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Number Theory / *Théorie des nombres*

A short proof of the canonical polynomial van der Waerden theorem

Une démonstration courte du théorème de van der Waerden polynomial canonique

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Abstract. We present a short new proof of the canonical polynomial van der Waerden theorem, recently established by Girão.

Résumé. Nous présentons une nouvelle démonstration courte du théorème de van der Waerden polynomial canonique, récemment établi par Girão.

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Girão [4] recently proved the following canonical version of the polynomial van der Waerden theorem. Here *canonical* [3] refers to the fact that the statement is independent of the number of colors. A set is *rainbow* if all elements have distinct colors. We write $[N] := \{1, \dots, N\}$.

Theorem 1 ([4]). *Let $p_1, \dots, p_k$ be distinct polynomials with integer coefficients and $p_i(0) = 0$ for each i. For all sufficiently large N, every coloring of $[N]$ contains a sequence $x + p_1(y), \dots, x + p_k(y)$ (for some $x, y \in \mathbb{N}$) that is monochromatic or rainbow.*

Girão's proof uses a color-focusing argument. Here we give a new short proof of Theorem 1, deducing it from the polynomial Szemerédi theorem of Bergelson and Leibman [1].

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Theorem 2 ([1]). *Let $p_1, \dots, p_k$ be distinct polynomials with integer coefficients and $p_i(0) = 0$ for each i. Let $\varepsilon > 0$. For all N sufficiently large, every $A \subset [N]$ with $|A| \ge \varepsilon N$ contains $x + p_1(y), \dots, x + p_k(y)$ for some $x, y \in \mathbb{N}$.*

Our proof of Theorem 1 follows the strategy of Erdős and Graham [2], who deduced a canonical van der Waerden theorem (i.e., for arithmetic progressions) using Szemerédi's theorem [7].

We quote the following result, proved by Linnik [6] in his elementary solution of Waring's problem (see [5, Theorem 19.7.2]). Note the left-hand side below counts the number of solutions $f(y_1) + \cdots + f(y_{s/2}) = f(y_{s/2+1}) + \cdots + f(y_s)$ with $y_1, \dots, y_s \in [n]$.

Theorem 3 ([6]). *Fix a polynomial f of degree $d \ge 2$ with integer coefficients. Let $s = 8^{d-1}$. Then*

$$\int_0^1 \left| \sum_{y=1}^n e^{2\pi i \theta f(y)} \right|^s \mathrm{d}\theta = O\big(n^{s-d}\big)$$

for any $n \in \mathbb{N}$, where the constant in the big-O depends only on f.

Lemma 4. *Fix a polynomial f of degree $d \ge 2$ with integer coefficients. For every $A \subset \mathbb{N}$ and $n \in \mathbb{N}$, the number of pairs $(a, y) \in A \times [n]$ with $a + f(y) \in A$ is*

$$O\left(|A|^{1+\frac{1}{s}} n^{1-\frac{d}{s}}\right),$$

where $s = 8^{d-1}$.

Proof. We write

$$\widehat{1}_A(\theta) = \sum_{x \in A} e^{2\pi i \theta x} \quad \text{and} \quad F(\theta) = \sum_{y=1}^n e^{2\pi i \theta f(y)}.$$

Then the number of solutions to $z = a + f(y)$ with $a, z \in A$ and $y \in [n]$ is

$$\begin{aligned}
\int_0^1 \left|\widehat{1}_A(\theta)\right|^2 F(\theta)\,\mathrm{d}\theta &\le \left(\int_0^1 \left|\widehat{1}_A(\theta)\right|^{\frac{2s}{s-1}} \mathrm{d}\theta\right)^{1-\frac{1}{s}} \left(\int_0^1 |F(\theta)|^s\,\mathrm{d}\theta\right)^{\frac{1}{s}} && \text{[Hölder]}\\
&\le \left(|A|^{\frac{2}{s-1}} \int_0^1 \left|\widehat{1}_A(\theta)\right|^2 \mathrm{d}\theta\right)^{1-\frac{1}{s}} \cdot O\left(n^{1-\frac{d}{s}}\right) && \left[\left|\widehat{1}_A(\theta)\right| \le |A| \text{ and Theorem 3}\right]\\
&= \left(|A|^{\frac{2}{s-1}} |A|\right)^{1-\frac{1}{s}} \cdot O\left(n^{1-\frac{d}{s}}\right) && \text{[Parseval]}\\
&= O\left(|A|^{1+\frac{1}{s}} n^{1-\frac{d}{s}}\right).
\end{aligned}$$

□

Lemma 5. *Fix a polynomial f of degree $d \ge 1$ with integer coefficients. Let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. Suppose that $|A \cap [x, x+L)| \le \varepsilon L$ for every $L \ge n^d$ and $x \in \mathbb{N}$. Then the number of pairs $(a, y) \in A \times [n]$ with $a + f(y) \in A$ is $O(\varepsilon^{1/s} |A|\, n)$, where $s = 8^{d-1}$.*

Proof. If $d = 1$, then for every $x \in A$, the number of $y \in [n]$ so that $x + f(y) \in A$ is $O(\varepsilon n)$ by the local density condition on A. Summing over all $x \in A$ yields the desired bound $O(\varepsilon |A| n)$ on the number of pairs. From now on assume $d \ge 2$.

Let $m = O(n^d)$ so that $|f(y)| \le m$ for all $y \in [n]$. Let $A_i = A \cap [im, (i+2)m)$. Then $|A_i| = O(\varepsilon m)$. Every pair $a, a + f(y) \in A$ with $y \in [n]$ is contained in some A_i, and, by Lemma 4, the number of pairs contained in each A_i is

$$O\left(|A_i|^{1+\frac{1}{s}} n^{1-\frac{d}{s}}\right) = O\left((\varepsilon m)^{\frac{1}{s}} |A_i| n^{1-\frac{d}{s}}\right) = O\left(\varepsilon^{1/s} |A_i| n\right).$$

Summing over all integers i yields Lemma 5 (each element of A lies in precisely two different A_i's). □

Proof of Theorem 1. Choose a sufficiently small $\varepsilon > 0$ (depending on $p_1, \dots, p_k$). Consider a coloring of $[N]$ without monochromatic progressions $x+p_1(y), \dots, x+p_k(y)$. By Theorem 2, every color class has density at most ε on every sufficiently long interval.

Let $D = \max_{i \neq j} \deg(p_i - p_j)$. Let n be an integer on the order of $N^{1/D}$ so that $x + p_1(y), \dots, x + p_k(y) \in [N]$ only if $y \in [n]$. We apply Lemma 5 with A a fixed color class and $f = p_i - p_j$; for every choice of $x + p_i(y) = a_1 \in A$ and $x + p_j(y) = a_2 \in A$, we have that $a_2 + f(y) = a_1$, so (a_2, y) is a solution of the form in Lemma 5. Summing over all $i \neq j$, we see that the number of pairs $(x, y) \in \mathbb{N} \times [n]$ where at least two of $x + p_1(y), \dots, x + p_k(y)$ lie in A is $O(\varepsilon^{1/8^{D-1}} |A| n)$. Summing over all color classes A, we see that the number of non-rainbow progressions $x+p_1(y), \dots, x+p_k(y) \in [N]$ is $O(\varepsilon^{1/8^{D-1}} N n)$. Since the total number of sequences $x+p_1(y), \dots, x+p_k(y) \in [N]$ is on the order of Nn, some such sequence must be rainbow, as long as $\varepsilon > 0$ is small enough and N is large enough. □

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