R}^2$ such that for every $(\alpha_0, -\alpha_0) \in \mathcal{D}$, there exists $r > 0$ such that $u|_{B_r((\alpha_0, -\alpha_0))} \in H^{4,\delta}$, where $B_r((\alpha, -\alpha))$ is open ball of radius $r$ around $(\alpha, -\alpha)$, and $u$ solves (3)-(4) uniquely in $\mathcal{O}$.

To prove Theorem 7, for each fixed point $(\alpha_0, -\alpha_0) \in \mathcal{D}$, by Theorem 6 we find a unique solution $u_{\alpha_0}$ of a translated version of the problem (5) defined in some neighbourhood $N_{\alpha_0}$ of $(\alpha_0, -\alpha_0)$. By using the uniqueness we can glue “local” solutions to get a solution $u$ as in the assertion.

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