INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Guillaume Pagel

On the modular Jones polynomial

Volume 358, issue 8 (2020), p. 901-908.

<https://doi.org/10.5802/crmath.106>

Comptes Rendus Mathématique
2020, 358, nº 8, p. 901-908
https://doi.org/10.5802/crmath.106

Knot Theory / *Théorie des nœuds*

On the modular Jones polynomial

Sur le polynôme de Jones modulaire

Guillaume Pagel[a]

[a] Univ. Littoral Côte d'Opale, UR 2597, LMPA, Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France
E-mail: guillaume.pagel@univ-littoral.fr

Abstract. A major problem in knot theory is to decide whether the Jones polynomial detects the unknot. In this paper we study a weaker related problem, namely whether the Jones polynomial reduced modulo an integer m detects the unknot. The answer is known to be negative for $m = 2^r$ with $r \geq 1$ and $m = 3$. Here we show that if the answer is negative for some m, then it is negative for m^r with any $r \geq 1$. In particular, for any $r \geq 1$, we construct nontrivial knots whose Jones polynomial is trivial modulo 3^r.

Résumé. Un problème majeur en théorie des nœuds est de décider si le polynôme de Jones détecte le nœud trivial. Dans cet article nous étudions une question similaire plus faible, c'est-à-dire si le polynôme de Jones réduit modulo un entier m détecte le nœud trivial. On sait que la réponse est négative pour $m = 2^r$ et $m = 3$. On montre ici que si cette affirmation est fausse pour un entier m, alors elle l'est aussi pour m^r avec $r \geq 1$. En particulier, on construit des nœuds non-triviaux avec un polynôme de Jones trivial modulo 3^r.

Keywords. Knot, Jones polynomial, Kauffman bracket, m-trivial knot, connected sum, Legendre formula, modular arithmetic.

Mots-clés. Nœud, Polynôme de Jones, crochet de Kauffman, nœud m-trivial, somme connexe, formule de Legendre, Arithmétique modulaire.

2020 Mathematics Subject Classification. 00X99.

Manuscript received 31st July 2020, revised 18th August 2020, accepted 20th August 2020.

Version française abrégée

Petite introduction

L'un des problèmes majeurs de la théorie des nœuds est de développer des méthodes pour déterminer le plus simplement possible si un nœud donné est isotope au nœud trivial ou non. L'une de ces méthodes est l'utilisation d'un invariant, l'un des plus connus étant le polynôme de Jones. Une question encore ouverte à l'heure actuelle est de savoir si celui-ci *détecte* le nœud trivial, c'est-à-dire que seul le nœud trivial ait un polynôme de Jones égal à 1.

Dans cet article, on propose d'étudier un problème proche, à savoir si il existe des nœuds non-triviaux dont le polynôme de Jones est trivial modulo un entier m, que l'on appellera nœuds m-triviaux par la suite.

Définition 1 (Nœud m-trivial). *On dit qu'un nœud non-trivial K est m-*trivial *si son polynôme de Jones $V(K)$ vérifie $V(K) \equiv 1[m]$.*

Dans le papier de S. Eliahou et J. Fromentin [3], on a une construction de nœuds premiers[1] m-triviaux pour m un entier s'écrivant comme une puissance de 2, ainsi que l'existence de nœuds 3-triviaux, mais pas d'informations quant aux autres entiers. Afin d'apporter quelques réponses à cette question, on propose de montrer le théorème suivant :

Théorème 2. *Si il existe un nœud m-trivial pour un certain $m \geq 2$, alors quel que soit $r \geq 1$ il existe une infinité de nœuds deux à deux distincts non-premiers m^r-triviaux.*

Dans la sous-section suivante, on donne les étapes clés de la démonstration constructive de ce théorème.

Résumé de la preuve

Pour arriver à ce résultat, on aura besoin de quelques propriétés du polynôme de Jones ainsi que des coefficients binômiaux. On commence par rappeler la définition d'un nœud :

Définition 3 (Nœud). *On appelle* nœud *l'image du cercle S^1 par un plongement dans $\mathbb{R}^3$ à déformation près. Le* nœud trivial *est donné par le plongement canonique.*

On peut voir une représentation d'un nœud non-trivial dans la figure 1, que nous appellerons γ par la suite. Cette définition peut être généralisée au plongement de plusieurs cercles, ce qui donnera un entrelac.

Lorsque l'on effectue une projection du nœud sur un plan, on crée un *diagramme* du nœud. A partir de celui-ci on peut calculer son polynôme de Jones via le *crochet de Kauffman* suivant le *modèle des états* introduit par L.H. Kauffman dans [6]. En résumé, il s'agit de modifier chaque croisement du nœud selon deux possibilités, ce qui donne deux nouveaux diagrammes pondérés chacun par un coefficient, appelés *états* du nœud. On peut résumer formellement ce modèle par trois règles (voir $(*)$ dans la section 2).

Définition 4 (Polynôme de Jones). *Pour K un nœud orienté, on définit le* polynôme de Jones *$V(K)$ appartenant à $\Lambda = \mathbb{Z}[t^{-1}, t]$ en utilisant le crochet de Kauffman comme :*

$$V(K) = \left((-\tau^3)^{-w(K)}\langle K\rangle\right)_{\tau = t^{-\frac{1}{4}}}$$

où $\langle\cdot\rangle$ désigne le crochet de Kauffman et $w(K)$ l'entortillement de K, défini comme la différence entre le nombre de croisements positifs et négatifs (voir les figures 2a et 2b).

Il faut préciser qu'en général le polynôme de Jones vit dans $\mathbb{Z}\left[\sqrt{t}, \sqrt{t^{-1}}\right]$, mais dans le cas des nœuds nous pouvons considérer Λ à la place [5, Theorem 2]. Cet invariant possède beaucoup de propriétés, ainsi le calcul du polynôme de Jones de la somme connexe de deux nœuds revient à une multiplication [5, Theorem 6] :

Proposition 5. *Pour K_1 et K_2 deux nœuds ayant comme polynôme de Jones V_1 et V_2 respectivement, le polynôme de Jones de la somme connexe $K_1 \# K_2$ est $V_1 V_2$.*

On utilisera la notation # pour la somme connexe, et on écrirera $\#(K, n)$ pour la somme connexe de n fois le nœud K. Cette propriété est une des deux clés pour montrer le théorème 2. On a à présent besoin de propriétés d'arithmétique modulaire.

[1] On dit d'un nœud qu'il est *premier* si il n'est pas trivial et si l'on ne peut pas l'écrire comme somme connexe de deux nœuds non-triviaux.

Proposition 6. *Pour $n, k \geq 2$ et $i \in [\![1, k-1]\!]$, la puissance n^{k-i} divise $\binom{n^{k-1}}{i}$.*

La preuve se base d'une part sur la formule de Legendre [1, Theorem 1.2.3 p. 6] (voir [7, XVI p. 8] pour l'original) et sur l'étude des p-valuations des coefficients binomiaux. Cette propriété donne alors le lemme suivant :

Lemme 7. *Soient P et Q deux polynômes à coefficients entiers tels que l'on ait $P = 1 + nQ$ pour un certain $n \in \mathbb{N}$. Alors pour tout $k \geq 1$ on a $P^{n^{k-1}} \equiv 1\,[n^k]$.*

La preuve du théorème 2 devient alors une formalité, il suffit de combiner la propriété 5 et le lemme 7 pour obtenir le résultat souhaité.

Conséquence

La conséquence principale de ce théorème est l'existence quel que soit r de nœuds non-premiers 2^r-triviaux et 3^r-triviaux, ce qui vient compléter les premières découvertes de S. Eliahou et J. Fromentin dans [3]. On peut voir en exemple le nœud $\#(\gamma, 3)$ qui est 9-trivial (figure 3).

Cependant on n'a aucune information sur d'autres modules. Découvrir un nœud 6-trivial serait en particulier très intéressant, étant à la fois 2-trivial et 3-trivial il pourrait permettre de déterminer si la propriété m-trivial est multiplicative par rapport à m.

Cette même propriété est définie sur le polynôme de Jones, on peut imaginer une définition similaire sur le crochet de Kauffman, donnant des résultats certainement plus forts et en lien avec ceux de cet article. Cette approche a déjà été utilisée dans [3], mais seulement pour des enchevêtrements algébriques.

Il reste également à déterminer le nombre minimal de croisements nécessaires pour qu'un nœud puisse être m-trivial, et de constater si ces nœuds m-triviaux « minimaux » sont premiers.

1. Introduction

One of the major aims of knot theory is to determine as simply as possible whether a given knot is isotopic to the unknot. The Jones polynomial is a knot invariant living in the ring of Laurent polynomials over the integers. A long-standing question is to determine whether the Jones polynomial can *detect* the unknot, meaning that the unknot is the only knot with Jones polynomial equal to 1. In case of links we know that this invariant does not detect the unlink with at least two components: this was proved first by M. Thistlethwaite [9] for links with 2 and 3 components, then generalised by S. Eliahou, L.H. Kauffman and M. Thistlethwaite [4], but leave unanswered the case of knots. However we know how to construct mutant knots that aren't distinguished by the Jones polynomial [8].

The idea here is to study the Jones polynomial in a modular way, in order to better understand structures formed by knots. Also thanks to the modulo operation, some of the coefficients of the polynomial will disappear, and sometimes the Jones polynomial modulo an integer m will become trivial. Nontrivial knots with this property will be called m-trivial.

Definition 1 (m-trivial knot). *We say that a nontrivial knot K is* m-trivial *if its Jones polynomial $V(K)$ satisfy $V(K) \equiv 1[m]$.*

A modular version of the Jones polynomial problem is then:

Problem 2. *Given any integer $m \geq 2$, do there exist m-trivial knots?*

For all $r \geq 1$, the existence of 2^r-trivial knots has been established by S. Eliahou and J. Fromentin in [3]. They also mention the existence of 3-trivial knots. Essentially nothing else is known, except that there are no 5-trivial knots up to 16 crossings.

In this paper we provide some answers to this problem, in fact we obtain the following theorem:

Theorem 3. *If there exists an m-trivial knot for some $m \geq 2$, then for all $r \geq 1$ there exists infinitely many pairwise distinct m^r-trivial knots.*

This result allows us to give a positive answer to Problem 2 for integers of the form 3^r, and gives a new proof for $m = 2^r$. Furthermore, the proof of Theorem 3 is constructive and gives an explicit way to obtain these knots. Let us remark here that the knots constructed in [3] are prime, which is not the case in this paper. To achieve that, we will need some results in modular arithmetic. Here is the structure of this paper: in Section 2, we recall some properties of the Jones polynomial, then in Section 3 we give a proof of Theorem 3. In Section 4, we prove some arithmetic results we used in the previous section. Finally in Section 5 we finish with some open questions.

2. Knots and the Jones polynomial

We start this section with a formal definition of a *knot*:

Definition 4 (Knot). *A* knot *is the image of an embedding of the circle S^1 into $\mathbb{R}^3$ up to deformation. The* unknot *is given by the canonical embedding.*

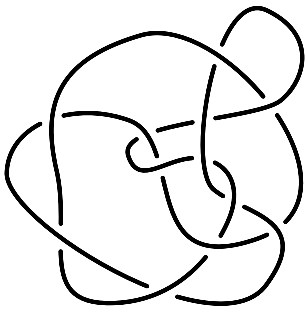

Figure 1. A representation of γ, a 3-trivial prime knot with 12 crossings [2, knot 12n659].
Une représentation de γ, un nœud 3-trivial premier avec 12 croisements [2, knot 12n659].

An example of a nontrivial knot can be seen in Figure 1. We can consider the embedding of a disjoint union of several circles, in that case we will obtain another object called *link*. The number of circles embedded gives the number of *components* of the link, knots are particular links with one component.

The Jones polynomial discovered by V.F.R. Jones [5] can be contructed via the *states model* introduced by L.H. Kauffman [6]. After we project the knot on a plane, the main idea is to split each crossing recursively. This operation create new *states* weighted with a coefficient on each step. More precisely, the *Kauffman bracket* follows the rules:

$$\begin{cases} \langle \bigcirc \rangle = 1, \\ \langle K \sqcup \bigcirc \rangle = -(\tau^{-2} + \tau^{2})\langle K \rangle, \\ \langle \times \rangle = \tau \langle)(\rangle + \tau^{-1} \langle \asymp \rangle. \end{cases} \qquad (*)$$

The second rule means that if we have a diagram isotopic to a circle next to an other diagram K without any crossing between them, then we can replace this circle by a coefficient. The third rule explains how to split each crossing locally, and the first one treats the case of the unknot. Once the bracket is computed, the only thing left to do is to normalize and change the variable.

Definition 5 (Jones polynomial). *For an oriented knot K, we can construct the* Jones polynomial *$V(K)$ living in $\Lambda = \mathbb{Z}[t, t^{-1}]$ using the Kauffman bracket as:*

$$V(K) = \left((-\tau^3)^{-w(K)}\langle K\rangle\right)_{\tau = t^{-\frac{1}{4}}}$$

where $\langle\cdot\rangle$ denote the Kauffman bracket and $w(K)$ is the writhe *of K, defined as the difference between the number of positive and negative crossings (see Figure 2).*

We have to specify that the Jones polynomial naturally lives in $\mathbb{Z}\left[\sqrt{t}, \sqrt{t^{-1}}\right]$, but in the case of knots we can consider the domain Λ instead [5, Theorem 2].

(a) Positive crossing
Croisement positif

(b) Negative crossing
Croisement négatif

Figure 2. Each crossing of an oriented knot can be identified to one of Figure 2a or 2b.
Tout croisement d'un nœud orienté peut être identifié à l'une des figures 2a ou 2b.

The computation by this method is simple but involves 2^l terms, with l the number of crossings.

Example 6. The Jones polynomial of the knot γ depictured in Figure 1 is:

$$V(\gamma) = 1 - 3t + 6t^2 - 9t^3 + 12t^4 - 12t^5 + 12t^6 - 9t^7 + 6t^8 - 3t^9.$$

We observe that this knot is 3-trivial.

The Jones polynomial has very interesting properties, we recall here a well known one [5, Theorem 6]:

Proposition 7. *For two knots K_1 and K_2, the Jones polynomial of the connected sum of K_1 and K_2 is V_1V_2 where V_1 and V_2 are the Jones polynomial of K_1 and K_2 respectively.*

The connected sum is the same as the topological one, i.e. we cut each knot in one point and glue the endpoints created on one knot to the other one without crossing. The end result is independent of the cutting points chosen. In the sequel, this operation will be denote by #, and the connected sum of n times the knot K will be denoted as $\#(K, n)$.

As a consequence of Proposition 7, the existence of p-trivial knots with p prime leads to the existence of p-trivial *prime* knots. We define this property below:

Definition 8 (Prime knot). *A knot is* prime *if it is not trivial and if it cannot be written as a connected sum of two non-trivial knots.*

For example, the knot γ represented in Figure 1 is prime. Now, we can properly state:

Proposition 9. *For p a prime number, if there exist a p-trivial knot, then there exist a p-trivial prime knot.*

Proof. Suppose that there exists a p-trivial non-prime knot K. We may assume $K = K_1\#K_2$ with K_1, K_2 both non-trivial knots and K_1 prime. As K is p-trivial, its Jones polynomial in the ring $\mathscr{R} = \mathbb{Z}/p\mathbb{Z}\left[\sqrt{t}, \sqrt{t}^{-1}\right]$ is $V(K) =_{\mathscr{R}} 1$. By Proposition 7, we have $V(K_1)\,V(K_2) =_{\mathscr{R}} 1$. As p is prime, the ring $\mathscr{R}$ is an integral domain, so we deduce that $V(K_1)$ and $V(K_2)$ are constant over $\mathscr{R}$. By [5, Theorem 15], we know that the Jones polynomial of any knot evaluated at $t = 1$ is 1. It follows here that $V(K_1) =_{\mathscr{R}} V(K_2) =_{\mathscr{R}} 1$. Hence K_1 is a p-trivial prime knot, as desired. □

3. Proof and consequences of Theorem 3

In this section we prove Theorem 3 and study its consequences. We use a lemma we will prove in the next section.

Lemma 10. *Let P and Q be two polynomials over the integers such that $P = 1 + nQ$ for some $n \geq 2$. Then we have:*

$$P \equiv 1\,[n]$$
$$P^{n^{k-1}} \equiv 1\left[n^k\right]$$

We can now prove Theorem 3 by construction:

Proof of Theorem 3. Let K be a m-trivial knot. We denote by $V(K) = 1+mP$ the Jones polynomial of K. Hence the connected sum of m^{r-1} times the knot K will be:

$$V\left(\#\left(K, m^{r-1}\right)\right) = (V(K))^{m^{r-1}} = (1+mP)^{m^{r-1}}$$

according to Proposition 7. By Lemma 10, the knot $\#\left(K, m^{r-1}\right)$ is m^r-trivial.

Also for $s \geq r$, m^s-trivial knots are in particular m^r-trivial knots, and as their Jones polynomial are different they are pairwise distinct. □

As an immediate consequence we have the following result:

Corollary 11. *For all integers $r \geq 2$ there exist 3^r-trivial and 2^r-trivial non-prime knots.*

Example 12. As we saw in example 6, the knot γ in Figure 1 is 3-trivial. We can construct a 9-trivial knot in the form of $\#(\gamma, 3)$ represented in Figure 3. Its Jones polynomial is:

$$-27t^{27} + 162t^{26} - 567t^{25} + \cdots - 41310t^{15} + 40257t^{14} + \cdots + 45t^2 - 9t + 1.$$

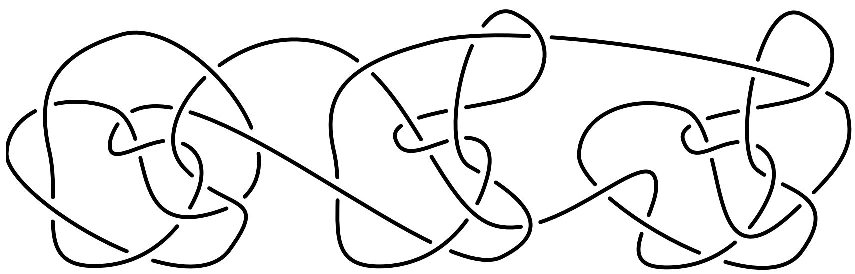

Figure 3. This figure represents the knot $\#(\gamma, 3)$. As γ (Figure 1) is 3-trivial, this one is 9-trivial.
Cette figure représente le nœud $\#(\gamma, 3)$. Comme γ (figure 1) est 3-trivial, celui-ci est 9-trivial.

At the time of writing, m-trivial knots are only known for 2^r or 3^r and $r \geq 1$. It remains an open problem to extend this result to other moduli m. The naive approach described in the next proposition shows that we cannot obtain a composite module directly by connected sum.

Proposition 13. *Let K_1 and K_2 be m_1-trivial and m_2-trivial knots respectively where $m_1 \neq m_2$. Then for all $n_1, n_2 \geq 1$, the connected sum of $\#(K_1, n_1)$ and $\#(K_2, n_2)$ is not $m_1 m_2$-trivial.*

Proof. See Proposition 19. □

The best composition we can obtain this way is the greatest common divisor of the moduli involved.

Corollary 14. *For $K_1, K_2, \ldots, K_n$ knots which are m_1-trivial, ..., m_n-trivial respectively and $k_1, k_2, \ldots, k_n \geq 1$, we have:*

$$V\left(\mathop{\#}_{i=1}^{n} \#\left(K_i, m_i^{k_i-1}\right)\right) \equiv 1 \left[\gcd\left(m_1^{k_1}, \ldots, m_n^{k_n}\right)\right]$$

4. Arithmetic properties

To reach Theorem 3, we used some arithmetic properties, in particular Lemma 10. The aim of this section is to prove this result.

Notation 15. For p a prime number, we denote by $v_p(n)$ the p-adic valuation of an integer n.

With this notation, the prime factor decomposition of an integer n is $n = \prod_{i=0}^{k} p_i^{v_{p_i}(n)}$. We start by showing the following:

Proposition 16. *For p a prime number and k a positive integer, the p-valuation of $k!$ is smaller than or equal to $k-1$.*

Proof. We denote by $s_p(k)$ the sum of the digits of k written in the base-p expansion. The alternate form of Legendre's formula [1, Theorem 1.2.3 p. 6] (see [7, XVI p. 8] for the original) immediately gives:

$$v_p(k!) = \frac{k - s_p(k)}{p-1}$$

It remains to establish the desired bound. As $k \geq 1$ we have $s_p(k) \geq 1$ and as $p > 1$ the expected result is obvious. □

The previous proposition allow us to establish the following divisibility result involving binomial coefficients at powers of n:

Proposition 17. *For $n, k \geq 2$ integers and $i \in [\![1, k-1]\!]$, we have that n^{k-i} divide $\binom{n^{k-1}}{i}$.*

Proof. For p a prime number dividing n, we study the p-valuation of $\frac{n^{k-1}}{\gcd(n^{k-1}, i!)}$. By Proposition 16, we know that $v_p(i!) \leq i-1 \leq v_p(n)(i-1)$ as $v_p(n) \geq 1$, thus:

$$v_p\left(\frac{n^{k-1}}{\gcd\left(n^{k-1}, i!\right)}\right) = v_p(n)(k-1) - v_p\left(\gcd\left(n^{k-1}, i!\right)\right) \geq v_p(n)(k-1) - v_p(i!) \geq v_p(n)(k-i)$$

We conclude that n^{k-i} divides $\frac{n^{k-1}}{\gcd(n^{k-1}, i!)}$, so it divides $\binom{n^{k-1}}{i}$ too. □

Proposition 17 has an interesting consequence on powers of specific polynomials. Let us prove now Lemma 10:

Proof of Lemma 10. The case $n \leq 1$ is trivial. Assuming $n \geq 2$, we develop the product:

$$P^{n^{k-1}} = (1+nQ)^{n^{k-1}} = \sum_{i=0}^{n^{k-1}} \left[\binom{n^{k-1}}{i}(nQ)^i\right] = 1 + \sum_{i=1}^{k-1}\left[\binom{n^{k-1}}{i}(nQ)^i\right] + n^k R_0$$

with R_0 a remainder polynomial. The only thing left to do is to take enough power of n from the combinatorial coefficient to have a factor n^k appear. However, by Proposition 17, we know that $\binom{n^{k-1}}{i}$ is divisible by n^{k-i} for i in $[\![1, k-1]\!]$, so:

$$1 + \sum_{i=1}^{k-1}\left[\binom{n^{k-1}}{i}(nQ)^j\right] + n^k R_0 = 1 + \sum_{i=1}^{k-1}\left[n^k R_i\right] + n^k R_0 \equiv 1\left[n^k\right]$$

where R_i are some polynomials. □

If we take for example a polynomial of the form $P = 1+3Q$, a direct computation gives $P^3 = 1+9Q+9Q^2+27Q^3$ which is congruent to 1 modulo 9.

Remark 18. We can generalize Lemma 10 for any ring R. In fact for any $A = I + nB$ living in R, since the neutral element I commutes with all elements, the binomial expansion works even if this ring is not commutative.

The following proposition explains why we can't generalize directly the proof of Theorem 3.

Proposition 19. *Let P_1 and P_2 be two polynomials of the form $P_1 = 1+n_1Q_1$, $P_2 = 1+n_2Q_2$ where Q_1 and Q_2 are some polynomials with at least two coprime coefficients and n_1, n_2 are different integers. Then:*

$$\forall\ a,b \geq 0, P_1^a P_2^b \not\equiv 1[n_1n_2]$$

Proof. We expand using the binomial formula:

$$\begin{aligned}
P_1^a P_2^b &= \left(\sum_{i=0}^{a}\binom{a}{i}(n_1Q_1)^i\right)\left(\sum_{j=0}^{b}\binom{b}{j}(n_2Q_2)^b\right)\\
&= 1+\underbrace{\sum_{i=1}^{a}\binom{a}{i}(n_1Q_1)^i}_{=n_1^aQ_1^a+R_1}+\underbrace{\sum_{j=1}^{b}\binom{b}{j}(n_2Q_2)^j}_{=n_2^bQ_2^b+R_2}+\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}\binom{a}{i}\binom{b}{j}(n_1Q_1)^i(n_2Q_2)^j}_{=n_1n_2R}\\
&= 1+n_1^aQ_1^a+n_2^bQ_2^b+R_1+R_2+n_1n_2R
\end{aligned}$$

Here R_1, R_2 and R are some polynomials. We clearly see that the terms $n_1^aQ_1^a$ and $n_2^bQ_2^b$ are not 0 modulo n_1n_2, hence $P_1^aP_2^b$ is not equivalent to 1 modulo n_1n_2. □

5. Further work

The main consequence of Theorem 3 is the existence for all r of 2^r-trivial and 3^r-trivial non-prime knots. However, we do not have any information on other moduli. It will be interesting to find a 6-trivial one, being 2-trivial and 3-trivial at the same time it may help to determine if the m-trivial property is multiplicative.

Although this property is defined on the Jones polynomial, we can imagine a similar definition on the Kauffman bracket. This might yield stronger results, surely linked with the one on the Jones polynomial. This approach was already used in [3] but with algebraic tangles only.

Another problem is determining the minimal number of crossing needed for a knot to be m-trivial, and also whether these "minimal" m-trivial knots are prime when m is composite.

References

[1] G. Boros, V. Moll, *Irresistible integrals: symbolics, analysis and experiments in the evaluation of integrals*, Cambridge University Press, 2004.

[2] J. C. Cha, C. Livingston, "Knotinfo: Table of knot invariants", 2011, https://knotinfo.math.indiana.edu/.

[3] S. Eliahou, J. Fromentin, "A remarkable 20-crossing tangle", *J. Knot Theory Ramifications* **26** (2017), no. 14, p. 1750091.

[4] S. Eliahou, L. H. Kauffman, M. Thistlethwaite, "Infinite families of links with trivial Jones polynomial", *Topology* **42** (2003), no. 1, p. 155-169.

[5] V. F. R. Jones, "A polynomial invariant for knots via von Neumann algebras", *Bull. Am. Math. Soc.* **12** (1985), no. 1, p. 103-112.

[6] L. H. Kauffman, "State models and the Jones polynomial", in *New developments in the theory of knots*, Advanced Series in Mathematical Physics, vol. 11, World Scientific, 1990, p. 162-174.

[7] A.-M. Legendre, *Essai sur la théorie des nombres*, Courcier, 1808.

[8] H. R. Morton, P. Traczyk, "The Jones polynomial of satellite links around mutants", in *New developments in the theory of knots*, Advanced Series in Mathematical Physics, vol. 11, World Scientific, 1990, p. 625-630.

[9] M. Thistlethwaite, "Links with trivial Jones polynomial", *J. Knot Theory Ramifications* **10** (2001), no. 4, p. 641-643.