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Number Theory / *Théorie des nombres*

# Green's problem on additive complements of the squares

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**Abstract.** Let  $A$  and  $B$  be two subsets of the nonnegative integers. We call  $A$  and  $B$  additive complements if all sufficiently large integers  $n$  can be written as  $a + b$ , where  $a \in A$  and  $b \in B$ . Let  $S = \{1^2, 2^2, 3^2, \dots\}$  be the set of all square numbers. Ben Green was interested in the additive complement of  $S$ . He asked whether there is an additive complement  $B = \{b_n\}_{n=1}^\infty \subseteq \mathbb{N}$  which satisfies  $b_n = \frac{\pi^2}{16}n^2 + o(n^2)$ . Recently, Chen and Fang proved that if  $B$  is such an additive complement, then

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n^{1/2} \log n} \geq \sqrt{\frac{2}{\pi}} \frac{1}{\log 4}.$$

They further conjectured that

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n^{1/2} \log n} = +\infty.$$

In this paper, we confirm this conjecture by giving a much more stronger result, i.e.,

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n} \geq \frac{\pi}{4}.$$

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## 1. Introduction

Two subsets  $A$  and  $B$  of nonnegative integers are said to be additive complements if their sum

$$a + b \quad (a \in A, b \in B)$$

contains all sufficiently large integers. If  $A$  and  $B$  are additive complements, we also call  $B$  an additive complement of  $A$ . For any set  $L$  of nonnegative integers, let  $L(x)$  be the number of elements in  $L$  which are no greater than  $x$ . As usual,  $[x]$  and  $\{x\}$  denote the integral part and fractional part of  $x$  respectively.

Let  $S = \{1^2, 2^2, 3^2, \dots\}$  be the set of all square numbers. Given a positive integer  $N$ , let  $T = \{t_1, t_2, t_3, \dots, t_l\}$  be a subset of nonnegative integers such that every positive integer  $n \leq N$

can be represented by the sum of the elements of  $S$  and  $T$ . It is sure that  $l\sqrt{N} \geq N$ . In [6], Erdős asked whether there exists a positive constant  $c$  such that

$$l\sqrt{N} > (1 + c)N$$

for all sufficiently large  $N$ . It was answered affirmative by Moser [8] with  $c = 1.06$ . Later the constant was improved by Balasubramanian [1] to 1.15, by Balasubramanian and Soundararajan [3] to 1.245. The best result of the constant  $c$  up to now is  $\frac{4}{\pi}$  which was obtained by Cilleruelo [5], Habsieger [7], Balasubramanian and Ramana [2] respectively.

Based on the above rich literature, Ben Green posed a problem to Fang about the additive complements of the squares during her visit to the Mathematical Institute, University of Oxford in 2016 [4]. He asked whether there is an additive complement  $B = \{b_n\}_{n=1}^\infty \subseteq \mathbb{N}$  of  $S$  satisfies

$$b_n = \frac{\pi^2}{16}n^2 + o(n^2), \tag{1}$$

or equivalently

$$B(n) = \frac{4}{\pi}\sqrt{N} + o(\sqrt{N}).$$

Chen and Fang [4] investigated this problem. They proved that if for any  $0 < \alpha < \sqrt{\frac{2}{\pi}} \frac{1}{\log 4}$  and  $\gamma > 0$ , we have

$$b_n \geq \frac{\pi^2}{16}n^2 - \alpha n^{1/2} \log n - \gamma n^{1/2}, \quad n = 1, 2, 3, \dots,$$

then  $B$  is not an additive complement of  $S$ . From which they deduced that if  $B$  is an additive complement of  $S$ , then

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n^{1/2} \log n} \geq \sqrt{\frac{2}{\pi}} \frac{1}{\log 4}.$$

Motivated by this result, they made the following conjecture.

**Conjecture.** [4] *If  $B = \{b_n\}_{n=1}^\infty$  is an additive complement of  $S$ , then*

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n^{1/2} \log n} = +\infty.$$

We confirm this conjecture by establishing the following stronger result.

**Theorem 1.** *If  $B = \{b_n\}_{n=1}^\infty$  is an additive complement of  $S$ , then*

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n} \geq \frac{\pi}{4}.$$

## 2. Proof of the Theorem 1

**Proof.** Suppose that

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16}n^2 - b_n}{n} = \beta < \frac{\pi}{4}.$$

Then there exists a number  $n_1 > 0$  such that

$$\frac{\frac{\pi^2}{16}n^2 - b_n}{n} \leq \beta + \frac{1}{2} \left( \frac{\pi}{4} - \beta \right) = \frac{\pi}{4} - \delta < \frac{\pi}{4}$$

for all  $n > n_1$ , where

$$\delta = \frac{1}{2} \left( \frac{\pi}{4} - \beta \right).$$

That means

$$b_n \geq \frac{\pi^2}{16}n^2 - \left( \frac{\pi}{4} - \delta \right)n = \frac{\pi^2}{16} \left( n - \frac{2}{\pi} + \frac{8}{\pi^2} \delta \right)^2 - \left( \frac{1}{2} - \frac{2}{\pi} \delta \right)^2$$

for all  $n > n_1$ . Thus there exists an integer  $n_2 \geq n_1$ , such that

$$b_n \geq \frac{\pi^2}{16} \left( n - \frac{2}{\pi} + \frac{4}{\pi^2} \delta \right)^2$$

for all  $n > n_2$ . This implies that

$$B(n) \leq \frac{4}{\pi} \sqrt{n} + \frac{2}{\pi} - \delta_1$$

for all  $n > n_2$ , where  $\delta_1 = \frac{4}{\pi^2} \delta$  is a positive constant. Let  $R(n) = \#\{(m, b) : n = m^2 + b, b \in B\}$  be the representation function of  $n$ . For any positive integer  $N$ , we have

$$\begin{aligned} \sum_{n=1}^N R(n) &= \sum_{n^2+b \leq N} 1 \\ &= \sum_{n \leq \sqrt{N}} \sum_{b \leq N-n^2} 1 \\ &= \sum_{n \leq \sqrt{N}} B(N-n^2) \\ &\leq \sum_{n \leq \sqrt{N}} \left( \frac{4}{\pi} \sqrt{N-n^2} + \frac{2}{\pi} - \delta_1 \right) + O(1) \\ &= \frac{4}{\pi} \sum_{n \leq \sqrt{N}} \sqrt{N-n^2} + \left( \frac{2}{\pi} - \delta_1 \right) \sqrt{N} + O(1), \end{aligned} \tag{2}$$

where the implied constant depends only on  $n_2$ .

Now we consider the summation  $\sum_{n \leq \sqrt{N}} \sqrt{N-n^2}$ . For square integer  $N = K^2$ , Euler–Maclaurian formula with  $f(t) = \sqrt{N-t^2}$  shows that

$$\begin{aligned} \sum_{n \leq \sqrt{N}} \sqrt{N-n^2} &= \sum_{n=0}^K f(n) - f(0) \\ &= \frac{f(K) - f(0)}{2} + \int_0^K f(t) dt + \int_0^K f'(t) \left( \{t\} - \frac{1}{2} \right) dt \\ &= -\frac{K}{2} + \frac{\pi}{4} K^2 - \int_0^K \frac{t \left( \{t\} - \frac{1}{2} \right)}{\sqrt{K^2 - t^2}} dt \\ &= \frac{\pi}{4} N - \frac{\sqrt{N}}{2} - \sum_{k=0}^{K-1} \int_k^{k+1} \frac{t \left( \{t\} - \frac{1}{2} \right)}{\sqrt{K^2 - t^2}} dt. \end{aligned} \tag{3}$$

Note that  $\frac{t}{\sqrt{N-t^2}}$  is a monotone increasing function on  $[0, \sqrt{N}]$ . We have

$$\begin{aligned} \int_k^{k+1} \frac{t \left( \{t\} - \frac{1}{2} \right)}{\sqrt{K^2 - t^2}} dt &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\left( k + \frac{1}{2} + x \right) \left( \{k + \frac{1}{2} + x\} - \frac{1}{2} \right)}{\sqrt{K^2 - \left( k + \frac{1}{2} + x \right)^2}} dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x \left( k + \frac{1}{2} + x \right)}{\sqrt{K^2 - \left( k + \frac{1}{2} + x \right)^2}} dx \\ &= \int_0^{\frac{1}{2}} x \left( \frac{k + \frac{1}{2} + x}{\sqrt{K^2 - \left( k + \frac{1}{2} + x \right)^2}} - \frac{k + \frac{1}{2} - x}{\sqrt{K^2 - \left( k + \frac{1}{2} - x \right)^2}} \right) dx \geq 0. \end{aligned} \tag{4}$$

Combining (3) with (4) gives

$$\sum_{n \leq \sqrt{N}} \sqrt{N - n^2} \leq \frac{\pi}{4} N - \frac{\sqrt{N}}{2}. \quad (5)$$

Hence

$$\sum_{n=1}^N R(n) \leq N - \left( \frac{2}{\pi} - \left( \frac{2}{\pi} - \delta_1 \right) \right) \sqrt{N} + O(1) = N - \delta_1 \sqrt{N} + O(1) \quad (6)$$

for square integers  $N$  with  $N > n_2$ . Recall that  $B$  is an additive complement of the squares, so there is an integer  $n_3 > 0$  such that

$$R(n) \geq 1$$

for all  $n \geq n_3$ . It yields that

$$\sum_{n=1}^N R(n) \geq \sum_{n=n_3}^N R(n) \geq N - n_3.$$

This obviously contradicts to equation (6) when  $N$  is a sufficiently large square integer.  $\square$

**Remark 2.** As one can see that the idea in the proof of our Theorem 1 is simple but very effective. We use nothing but the trivial estimate on  $R(n)$ , i.e.,  $R(n) \geq 1$  for all sufficiently large integers. At the end of this short note, we formulate a conjecture similar to the one of Chen and Fang: If  $B = \{b_n\}_{n=1}^{\infty}$  is an additive complement of  $S$ , then

$$\limsup_{n \rightarrow \infty} \frac{\frac{\pi^2}{16} n^2 - b_n}{n} = +\infty.$$

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