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**Contact unimodal map germs from the plane to the plane**


<https://doi.org/10.5802/crmath.114>
Singularity Theory / Théorie des singularités

Contact unimodal map germs from the plane to the plane

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\textbf{Abstract.} In this article, we correct the classification of unimodal map germs from the plane to the plane of Boardman symbol (2, 2) given by Dimca and Gibson. Also, we characterize this classification of unimodal map germs in terms of certain invariants. Moreover, on the basis of this characterization we present an algorithm to compute the type of unimodal map germs of the Boardman symbol (2, 2) without computing the normal form and give its implementation in the computer algebra system \textsc{Singular} \cite{2010MathSoft}.  

\textbf{2020 Mathematics Subject Classification.} 58Q05, 14H20. 

\textbf{Funding.} The research of the first author is supported by Higher Education Commission, Pakistan by the Project Number 7495/Punjab/NRPU/R&D/HEC/2017.

\textit{Manuscript received 21st July 2020, accepted 8th September 2020.}

1. Introduction

After the classical work of Whitney \cite{14} the singularities of maps from plane to the plane became a subject of great interest due to its nice visual interpretations. Different authors concentrated to study the degenerate singularities of low codimension contained in families of maps from the plane to the plane \cite{2, 6, 7}. Rieger gave the classification of all \(\mathcal{A}\)-simple and \(\mathcal{A}\)-unimodal map germs from the plane to the plane of corank at most 1 in \cite{12, 13}. These classifications are characterized in \cite{3, 5, 11} in terms of certain invariants. In \cite{9}, Dimca and Gibson gave the classification of all map germs from the plane to the plane of Boardman symbol (2, 1) and (2, 2) with respect to \(\mathcal{K}\)-equivalence. A classifier for the classification of Dimca and Gibson, in the case of Boardman symbol (2, 1) is given in \cite{4}.
Let \( C \) be the field of complex numbers and \( A(2, 2) = (x, y)C[[x, y]]^2 \). Let \( \mathcal{K} = \text{Aut}_C (C[[x, y]]) \times \text{Gl}_2(C[[x, y]]) \) acting on \( A(2, 2) \) by

\[
((\varphi, M), f) \mapsto \varphi^{-1} \circ Mf.
\]

This is equivalent to consider \( A(2, 2) \) as the set of map germs \((C^2, 0) \rightarrow (C^2, 0)\) under the action of the group \( \mathcal{K} = \text{Aut}_C (C^2, 0) \times \text{Gl}_2(C^2, 0) \).

The map germs \( f, g \in A(2, 2) \) are called \( \mathcal{K} \)-equivalent \( (f \sim \mathcal{K} g) \) if they are in the same orbit under the action of \( \mathcal{K} \). In the classification of map germs with respect to the action of the group \( \mathcal{K} \) the tangent spaces to the orbit under the action of this group and their codimension play an important role (cf. [1]). Given \( f \in A(2, 2) \) the orbit map \( \theta_f : \mathcal{K} \rightarrow A(2, 2) \) is defined by \( \theta_f(\varphi, M) = \varphi^{-1} \circ Mf \). The corresponding tangent map has as image the tangent space to the orbit at \( f = (f_1, f_2) \):

\[
T_{\theta_f, id} = \langle \langle x, y \rangle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{C[[x, y]]} + (f_1, f_2)C[[x, y]]^2.
\]

In this article we characterize the classification of maps from the plane to the plane of Boardman symbol \((2, 2)\) given by Dimca and Gibson, in terms of invariants and on the basis of this characterization we give the implementation of this classification in the computer algebra system Singular [8, 10].

We use the following invariants for our characterization:

Let \( f \in A(2, 2) \) be a map germ then the codimension of the tangent space, \( c \equiv \dim_C \frac{\langle x, y \rangle C[[x, y]]}{I_{\theta_f, id}} \), a number closely connected with the order of \( k \)-determinacy, \( \sigma = \min \{ s : \langle x, y \rangle^s C[[x, y]]^2 \subseteq T_{\theta_f, id} \} \), the Milnor number, \( \mu = \dim_C \frac{C[[x, y]]}{f} - 1 \) and the double fold number \( d \), which is the number of times 2 occurs in the sequence \( \dim \frac{\langle x, y \rangle^k}{I_{\theta_f, id} \cap \langle x, y \rangle^s} \), \( k = 1, 2, \ldots \), are the invariants used in the classification. An other important invariant is Boardman symbol \( \Sigma f \), which is define as follows:

**Definition 1.** Let \( I = (f_1, \ldots, f_p) \) be an ideal in \( C[[x_1, \ldots, x_n]] \). Let the \( s \)-th Jacobian extension of \( I \) to be the ideal \( \Delta^s(I) = I + I_1 \), where \( I_1 \) is the ideal in \( C[[x_1, \ldots, x_n]] \) generated by all \( n - s + 1 \times n - s + 1 \) minors of the Jacobian matrix \( \left( \frac{\partial f_i}{\partial x_j} \right)_{i=1,\ldots,p} \). Then one has a sequence of inclusions

\[
I = \Delta^0(I) \subset \Delta^1(I) \subset \cdots \subset \Delta^n(I).
\]

If \( I \) is a proper ideal then the critical Jacobian extension of \( I \) is the last ideal \( \Delta^1(I) \) in the sequence which is proper. This ideal \( \Delta^1(I) \) has in turn its critical Jacobian extension \( \Delta^2(\Delta^1(I)) \) and so on. The sequence of integers \((i_1, i_2, \ldots)\) obtained in this way is called Boardman symbol of the ideal \( I \).

**Definition 2.** The Boardman symbol of a map germ \( f : (C^n, 0) \rightarrow (C^p, 0) \) is the Boardman symbol of the ideal \( I_f \), generated by the components \( f_1, f_2, \ldots, f_p \) of \( f \).

### 2. Classification of unimodal map germs from the plane to the plane

Table 1 contains all map germs from the plane to the plane of Boardman symbol \((2, 2)\) given by Dimca and Gibson [9].

In the following proposition, we correct the classification of Dimca and Gibson by showing that \( Y_1 \) is contact equivalent to \( Y_2 \), \( Z_0 \) is contact equivalent to \( Z_2 \) and \( Q_4 \) is contact equivalent to \( X_1 \).

**Proposition 3.** The following hold:

1. \((x^3 + y^5 + xy^3, x^2 y + y^4) \sim_{\mathcal{K}} (x^3 + y^5, x^2 y + y^5)\).
2. \((x^3 + 12xy^3, x^2 y + y^4) \sim_{\mathcal{K}} (x^3 + 12xy^3 + y^6, x^2 y + y^4)\).
3. \((x^3 + xy^3 + y^4, x^2 y + y^4) \sim_{\mathcal{K}} (x^3 + y^4, x^2 y + y^4)\).
Table 1.

<table>
<thead>
<tr>
<th>Type</th>
<th>Normal form</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{H}$</td>
<td>$(x^3 + 3x^2 y, 3xy^2 + \lambda y^3)$</td>
<td>$\lambda \neq 0, 1, 9$</td>
</tr>
<tr>
<td>$I$</td>
<td>$(x^3, xy^2 + y^3)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\mathfrak{S}$</td>
<td>$(x^3, y^3)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$K_p$</td>
<td>$(x^3 + x^2 y + y^p, xy^2)$</td>
<td>$p \geq 4$</td>
</tr>
<tr>
<td>$L_{p,q}$</td>
<td>$(x^2 y + y^p, xy^2 + x^q)$</td>
<td>$p \geq q \geq 4$</td>
</tr>
<tr>
<td>$M_p$</td>
<td>$(x^3 + y^p, xy^2)$</td>
<td>$p \geq 4$</td>
</tr>
<tr>
<td>$N$</td>
<td>$(x^3 + \lambda xy^3, x^2 y + y^4)$</td>
<td>$\lambda \neq 1, 12$</td>
</tr>
<tr>
<td>$P_p$</td>
<td>$(x^3 + xy^3 + xy^p, x^2 y + y^4)$</td>
<td>$p \geq 4$</td>
</tr>
<tr>
<td>$Q_p$</td>
<td>$(x^3 + xy^3 + y^p, x^2 y + y^4)$</td>
<td>$p \geq 4$</td>
</tr>
<tr>
<td>$R_p$</td>
<td>$(x^3 + xy^3, x^2 y + y^p)$</td>
<td>$p \geq 5$</td>
</tr>
<tr>
<td>$X_0$</td>
<td>$(x^3 + y^3, x^2 y)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$X_1$</td>
<td>$(x^3 + y^4, x^2 y + y^4)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>$(x^3 + y^5, x^2 y)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>$(x^3 + y^5 + xy^4, x^2 y)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>$(x^3 + y^5, x^2 y + y^4)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>$(x^3 + 12xy^3, x^2 y + y^4)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>$(x^3 + 12xy^3 + y^3, x^2 y + y^4)$</td>
<td>$-$</td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$(x^3 + 12xy^3 + y^3, x^2 y + y^4)$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

**Proof.** (1). Let $f = (x^3 + y^5 + xy^4, x^2 y)$ and $g = (x^3 + y^5, x^2 y + y^4)$. Note that $I_g = (x^3 + y^5, x^2 y - x^3)$. Then the transformation

$$x \rightarrow x, \quad y \rightarrow x + y,$$

transform $I_g$ into $(x^3 + y^5 + 5xy^4 + x^5, x^2 y)$. Since

$$x^3 + y^5 + 5xy^4 + x^5 - x^2(x^3 + y^5 + 5xy^4 + x^5) = x^3 + y^5 + 5xy^4 - x^7,$$

therefore $I_g \sim_{\mathcal{K}} (x^3 + y^5 + 5xy^4 - x^7, x^2 y)$. Iterating in a similar way, we get

$$I_g \sim_{\mathcal{K}} (x^3 + y^5 + 5xy^4, x^2 y).$$

Applying $\mathbb{C}^*$-action on $(x^3 + y^5 + 5xy^4, x^2 y)$, we get

$$I_g \sim_{\mathcal{K}} (x^3 + y^5 + xy^4, x^2 y) = I_f.$$

This gives $f \sim_{\mathcal{K}} g$.

(2). Let $f = (x^3 + 12xy^3, x^2 y + y^4)$ and $g = (x^3 + 12xy^3 + y^3, x^2 y + y^4)$. We find $g$ in the deformation $f + t(y^6, 0)$ of $f$. Computing the module of first order infinitesimal deformations and the tangent space of $f$ of the orbit of $f$ under the action of the contact group.

$$T^1 = \mathbb{C}[[x, y]]^2 \left\langle (x^3 + 12xy^3, 0), (0, x^3 + 12xy^3), (x^2 y + y^4, 0), (0, x^2 y + y^4), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

_C. R. Mathématique, 2020, 358, n° 8, 923-930_
We find that \((y^6,0)\) is zero in \(T^1\), mainly it is easy to check that
\[
(y^6,0) = \frac{1}{9} y^3 \frac{\partial f}{\partial x} - \frac{2}{99} y (x^3 + 12x y^3) - \frac{1}{3} y^2 (x^2 y + y^4,0) + \frac{2}{99} x (0, x^2 y + y^4);
\]
\[
- \frac{1}{9} y^3 \frac{\partial f}{\partial x} + f \left( - \frac{2}{99} y, - \frac{2}{99} x \right).
\]

Define \(\varphi: \mathbb{C}[[x,y]] \rightarrow \mathbb{C}[[x,y]]\) by \(\varphi(x) = x + \frac{1}{5} y^3\), \(\varphi(y) = y \) and \(M = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) - \left( \begin{array}{c} 0 \\ -\frac{2}{99} y \end{array} \right) \)
then
\[
\varphi(f) = (f + (y^6,0))M + \left( \begin{array}{c} \frac{1}{729} y^9 + \frac{1}{27} x y^6, -\frac{7}{891} y^7 \end{array} \right).
\]

Note that \(f\) is homogeneous of degree 9 with respect to \(\deg((x) = 3, \deg((y) = 2, \deg((1,0) = 0 \) and \(\deg((0,1) = 1\). Using this, we obtain \(\varphi(f) = g M + (\frac{1}{27} x y^6, -\frac{7}{891} y^7) + \) terms of higher degree. Since all homogeneous elements of degree \(\geq 12 = \deg(y^6,0)\) are zero in \(T^1\) and \(\deg((\frac{1}{27} x y^6, -\frac{7}{891} y^7)) = 15\), we can repeat this procedure. Assume we have an automorphism \(\varphi_k : \mathbb{C}[[x,y]] \rightarrow \mathbb{C}[[x,y]]\), a matrix \(M_k \in \text{GL}_2(\mathbb{C}[[x,y]])\) such that \(\varphi_k(f) = g M_k + u_k + \) terms of higher degree, where \(u_k\) is homogeneous and \(\deg((u_k) = k \geq 15\), then we can write
\[
u_k = a_k \frac{\partial f}{\partial x} + b_k \frac{\partial f}{\partial y} + c_k (x^3 + 12x y^3,0) + d_k (0,x^3 + 12x y^3) + e_k (x^2 y + y^4,0) + f_k (0, x^2 y + y^4)
\]
\[
= a_k \frac{\partial f}{\partial x} + b_k \frac{\partial f}{\partial y} + f \left( \begin{array}{c} c_k \\ d_k \\ e_k \\ f_k \end{array} \right),
\]

where \(a_k,b_k,c_k,d_k,e_k,f_k\) are homogeneous and \(\deg((a_k) = k - 6, \deg((b_k) = k - 7, \deg((c_k) = \deg((d_k) = \deg((e_k) = \deg((f_k) = k - 9\). We define \(\varphi_{k+1}(x) = \varphi_k(x) - a_k, \varphi_{k+1}(y) = \varphi_k(y) - b_k, M_{k+1} = M_k - \left( \begin{array}{c} c_k \\ d_k \\ e_k \\ f_k \end{array} \right) \) and obtain \(\varphi_{k+1}(f) = g M_{k+1} + \) terms of degree \(\geq k + 1\). Induction implies that \(f \sim_k g\).

(3). The following SINGULAR computation shows that \(Q_4\) is contact equivalent to \(X_1\).

\[
\text{ring } R=0,(x,y),\text{ds};
\]
\[
\text{poly } p=x^3+xy^3+y^4;
\]
\[
\text{poly } q=x^2y+y^4;
\]
\[
\text{poly } p1=x^3+y^4;
\]
\[
\text{poly } q1=x^2y+y^4;
\]
\[
\text{matrix } G[2][1]=p1,q1;
\]
\[
\text{matrix } T[2][2]=16777216/244140625,301989888/6103515625*y, -4194304/244140625,4194304/48828125;
\]
\[
\text{map } phi=R,256/390625*x^2-2048/234375*x*y+2048/15625*y^2+256/625*x, -128/390625*x^2+1664/234375*x*y-512/46875*y^2-64/625*x+64/125*y;
\]
\[
\text{matrix } N=T*G;
\]
\[
\text{poly } p2=\text{jet}(phi(p)-N[1,1],4);
\]
\[
\text{poly } q2=\text{jet}(phi(q)-N[2,1],4);
\]
\[
p2;q2;
\]
\[
0
\]
\[
0
\]
The computation shows that \((x^3 + y^4, x^2 y + y^4)\), denoted by \(X_1\) is contact equivalent to \(\mathcal{G} = (x^3 + x^3 y^3 + y^4, x^2 y + y^4)\) terms of order \(\geq 5\). \(\mathcal{G}\) has 4-jet \((x^3 + x^3 y^3 + y^4, x^2 y + y^4)\), denoted by \(Q_4\). In the paper of Dimca and Gibson [9] it is proved (Lemma 3.6) that \(\mathcal{G}\) is contact equivalent to its 4-jet \((x^3 + x^3 y^3 + y^4, x^2 y + y^4)\). This implies that \(X_1\) is contact equivalent to \(Q_4\).

Table 2 contains the correct classification of unimodal map germs from the plane to the plane of Boardman symbol \((2, 2)\) given by Dimca and Gibson.

\[
\begin{array}{|c|c|c|}
\hline
\text{Type} & \text{Normal form} & \text{Conditions} \\
\hline
H & (x^3 + 3x^2 y, 3xy^2 + \lambda y^3) & \lambda \neq 0, 1, 9 \\
I & (x^3, xy^2 + y^3) & - \\
\mathcal{G} & (x^3, y^3) & - \\
K_p & (x^3 + x^2 y + y^p, xy^2) & p \geq 4 \\
L_{p,q} & (x^2 y + y^p, xy^2 + x^q) & p \geq q \geq 4 \\
M_p & (x^3 + y^p, xy^2) & p \geq 4 \\
N & (x^3 + \lambda xy^2, x^2 y + y^4) & \lambda \neq 1, 12 \\
P_p & (x^3 + xy^3 + xy^p, x^2 y + y^4) & p \geq 4 \\
Q_p & (x^3 + xy^3 + y^p, x^2 y + y^4) & p \geq 4 \\
R_p & (x^3 + xy^3, x^2 y + y^p) & p \geq 5 \\
X_0 & (x^3 + y^4, x^2 y) & - \\
Y_0 & (x^3 + y^3, x^2 y) & - \\
Y_1 & (x^3 + y^5 + xy^4, x^2 y) & - \\
Z_0 & (x^3 + 12xy^3 + x^2 y + y^4) & - \\
Z_1 & (x^3 + 12xy^3 + y^3, x^2 y + y^4) & - \\
\hline
\end{array}
\]

3. Characterization of Unimodal Map Germs by Invariants

In this section we characterize all unimodal map germs from the plane to the plane of Boardman symbol \((2, 2)\) in terms of certain invariants.

**Proposition 4.** Let \(f(x, y)\) be a map germ from \((C^2, 0) \to (C^2, 0)\) such that \(\sum f = (2, 2)\) and \(H_p\) denote the Hilbert polynomial of \(j^3(f)\). Then

1. if \(H_p = 0\) then \(f\) is unimodal of type \(\bar{H}\) or \(I\) or \(\mathcal{G}\); 
2. if \(H_p = 1\) then \(f\) is unimodal of type \(K_p\) or \(M_p\); 
3. if \(H_p = 2\) then \(f\) is unimodal of type \(L_{p,q}\) or \(f\) is one of the type with \(j^3(f) = (x^3, x^2 y)\); 
4. if \(H_p \neq 0, 1, 2\) then contact modality of \(f\) is \(\geq 2\).

**Proof.** From the classification of the pencils of binary cubic forms and the classification of Dimca and Gibson [9] it follows that \(f\) is unimodal if and only if \(H_p \in \{0, 1, 2\}\).

(1). If \(H_p = 0\) then we obtain that \(\bar{H}, I\) and \(\mathcal{G}\) have these properties. Now if \(c = 11\) and \(\sqrt{\text{det}(J(J^3(f)))}\) is the intersection of a smooth curve and \(D_4\)-singularity then \(f\) is of type \(H\) and if \(c = 11\) and \(\sqrt{\text{det}(J(J^3(f)))}\) is the intersection of a smooth curve and \(A_1\)-singularity then \(f\) is of type \(I\) and if \(c = 12\) and \(\sqrt{\text{det}(J(J^3(f)))}\) is the intersection of two smooth curves then \(f\) is of type \(\mathcal{G}\).
Table 3. The Invariants used for the Classification.

<table>
<thead>
<tr>
<th>Type</th>
<th>$c$</th>
<th>$\mu$</th>
<th>$d$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{H}$</td>
<td>11</td>
<td>8</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$I$</td>
<td>11</td>
<td>8</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>12</td>
<td>8</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$K_p$</td>
<td>$p + 7$</td>
<td>$p + 5$</td>
<td>1</td>
<td>$p$</td>
</tr>
<tr>
<td>$L_{p,q}$</td>
<td>$p + q + 4$</td>
<td>$p + q + 2$</td>
<td>$q - 2$</td>
<td>$p$</td>
</tr>
<tr>
<td>$M_p$</td>
<td>$p + 8$</td>
<td>$p + 5$</td>
<td>1</td>
<td>$p$</td>
</tr>
<tr>
<td>$N$</td>
<td>15</td>
<td>11</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$P_p$</td>
<td>$2p + 8$</td>
<td>$2p + 5$</td>
<td>$p - 1$</td>
<td>$p + 2$</td>
</tr>
<tr>
<td>$Q_p$</td>
<td>$2p + 5$</td>
<td>$2p + 2$</td>
<td>$p - 2$</td>
<td>$p$</td>
</tr>
<tr>
<td>$R_p$</td>
<td>$p + 10$</td>
<td>$p + 7$</td>
<td>2</td>
<td>$p$</td>
</tr>
<tr>
<td>$X_0$</td>
<td>14</td>
<td>10</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>17</td>
<td>12</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>16</td>
<td>12</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>16</td>
<td>11</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>15</td>
<td>11</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

(2). If $H_p = 1$ then we obtain that $K_p$ and $M_p$ have these properties. Now if $c = \sigma + d + 6$ and $\sqrt{\text{det}(f_j(f^3(f)))}$ is the intersection of three smooth curves then $f$ is of type $K_p$ and if $c = \sigma + d + 7$ and $\sqrt{\text{det}(f_j(f^3(f)))}$ is the intersection of two smooth curves then $f$ is of type $M_p$.

(3). If $H_p = 2$ then if $j^3(f)$ has 3 primary ideals then $f$ is of type $L_{p,q}$ with $\sigma = p$, $d = q - 2$ and $c = \sigma + d + 6$ and if $j^3(f)$ has 2 primary ideals then $f$ is one of the type with $j^3(f) = (x^3, x^2y)$.

(4). $\sum f = (2, 2)$ implies that the modality is at least 1. So the statement follows from the classification of Dimca and Gibson.

**Proposition 5.** Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $j^3(f) = (x^3, x^2y)$. Then

1. If $c \leq 18$ then $f$ is unimodal.
2. If $c \geq 19$ then $f$ is unimodal if and only if $f \sim_{x'} (x^3 + xy^3a(y) + y^db(y), x^2y + y'c(y))$ with $a(0) \neq 0, b(0) \neq 0, c(0) \neq 0$ and $q, r \geq 4$.

**Proof.** We obtain (1) because of [9, Lemma 3.5 (ii)] and (2) is a consequence of the classification. From the list it follows the codimension is $\geq 19$ only in this case.

**Proposition 6.** Let $f(x, y)$ be a map germ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $j^3(f) = (x^3, x^2y)$. Then

1. If $c = 13$ then $f$ is unimodal of type $Q_4$;
2. If $c = 14$ then $f$ is unimodal of type $X_0$;
3. If $c = 15$ then $f$ is unimodal of type $\tilde{N}$ or $Z_1$ or $Q_5$ or $R_3$;
4. If $c = 16$ then $f$ is unimodal of type $Y_1$ or $Z_0$ or $P_4$ or $R_6$;
5. If $c = 17$ then $f$ is unimodal of type $Y_0$ or $Q_6$ or $R_7$;
6. If $c > 17$ and $f \sim_{x'} (x^3 + xy^a a(y) + y^d b(y), x^2y + y' c(y))$ with $a(0) \neq 0, b(0) \neq 0, c(0) \neq 0$ and $q, r \geq 4$ then $f$ is unimodal of type $P_p, Q_p$ or $R_p$.

**Proof.** (1) and (2) are clear.

(3). If $c = 15$ then we obtain that $\tilde{N}, Z_1, Q_5$ and $R_3$ have these properties. If $\mu = 11$, $d = 2$ and $\sigma = 5$ then $f$ is of type $\tilde{N}$ or $Z_1$. To differentiate these two types, compute normal form up to order 4 to
obtain $\lambda$, if $\lambda = 12$ then $f$ is of type $Z_1$ and if $\lambda \neq 1, 12$ then $f$ is of type $\bar{N}$. Moreover if $\mu = 12, d = 2$ and $\sigma = 5$ then $f$ is of type $R_5$ and if $\mu = 12, d = 3$ and $\sigma = 5$ then $f$ is of type $Q_5$.

(4). If $c = 16$ then we observe that $Y_1, Z_0, P_4, R_6$ have these properties. If $\mu = 11, d = 2$ and $\sigma = 6$ then $f$ is of type $Z_9$ and if $\mu = 12, d = 3$ and $\sigma = 5$ then $f$ is of type $Y_1$. Moreover if $\mu = 13, d = 2$ and $\sigma = 6$ then $f$ is of type $R_6$ and if $\mu = 13, d = 3$ and $\sigma = 6$ then $f$ is of type $P_4$.

(5). If $c = 17$ then we observe that $Y_0, Q_6$ and $R_7$ have these properties. If $\mu = 12, d = 3$ and $\sigma = 6$ then $f$ is of type $Y_0$ and if $\mu = 14, d = 4$ and $\sigma = 6$ then $f$ is of type $Q_6$ and if $\mu = 14, d = 2$ and $\sigma = 7$ then $f$ is of type $R_7$.

(6). If $c > 17$ then we observe that $P_p, Q_p$ and $R_p$ have these properties. If $c$ is even then $f$ is of type $P_p$ or $R_p$, if $d = 2$ then $f$ is of type $R_p$ with $\sigma = p$ and if $d > 2$ then $f$ is of type $P_p$ with $\sigma = p + 2$. Similarly if $c$ is odd then $f$ is of type $Q_p$ or $R_p$, if $d = 2$ then $f$ is of type $R_p$ with $\sigma = p$ and if $d > 2$ then $f$ is of type $Q_p$ with $\sigma = p$.

We use the following algorithm to differentiate the type of $\bar{N}$ and $Z_1$.

**Algorithm 1** Normal form of $\bar{N}$ or $Z_1$

**Require:** Ideal $I = f, g$ and an integer $k > 0$.  

**Ensure:** Ideal $I = h, l$ required normal form.

1: Define a map $\varphi, \varphi(x) = \sum_{1 \leq i+j \leq k-2} a_{ij} x^i y^j, \varphi(y) = \sum_{1 \leq i+j \leq k-2} b_{ij} x^i y^j$ with parameters up to the degree $k - 2$.

2: Define a $2 \times 2$-matrix $T$ with polynomial entries of degree $k - 3$ and parameters as coefficients, i.e., $T = (t_{ij}), t_{ij} = \sum_{0 \leq i+m \leq k-3} a_{ij, l_1, l_2} x^i y^m, a_{ij, l_1, l_2}$ are parameters.

3: Define the expected normal form $(h_1, h_2)$ depending on parameters.

4: Define $J := \varphi(I) - T \left[ \begin{array}{c} h_1 \\ h_2 \end{array} \right]$.

5: Let $K$ be the ideal generated by all the coefficients of $J$ with respect to $x, y$.

6: Compute the minimal associated primes of $K$.

7: Let $h := \det(a_{ij, 0, 0}) \cdot (a_{10} b_{01} - a_{01} b_{10})$.

8: For the prime ideal $P \ni K$, compute $n = NF(h/P)$.

9: If $h \neq 0$, eliminate in $P$ the $a_{ij}, b_{ij}$ and $a_{ij, l_1, l_2}$. The elimination just contains the conditions on the parameters of $(h_1, h_2)$.

10: Use the conditions to produce the required normal form.

4. **Singular Examples**

We have implemented the Algorithm in the computer algebra system SINGULAR [8]. Code can be downloaded from mathcity.org/junaid.

We give some examples.

```plaintext
ring R=0,(x,y),ds;

> ideal f=2x3+4x2y+2xy2+x10-10x9y+45x8y2-120x7y3+210x6y4-252x5y5+210x4y6
-120x3y7+45x2y8-10xy9+y10,x3-x2y-xy2+y3;
> contactMapgerms(f);

f is unimodal of type K_10

> ideal f=x3+3x2y+3xy2+y3+77x4-154x3y+154xy3-77y4,
x3+x2y-xy2-y3+x4-4x3y+6x2y2-4xy3+y4;
```

C. R. Mathématique, 2020, 358, no 8, 923-930
> contactMapgerms(f);
f is unimodal of type tild(N)

> ideal f=x3+3x2y+3xy2+y3+xy3+y4,x2y+2xy2+y3+y7;
> contactMapgerms(f);
f is unimodal of type R_7

References