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Mathématique

Yayun Li, Qinghua Chen and Yutian Lei

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Volume 358, issue 6 (2020), p. 727-731

Published online: 2 October 2020

<https://doi.org/10.5802/crmath.91>



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Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569



Elliptical Partial Differential Equations / *Équations aux dérivées partielles elliptiques*

A Liouville theorem for the fractional Ginzburg–Landau equation

Yayun Li^a, Qinghua Chen^b and Yutian Lei^{*, b}

^a School of Applied Mathematics, Nanjing University of Finance & Economics, Nanjing, 210023, China

^b Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

E-mails: yayunli@nufe.edu.cn, leiyutian@njnu.edu.cn

Abstract. In this paper, we are concerned with a Liouville-type result of the nonlinear integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(1-|u|^2)}{|x-y|^{n-\alpha}} dy,$$

where $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$ with $k \geq 1$ and $1 < \alpha < n/2$. We prove that $u \in L^2(\mathbb{R}^n) \Rightarrow u \equiv 0$ on \mathbb{R}^n , as long as u is a bounded and differentiable solution.

2020 Mathematics Subject Classification. 45G05, 45E10, 35Q56, 35R11.

Funding. This research was supported by NNSF (11871278) of China and NSF of Jiangsu Education Commission (19KJB110016).

Manuscript received 18th March 2020, revised 27th April 2020 and 29th April 2020, accepted 26th June 2020.

If a harmonic function u is bounded on \mathbb{R}^n , then $u \equiv \text{Const.}$ (this is the Liouville theorem). Moreover, if u is integrable (i.e. $u \in L^s(\mathbb{R}^n)$ for some $s \geq 1$), then $u \equiv 0$ on \mathbb{R}^n .

In 1994, Brezis, Merle and Rivière [2] studied the quantization effects of the following equation

$$-\Delta u = u(1-|u|^2) \quad \text{on } \mathbb{R}^2. \quad (1)$$

Here $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a vector valued function. It is the Euler–Lagrange equation of the Ginzburg–Landau energy

$$E_{GL}(u) = \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4} \|1-|u|^2\|_{L^2(\mathbb{R}^2)}^2.$$

In particular, they proved the finite energy solution (i.e., u satisfies $\nabla u \in L^2(\mathbb{R}^2)$) is bounded (see also [4] and [6])

$$|u| \leq 1 \quad \text{on } \mathbb{R}^n. \quad (2)$$

* Corresponding author.

(Here $n = 2$.) Based on this result, they obtained a Liouville type theorem for finite energy solutions (cf. [2, Theorem 2]):

Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a classical solution of (1). If $\nabla u \in L^2(\mathbb{R}^2)$, then either $u \in L^2(\mathbb{R}^2)$ which implies $u \equiv 0$, or $1 - |u|^2 \in L^1(\mathbb{R}^2)$ which implies $u \equiv C$ with $|C| = 1$.

The boundedness and the integrability of solutions are the important conditions which ensure that the Liouville theorem holds. The Pohozaev identity plays a key role in the proof.

In this paper, we are concerned with the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(1 - |u|^2)}{|x - y|^{n-\alpha}} dy. \quad (3)$$

Here $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$, $k \geq 1$, $n \geq 3$, and $1 < \alpha < n/2$. We also apply the integral form of the Pohozaev identity (which was used for the Lane–Emden equations in [3], [5] and [12]) to establish a Liouville theorem.

Theorem 1. *Assume that $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is bounded and differentiable, and solves (3) with $\alpha \in (1, n/2)$. If $u \in L^2(\mathbb{R}^n)$, then $u(x) \equiv 0$.*

Proof. For convenience, we denote $B_R(0)$ by B_R here.

Step 1. We claim that the improper integral

$$\int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(z)(1 - |u(z)|^2)]}{|x - z|^{n-\alpha}} dz \quad (4)$$

is convergent at each $x \in \mathbb{R}^n$.

In fact, since $u \in L^2(\mathbb{R}^n)$, we can find $R = R_j \rightarrow \infty$ such that

$$R \int_{\partial B_R} |u(z)|^2 ds \rightarrow 0. \quad (5)$$

Since u is bounded, by the Hölder inequality, we obtain that for sufficiently large R , there holds

$$\begin{aligned} R \left| \int_{\partial B_R} \frac{u(z)(1 - |u(z)|^2)}{|x - z|^{n-\alpha}} ds \right| &\leq CR^{1-n+\alpha} \int_{\partial B_R} |u(z)| ds \\ &\leq CR^{1-n+\alpha} \left(R \int_{\partial B_R} |u(z)|^2 ds \right)^{\frac{1}{2}} R^{-\frac{1}{2} + \frac{n-1}{2}}. \end{aligned}$$

Let $R = R_j \rightarrow \infty$. Noting $\alpha < n/2$, and using (5) we get

$$R \int_{\partial B_R} \frac{u(z)(1 - |u(z)|^2)}{|x - z|^{n-\alpha}} ds \rightarrow 0 \quad (6)$$

when $R = R_j \rightarrow \infty$.

Next, we claim that the improper integral

$$I(\mathbb{R}^n) := \int_{\mathbb{R}^n} \frac{u(z)(1 - |u(z)|^2)(x - z) \cdot z}{|x - z|^{n-\alpha+2}} dz \quad (7)$$

absolutely converges for each $x \in \mathbb{R}^n$.

In fact, we observe that the defect points of $I(\mathbb{R}^n)$ are x and ∞ . When z is near ∞ , we have

$$|I(\mathbb{R}^n \setminus B_r)| \leq C \int_{\mathbb{R}^n \setminus B_r} \frac{|u(z)| dz}{|x - z|^{n-\alpha}} \leq C \left(\int_{\mathbb{R}^n} |u|^2 dz \right)^{\frac{1}{2}} \left(\int_r^\infty \rho^{n-2(n-\alpha)} \frac{d\rho}{\rho} \right)^{\frac{1}{2}}.$$

In view of $u \in L^2(\mathbb{R}^n)$ and $\alpha < n/2$, we get

$$|I(\mathbb{R}^n \setminus B_r)| < \infty. \quad (8)$$

When z is near x , we first take

$$s \in \left(\frac{n}{\alpha - 1}, \infty \right). \quad (9)$$

Clearly, $1 < \alpha < n/2$ implies $s > 2$. In addition,

$$u \in L^s(\mathbb{R}^n) \tag{10}$$

because u is bounded and $u \in L^2(\mathbb{R}^n)$. Note that

$$|I(B_\delta(x))| \leq C \int_{B_\delta(x)} \frac{|u(z)| dz}{|x-z|^{n-\alpha+1}} \leq C \left(\int_{\mathbb{R}^n} |u|^s dz \right)^{\frac{1}{s}} \left(\int_0^r \rho^{n-\frac{s}{s-1}(n-\alpha+1)} \frac{d\rho}{\rho} \right)^{1-\frac{1}{s}}.$$

By (9) and (10), we get

$$|I(B_\delta(x))| < \infty.$$

Combining this with (8), we prove that (7) is absolutely convergent.

Finally we prove that (4) is convergent. Integrating by parts yields

$$\begin{aligned} \int_{B_R} \frac{z \cdot \nabla [u(z)(1 - |u(z)|^2)]}{|x-z|^{n-\alpha}} dz &= R \int_{\partial B_R} \frac{u(z)(1 - |u(z)|^2)}{|x-z|^{n-\alpha}} ds - n \int_{B_R} \frac{u(z)(1 - |u(z)|^2)}{|x-z|^{n-\alpha}} dz \\ &\quad - (n-\alpha) \int_{B_R} \frac{u(z)(1 - |u(z)|^2)(x-z) \cdot z}{|x-z|^{n-\alpha+2}} dz. \end{aligned} \tag{11}$$

Letting $R = R_j \rightarrow \infty$ in (11) and using (3) and (6), we can see that

$$\int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(z)(1 - |u(z)|^2)]}{|x-z|^{n-\alpha}} dz = -nu(x) + (\alpha - n)I(\mathbb{R}^n),$$

and hence it is convergent at each $x \in \mathbb{R}^n$.

Step 2. Proof of Theorem 1. For any $\lambda > 0$, from (3) it follows

$$u(\lambda x) = \lambda^\alpha \int_{\mathbb{R}^n} \frac{u(\lambda z)(1 - |u(\lambda z)|^2)}{|x-z|^{n-\alpha}} dz.$$

Differentiating both sides with respect to λ yields

$$\begin{aligned} x \cdot \nabla u(\lambda x) &= \alpha \lambda^{\alpha-1} \int_{\mathbb{R}^n} \frac{u(\lambda z)(1 - |u(\lambda z)|^2)}{|x-z|^{n-\alpha}} dz \\ &\quad + \lambda^\alpha \int_{\mathbb{R}^n} \frac{(z \cdot \nabla u(\lambda z))(1 - |u(\lambda z)|^2) + u(\lambda z)[-2u(\lambda z)(z \cdot \nabla u(\lambda z))]}{|x-z|^{n-\alpha}} dz. \end{aligned}$$

Letting $\lambda = 1$ yields

$$x \cdot \nabla u(x) = \alpha u(x) + \int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(1 - |u|^2)]}{|x-z|^{n-\alpha}} dz. \tag{12}$$

Since u is bounded and $u \in L^2(\mathbb{R}^n)$, it follows that $u \in L^4(\mathbb{R}^n)$, and hence

$$R \int_{\partial B_R} |u|^4 ds \rightarrow 0 \tag{13}$$

for some $R = R_j \rightarrow \infty$. Thus, integrating by parts and using (5) and (13), we respectively obtain

$$\int_{\mathbb{R}^n} u(x)(x \cdot \nabla u(x)) dx = \frac{-n}{2} \int_{\mathbb{R}^n} |u(x)|^2 dx, \tag{14}$$

and

$$\int_{\mathbb{R}^n} u(x)|u(x)|^2(x \cdot \nabla u(x)) dx = \frac{-n}{4} \int_{\mathbb{R}^n} |u(x)|^4 dx. \tag{15}$$

These results show that

$$\int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2)(x \cdot \nabla u(x)) dx < \infty. \tag{16}$$

Multiply (12) by $u(x)(1 - |u(x)|^2)$ and integrate over B_R . Letting $R = R_j \rightarrow \infty$, from $u \in L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n)$ and (16), we get

$$\int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2) \int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(1 - |u|^2)]}{|x-z|^{n-\alpha}} dz dx < \infty,$$

and

$$\int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2)(x \cdot \nabla u(x)) dx - \alpha \int_{\mathbb{R}^n} |u(x)|^2(1 - |u(x)|^2) dx = \int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2) \int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(1 - |u|^2)]}{|x - z|^{n-\alpha}} dz dx. \tag{17}$$

We use the Fubini theorem and (3) to handle the right hand side term. Thus,

$$\begin{aligned} & \int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2) \int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(1 - |u|^2)]}{|x - z|^{n-\alpha}} dz dx \\ &= \int_{\mathbb{R}^n} z \cdot \nabla [u(1 - |u|^2)] \int_{\mathbb{R}^n} \frac{u(x)(1 - |u(x)|^2)}{|x - z|^{n-\alpha}} dx dz \\ &= \int_{\mathbb{R}^n} (x \cdot \nabla [u(1 - |u|^2)]) u(x) dx \\ &= \int_{\mathbb{R}^n} u(x)(1 - |u|^2)(x \cdot \nabla u(x)) dx - \int_{\mathbb{R}^n} |u|^2(x \cdot \nabla |u(x)|^2) dx. \end{aligned} \tag{18}$$

Inserting this result into (17), and using (15) we have

$$\int_{\mathbb{R}^n} [\alpha |u(x)|^2 + (\frac{n}{2} - \alpha) |u(x)|^4] dx = 0. \tag{19}$$

In view of $\alpha \in (1, n/2)$, (19) leads to $|u| \equiv 0$. Theorem 1 is proved. □

Remark 2. Clearly, (19) implies

$$\int_{\mathbb{R}^n} [\alpha (|u(x)|^2 - |u(x)|^4) + \frac{n}{2} |u(x)|^4] dx = 0. \tag{20}$$

When u satisfies (2), (20) also implies $|u| \equiv 0$. For (1), the bound $|u| \leq 1$ for solutions $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$ was first proved by Brezis (cf. [1]). Ma also pointed out that (2) holds true (cf. [6]).

Remark 3. In 2016, Ma [7] proved (2) for the Ginzburg–Landau-type equation with fractional Laplacian

$$(-\Delta)^{\frac{\alpha}{2}} u = (1 - |u|^2)u \quad \text{on } \mathbb{R}^n \tag{21}$$

under the assumption

$$1 - |u|^2 \in L^2(\mathbb{R}^n), \tag{22}$$

where $n \geq 2$ and $0 < \alpha < 2$. The physical background of (21) can be found in [9] and [11]. Such an equation with $\alpha = 1$ was well studied in [8]. Recall the definition of fractional Laplacian on \mathbb{R}^n . Let $n \geq 2$ and $0 < \alpha < 2$. Write

$$E = L_\alpha \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n),$$

where $L_\alpha = \{u \in L^1_{\text{loc}}(\mathbb{R}^n); \int_{\mathbb{R}^n} \frac{|u(x)| dx}{1 + |x|^{n+\alpha}} < \infty\}$. For a vector value function $u \in E$ from \mathbb{R}^n to \mathbb{R}^k , define

$$(-\Delta)^{\frac{\alpha}{2}} u := C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy = C_{n,\alpha} \lim_{\epsilon \rightarrow 0^+} \int_{|x-y| \geq \epsilon} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy. \tag{23}$$

Here $C_{n,\alpha}$ is a positive constant.

Clearly, (22) and $u \in L^2(\mathbb{R}^n)$ are incompatible.

Remark 4. Another definition of the fractional order Laplacian involves the Riesz potential (cf. [10, Chapter 5]). Assume $\alpha \in (0, n)$, and $u, u(1 - |u|^2) \in \mathcal{S}'(\mathbb{R}^n)$, then (21) can be explained as (3). In fact, (3) is equivalent to

$$\widehat{u}(\xi) = (|x|^{\alpha-n} * [u(1 - |u|^2)])^\wedge(\xi) = C|\xi|^{-\alpha} [u(1 - |u|^2)]^\wedge(\xi), \tag{24}$$

where C is a positive constant. By the property of the Riesz potential, we have

$$[(-\Delta)^{\alpha/2} u]^\wedge(\xi) = C|\xi|^\alpha \widehat{u}(\xi), \tag{25}$$

where C is another positive constant. Therefore, the above equality (24) amounts to (21). In addition, let $u \in E$ be a solution of (21) with $0 < \alpha < 2$. From (23), it follows that (25) is still true. If the Fourier inversion formula of (24) holds, then u also solves (3) (if we omit the constants).

Remark 5. If u is a finite energy solution of (1), then [2] shows that

$$\int_{\mathbb{R}^n} |u|^2(1 - |u|^2) dx < \infty. \quad (26)$$

Therefore, we sometimes call u a finite energy solution of (3) if u satisfies (26). Moreover, if u is uniformly continuous, we can see that either $u \in L^2(\mathbb{R}^n)$ or $1 - |u|^2 \in L^1(\mathbb{R}^n)$ by the same argument of (3.9) and (3.10) in [2]. Therefore, if a bounded, uniformly continuous, differentiable function u is a finite energy solution of (3), then either $u \equiv 0$, or $|u(x)| \rightarrow 1$ when $|x| \rightarrow \infty$.

Acknowledgements

The authors thank Prof. P. Mironescu and the unknown referee very much for useful suggestions.

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