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Picard-Hayman behavior of derivatives of meromorphic functions

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Abstract. Let \( f \) be a transcendental meromorphic function on \( \mathbb{C} \), and \( P(z), Q(z) \) be two polynomials with \( \deg P(z) \geq \deg Q(z) \). In this paper, we prove that: if \( f(z) = 0 \Rightarrow f'(z) = a \) (a nonzero constant), except possibly finitely many, then \( f'(z) - P(z)/Q(z) \) has infinitely many zeros. Our result extends or improves some previous related results due to Bergweiler–Pang, Pang–Nevo–Zalcman, Wang–Fang, and the author, et. al.

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1. Introduction and Main Results

In 1959, Hayman [2, 3] proved the following result, which has come to be known as Hayman’s alternative.

**Theorem 1.** Let \( f \) be a transcendental meromorphic function in the complex plane \( \mathbb{C} \), and let \( k \in \mathbb{N}, a \in \mathbb{C} \) and \( b \in \mathbb{C}\{0\} \). Then either \( f - a \) or \( f^{(k)} - b \) has infinitely many zeros.

Considering \( g = f - a \), it suffices to take \( a = 0 \) in Theorem 1.

In the past years, a number of improvements and extensions of Theorem 1 have been obtained. Wang and Fang [8] proved the following result.

**Theorem 2.** Let \( f \) be a transcendental meromorphic function in the complex plane \( \mathbb{C} \) and \( k \in \mathbb{N} \). If

(i) all zeros of \( f \) have multiplicity at least \( k + 1 \) and all poles of \( f \) are multiple, or

(ii) all zeros of \( f \) have multiplicity at least 3,

then, for each \( b \in \mathbb{C}\{0\} \), \( f^{(k)} - b \) has infinitely many zeros.

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For \( k = 1 \), Nevo, Pang and Zalcman [5] proved the following result, in which they use an argument involving quasinormal families.

**Theorem 3.** Let \( f \) be a transcendental meromorphic function in the complex plane \( \mathbb{C} \). If all zeros of \( f \) are multiple, then, for each \( b \in \mathbb{C} \setminus \{0\} \), \( f' - b \) has infinitely many zeros.

It is a natural to ask: Does the above results still hold if we replace the nonzero constant \( b \) by a small function function \( R(z) (\neq 0) \) of \( f \)?

By using the theory of normal families, Bergweiler and Pang [1] (for \( k = 1 \)), and the author [9] (for \( k \geq 2 \)) obtained

**Theorem 4.** Let \( f \) be a transcendental meromorphic function in the complex plane \( \mathbb{C} \), and \( R(\neq 0) \) be a rational function. If

(i) all zeros of \( f \) have multiplicity at least \( k + 1 \) and all poles of \( f \) are multiple, or

(ii) all zeros of \( f \) have multiplicity at least \( k + 2 \), except possibly finitely many,

then \( f^{(k)} - R \) has infinitely many zeros.

For the case \( k = 1 \), using the theory of quasinormal families, Pang, Nevo and Zalcman [6] proved the following stronger result.

**Theorem 5.** Let \( f \) be a transcendental meromorphic function in the complex plane \( \mathbb{C} \). If all zeros of \( f \) are multiple except possibly finitely many, then, for each rational function \( R \neq 0 \), \( f' - R \) has infinitely many zeros.

Clearly, the condition “all zeros of \( f \) are multiple” in Theorem 5 is equivalent to the condition “\( f = 0 \Rightarrow f'(z) = 0 \)”. We note that the function \( f(z) = e^z - a (a \in \mathbb{C}) \) satisfying “\( f = 0 \Rightarrow f'(z) = a \)”, and \( f'(z) - R(z) \) has infinitely many zeros for rational function \( R(z) \neq -a \).

Inspired by this observation, we prove the following result by using the theory of normal family.

**Theorem 6.** Let \( f \) be a transcendental meromorphic function on \( \mathbb{C} \), and \( a \in \mathbb{C} \). If \( f(z) = 0 \Rightarrow f'(z) = a \), except possibly finitely many, then \( f'(z) - R(z) \) has infinitely many zeros, where \( R(z) = P(z)Q(z)(\neq 0) \), and \( P(z), Q(z) \) are two polynomials with \( \text{deg} P(z) > \text{deg} Q(z) \).

**Remark.** At present, we are not clear whether the condition \( \text{deg} P(z) > \text{deg} Q(z) \) in Theorem 6 can be omitted.

**Corollary 7.** Let \( f \) be a transcendental meromorphic function on \( \mathbb{C} \), and \( a \in \mathbb{C} \). If \( f(z) = 0 \Rightarrow f'(z) = a \), except possibly finitely many, then for nonconstant polynomial \( P(z) \), \( f'(z) - P(z) \) has infinitely many zeros.

### 2. Some Lemmas

First we recall some definitions. If there exists a curve \( \Gamma \subset \mathbb{C} \) tending to \( \infty \) such that \( f(z) \rightarrow a \) as \( z \rightarrow \infty \) and \( z \in \Gamma \), we call that \( a \) is an asymptotic value of \( f \).

A meromorphic function \( f \) is called a Julia exceptional function if \( f^d(z) = O(1/|z|) \) as \( z \rightarrow \infty \). Here, as usual, \( f^d(z) = |f'(z)|/(1 + |f(z)|^2) \) is the spherical derivative of \( f \). It follows easily from the Ahlfors-Shimizu characteristic function that if \( f \) is a Julia exceptional function, then \( T(r,f) = O((\log r)^2) \) as \( r \rightarrow \infty \).

The following result is due to Lehto and Virtanen [4].

**Lemma 8.** A transcendental Julia exceptional function does not have an asymptotic value.

**Lemma 9 (see [1]).** Let \( f \) be a transcendental meromorphic function, and let \( R \) be a rational function satisfying \( R(z) \sim cz^d \) as \( z \rightarrow \infty \), with \( c \in \mathbb{C} \setminus \{0\} \) and \( d \in \mathbb{Z} \). Suppose that \( f' - R \) has only finitely many zeros and \( T(r,f) = O((\log r)^2) \) as \( r \rightarrow \infty \). Set \( g := f(z)/z^{d+1} \). Then \( g \) has an asymptotic value.
Lemma 10 (see [2, 3]). Let $f$ be a meromorphic function in the complex plane, and $k$ a positive integer. If $f \neq 0$ and $f^{(k)}(z) \neq 1$, then $f(z)$ is a constant.

The next is a local version of Zalcman’s lemma due to Pang and Zalcman [7].

Lemma 11. Let $k$ be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in a domain $D$, such that each function $f \in \mathcal{F}$ has only zeros of multiplicity at least $k$, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then, for each $0 \leq \alpha \leq k$, there exist a sequence of points $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^a} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k$, such that $g^\#(\zeta) \leq g^\#(0) = kA + 1$. Moreover, $g$ has order at most 2.

3. Proof of Theorem 6

By Theorem 5, we only need to consider the case $a \neq 0$.

Since $R(z) = P(z)/Q(z)(\neq 0)$, where $P(z)$ and $Q(z)$ are two polynomials with $\deg P(z) > \deg Q(z)$. We may assume that $R(z) \sim cz^d$ as $z \to \infty$, where $c \in \mathbb{C}\setminus\{0\}$ and $d$ is a positive integer. Define

$$g(z) := \frac{f(z)}{z^{d+1}}.$$

Suppose that $f' - R$ has only finitely many zeros. If $g$ is a Julia exceptional function, then $T(r, g) = O((\log r)^2)$ and hence $T(r, f) = O((\log r)^2)$ as $r \to \infty$. Lemma 9 implies that $g$ has an asymptotic value. But, by Lemma 8, $g$ has no asymptotic value, a contradiction.

Thus $g$ is not a Julia exceptional function. Hence, by the definition of Julia exceptional function, there exists $\{a_n\}$ such that $a_n \to \infty$ and $a_n g^\#(a_n) \to \infty$ as $n \to \infty$.

Let $D = \{z \in \mathbb{C} : |z - 1| < 1/2\}$, and set

$$\mathcal{G} = \{g_n(z) := g(a_n z)z^{d+1} = \frac{f(a_n z)}{a_n^{d+1}}, z \in D\}.$$

The family $\mathcal{G}$ is not normal at $z = 1$. Indeed, by computation, we have

$$g_n(1) = \frac{|a_n g'(a_n) + (d + 1) g(a_n)|}{1 + |g(a_n)|^2} \geq |a_n| g^\#(a_n) - \frac{|d+1|}{2} \to \infty$$

as $n \to \infty$. So, by Marty’s criterion, $\mathcal{F}$ is not normal at $z = 1$.

If $g_n(z) = 0$, that is, $f(a_n z) = 0$, then from the hypotheses of theorems, we have $f'(a_n z) = a$. Thus there exists $M \geq 1$ such that (for large $n$)

$$|g_n'(z)| = \left| \frac{f'(a_n z)}{a_n^d} \right| \leq \frac{a}{a_n^d} \leq M$$

whenever $g_n(z) = 0$. Then, applying Lemma 11, we can find $z_n \in D$, $z_n \to 1$, $\rho_n \to 0^+$, and $g_n \in \mathcal{G}$ such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n} \to \frac{f(a_n(z_n + \rho_n \zeta))}{\rho_n a_n^{d+1}} \to G(\zeta)$$

locally uniformly with respect to the spherical metric, where $\mathcal{G}$ is a nonconstant meromorphic function in $\mathbb{C}$.

Claim. $G(\zeta) \neq 0$ on $\mathbb{C}$.
Suppose that there exists a point \( \zeta_0 \) such that \( G(\zeta_0) = 0 \). Then, there exist \( \zeta_n, \zeta_n \to \zeta_0 \), such that \( G_n(\zeta_n) = 0 \) (for \( n \) sufficiently large) since \( G \) is not constant, and thus \( f(a_n(z_n + \rho_n\zeta_n)) = 0 \). By the assumption of theorem, we have \( f'(a_n(z_n + \rho_n\zeta_n)) = a \).

From (1), we have
\[
G_n'(\zeta) = g_n'(z_n + \rho_n\zeta) = \frac{f'(a_n(z_n + \rho_n\zeta))}{a_n^d} \to G'(\zeta),
\]
uniformly on compact subsets of \( \mathbb{C} \) disjoint from the poles of \( G \). It follows that \( G'(\zeta_0) = \lim_{n \to \infty} G_n'(\zeta_n) = \lim_{n \to \infty} a/a_n^d = 0 \). Thus, all zeros of \( G \) are multiple.

Let \( \zeta_0 \) be a zero of \( G \) with multiplicity \( m \geq 2 \), then \( G^{(m)}(\zeta_0) \neq 0 \). By (1) and Rouché’s theorem, there exist \( \delta > 0 \), and \( m \) sequences \( \{c_n^{(i)}\} (i = 1, 2, \ldots, m) \) on \( D_\delta(\zeta_0) = \{ \zeta : |\zeta - \zeta_0| < \delta \} \), tending to \( \zeta_0 \), such that
\[
G_n(c_n^{(i)}) = 0, \quad (i = 1, 2, \ldots, m).
\]
From (1), we have \( f(a_n(z_n + \rho_n\zeta_n^{(i)})) = 0 \) (i = 1, 2, m). It follows that \( f'(a_n(z_n + \rho_n\zeta_n^{(i)})) = a’\), and thus \( G_n'(\zeta_n^{(i)}) = a’/a_n^d \neq 0 \) (i = 1, 2, m). This means that each \( c_n^{(i)} \) is a simple zero of \( G_n \), which rules out the possibility that each two of \( c_n^{(i)} \) (i = 1, 2, m) might coincide. So, \( \zeta_n^{(i)} \) are \( m \) distinct zeros of \( G_n'(\zeta) - a/a_n^d \) on \( D_\delta(\zeta_0) \). Noting that \( \zeta_n^{(i)} \to \zeta_0 \) and
\[
G_n'(\zeta) - \frac{a}{a_n^d} \to G'(\zeta),
\]
Rouché’s theorem implies that \( \zeta_0 \) is the zero of \( G'(\zeta) \) with multiplicity at least \( m \). We get \( G^{(m)}(\zeta_0) = 0 \), a contradiction. We thus proved our claim.

Since \( R(z) \sim cz^d \) as \( z \to \infty \), by (2), we have
\[
G_n'(\zeta) - \frac{R(a_n(z_n + \rho_n\zeta))}{a_n^d}\to G'(\zeta) - c
\]
uniformly on compact subsets of \( \mathbb{C} \) disjoint from the poles of \( G \). On the other hand, for \( n \) sufficiently large
\[
G_n'(\zeta) - \frac{R(a_n(z_n + \rho_n\zeta))}{a_n^d} = f'(a_n(z_n + \rho_n\zeta)) - R(a_n(z_n + \rho_n\zeta)) \neq 0.
\]
By (3) and Hurwitz’s theorem, either \( G'(\zeta) \neq c \) or \( G'(\zeta) \equiv c \) on \( \mathbb{C} \setminus G^{-1}(\infty) \). Clearly, these also hold on \( \mathbb{C} \). If \( G' \equiv c \), then \( G \) is a polynomial with degree 1. This is impossible since \( G \neq 0 \). Hence \( G' \neq c \).

So, by Lemma 10, \( G(\zeta) \) must be a constant, a contradiction. This completes the proof of Theorem 6.

\[\Box\]

References