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## *Mathématique*


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Volume 358, issue 8 (2020), p. 961-970.

<https://doi.org/10.5802/crmath.119>

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Number Theory / *Théorie des nombres*

# Monotonicity and sharp inequalities related to complete $(p, q)$ -elliptic integrals of the first kind

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*Dedicated to Professir Sen-Lin Xu retired from USTC in China on his 80th birthday anniversary*

**Abstract.** With the aid of the monotone L'Hôpital rule, the authors verify monotonicity of some functions involving complete  $(p, q)$ -elliptic integrals of the first kind and the inverse of generalized hyperbolic tangent function, derive several sharp inequalities of complete  $(p, q)$ -elliptic integrals of the first kind, and generalize some known sharp approximation of complete elliptic integrals of the first kind.

**2020 Mathematics Subject Classification.** 33E05, 33C75.

**Funding.** The first author was partially supported by the Project for Combination of Education and Research Training at Zhejiang Institute of Mechanical and Electrical Engineering under Grant No. S027120206.

*Manuscript received 17th March 2020, revised 15th September 2020 and 13th October 2020, accepted 14th October 2020.*

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### 1. Motivation and main results

For  $z \in \mathbb{C}$  and  $n \in \{0\} \cup \mathbb{N}$ , the rising factorial  $(z)_n$  is defined [17] by

$$(z)_n = \prod_{\ell=0}^{n-1} (z + \ell) = \begin{cases} z(z+1)\cdots(z+n-1), & n \geq 1; \\ 1, & n = 0. \end{cases}$$

It can also be called the Pochhammer symbol or shifted factorial. The hypergeometric function  $F(a, b; c; z)$  is defined [23, p. 108, (5.3)] by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1 \tag{1}$$

for  $a, b, c \in \mathbb{C}$  with  $c \neq 0, -1, -2, \dots$

The complete elliptic integrals of the first and second kinds  $\mathcal{K}(r)$  and  $\mathcal{E}(r)$  can be expressed by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-r^2 \sin^2 \phi}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right)$$

and

$$\mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2 \sin^2 \phi} d\phi = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right).$$

See [14, Section 3.4] and [23, p. 128, Exercise 5.2].

Let  $F_{p,q} : [0, 1] \rightarrow [0, \frac{\pi_{p,q}}{2}]$  be defined [10, 21] by

$$F_{p,q}(x) = \arcsin_{p,q}(x) = \int_0^x (1-t^q)^{-1/p} dt, \quad x \in [0, 1]$$

and let  $\pi_{p,q} = 2 \arcsin_{p,q}(1)$ . Then

$$\pi_{p,q} = \frac{2}{q} \int_0^1 \frac{t^{1/p-1}}{(1-t)^{1/q}} dt = \frac{2}{q} B\left(1 - \frac{1}{p}, \frac{1}{q}\right) = \frac{2\pi}{q \sin\left(\frac{\pi}{p}\right)},$$

where

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \Re(x), \Re(y) > 0$$

denotes the classical beta function. The inverse function  $F_{p,q}^{-1} : [0, \frac{\pi_{p,q}}{2}] \rightarrow [0, 1]$  is called generalized  $(p, q)$ -sine function, denoted by  $\sin_{p,q}$ . It is clear that  $\sin_{2,2} = \sin$ .

The complete  $(p, q)$ -elliptic integrals of the first and second kinds were defined in [12, 22] by

$$\mathcal{K}_{p,q}(r) = \int_0^{\pi_{p,q}/2} (1-r^q \sin_{p,q}^q t)^{1/p-1} dt \quad \text{and} \quad \mathcal{E}_{p,q}(r) = \int_0^{\pi_{p,q}/2} (1-r^q \sin_{p,q}^q t)^{1/p} dt$$

for  $p, q \in (1, \infty)$  and  $r \in [0, 1)$ . It is obvious that  $\mathcal{K}_{2,2}(r) = \mathcal{K}(r)$  and  $\mathcal{E}_{2,2}(r) = \mathcal{E}(r)$  are classical complete elliptic integrals of the first and second kinds.

For  $p, q \in (1, \infty)$  and  $r \in [0, 1)$ , the complete  $(p, q)$ -elliptic integrals of the first and second kinds can be represented [11, 12, 22] in terms of the hypergeometric functions  $F(a, b; c; z)$  by

$$\begin{cases} \mathcal{K}_{p,q}(r) = \frac{\pi_{p,q}}{2} F\left(1 - \frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right); \\ \mathcal{K}_{p,q}(0) = \frac{\pi_{p,q}}{2}, \quad \mathcal{K}_{p,q}(1) = \infty \end{cases} \tag{2}$$

and

$$\begin{cases} \mathcal{E}_{p,q}(r) = \frac{\pi_{p,q}}{2} F\left(-\frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right); \\ \mathcal{E}_{p,q}(0) = \frac{\pi_{p,q}}{2}, \quad \mathcal{E}_{p,q}(1) = 1. \end{cases} \tag{3}$$

In [3], the double inequality

$$\frac{\pi}{2} \left( \frac{\operatorname{arctanh} r}{r} \right)^{1/2} < \mathcal{H}(r) < \frac{\pi}{2} \frac{\operatorname{arctanh} r}{r} \tag{4}$$

was obtained, where  $\operatorname{arctanh} r$  denotes the inverse of hyperbolic tangent function. In [20], the double inequality

$$\left( \alpha + \frac{\pi}{2r} \right) \operatorname{arctanh} r < \mathcal{H}(r) < \left( \beta + \frac{\pi}{2r} \right) \operatorname{arctanh} r \tag{5}$$

was proved to be valid if and only if  $\alpha \leq 1 - \frac{\pi}{2}$  and  $\beta \geq 0$ . In [15] and [16, Section 9], among other things, the inequalities

$$\frac{\pi \arcsin r}{2r} < \mathcal{H}(r) < \frac{\pi}{4r} \ln \frac{1+r}{1-r} \tag{6}$$

and

$$\mathcal{E}(r) < \frac{16 - 4r^2 - 3r^4}{4(4 + r^2)} \mathcal{H}(r) \tag{7}$$

were derived from the Čebyšev integral inequality [18]. In [1], the double inequality

$$\frac{\pi}{2} \left( \frac{\operatorname{arctanh} r}{r} \right)^{\alpha_1} < \mathcal{H}(r) < \frac{\pi}{2} \left( \frac{\operatorname{arctanh} r}{r} \right)^{\beta_1} \tag{8}$$

was sharpened by  $\alpha_1 = \frac{3}{4}$  and  $\beta_1 = 1$ . In [7], among other things, it was obtained that

$$\frac{\pi}{2} - \frac{1}{2} \ln \frac{(1+r)^{r-1}}{(1-r)^{r+1}} < \mathcal{E}(r) < \frac{\pi-1}{2} + \frac{1-r^2}{4r} \ln \frac{1+r}{1-r}. \tag{9}$$

In [35], the double inequalities

$$\frac{\pi \sqrt{6 + 2\sqrt{1-r^2} - 3r^2}}{4\sqrt{2}} \leq \mathcal{E}(r) \leq \frac{\pi \sqrt{10 - 2\sqrt{1-r^2} - 5r^2}}{4\sqrt{2}} \tag{10}$$

and

$$\frac{\pi \sqrt{32 - r^4 - 32r^2}}{8\sqrt{2} \sqrt[4]{(1-r^2)^3}} \leq \mathcal{H}(r) \leq \frac{\pi \sqrt{r^4 - 32r^2 + 32}}{8\sqrt{2} \sqrt[4]{(1-r^2)^3}} \tag{11}$$

were established. In [24], we discussed monotonicity and some inequalities related to complete elliptic integrals of the second kind  $\mathcal{E}(r)$ .

We observe that

- (1) because  $\operatorname{arctanh} r = \frac{1}{2} \ln \frac{1+r}{1-r}$ , the upper bounds in (4) and (6) and the best possible bounds in (5) and (8) are the same one which cannot compare with the upper bound in (11) on  $(0, 1)$ ;
- (2) the lower bound in (8) is clearly better than the corresponding one in (4), the lower bounds in (5) and (6) cannot compare with each other on  $(0, 1)$ , the lower bounds in (5) and (8) cannot compare with each other on  $(0, 1)$ , the lower bound in (8) is better than the corresponding one in (6), and the lower bound in (11) cannot compare with the corresponding ones in (4) to (8);
- (3) the lower bound in (10) is better than the corresponding one in (9) and the upper bounds in (10) and (9) cannot compare with each other on  $(0, 1)$ .

These observations can be verified by plotting via the Wolfram Mathematica 11.1.

In [36], it was obtained that, for  $p \in (1, \infty)$ ,

$$\frac{\operatorname{arctanh}_p r}{r} < \mathcal{H}_p(r) < \frac{\pi_p}{2} \frac{\operatorname{arctanh}_p r}{r}, \tag{12}$$

where  $\mathcal{H}_p(r) = \mathcal{H}_{p,p}(r)$  and

$$\operatorname{arctanh}_p r = rF\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; r^p\right). \tag{13}$$

In [25], we investigated monotonicity and some inequalities related to generalized Grötzsch ring function

$$\frac{\pi}{2 \sin(\pi q)} \frac{\mathcal{K}_{1/q,1/q}((1-r^2)^q)}{\mathcal{K}_{1/q,1/q}(r^{2q})}, \quad q \in \left(0, \frac{1}{2}\right].$$

Let  $\gamma = 0.57721566\dots$  stands for Euler–Mascheroni’s constant, let  $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$  be the logarithmic derivative of the classical Euler gamma function which can be defined (see [13] and [23, Chapter 3]) by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0 \quad \text{or by} \quad \Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

and let

$$R(x, y) = \psi(x) - \psi(y) - 2\gamma, \quad x, y \in (0, \infty)$$

denote the Ramanujan constant function [6]. In [2, Theorem 2.2], the function  $\frac{\mathcal{K}(r)}{\ln\left[\frac{c}{\sqrt{1-r^2}}\right]}$  is showed to be decreasing if and only if  $1 < c \leq 4$  and to be increasing if and only if  $c \geq e^2$ .

In this paper, with the aid of the monotone L’Hôpital rule, we will use a new and concise method to generalize the above inequalities and monotonicity for functions involving  $\mathcal{K}(r)$  and  $\mathcal{E}(r)$  to those involving complete  $(p, q)$ -elliptic integrals  $\mathcal{K}_{p,q}(r)$  and  $\mathcal{E}_{p,q}(r)$ , to reveal monotonicity of several functions involving  $\mathcal{K}_{p,q}(r)$ ,  $\mathcal{E}_{p,q}(r)$ , and the inverse of generalized hyperbolic tangent function, and to improve inequalities (4), (5), (8), and (12).

Our main results can be stated as the following theorems.

**Theorem 1.** For  $r \in (0, 1)$  and  $p, q \in (1, \infty)$ , let  $F(r) = \frac{\mathcal{K}_{p,q}(r)}{\ln\left[\frac{c}{(1-r^q)^{1/q}}\right]}$ . Then the function  $F(r)$

- (1) increases on  $(0, 1)$  if and only if  $c \geq \exp\left(\frac{q(p-1)+p}{q(p-1)}\right)$ ;
- (2) decreases on  $(0, 1)$  if and only if  $1 \leq c \leq \exp\left(\frac{R(1-1/p, 1/q)}{q}\right)$ ;

and, consequently, when  $1 \leq c \leq \exp\left(\frac{R(1-1/p, 1/q)}{q}\right)$ ,

$$\ln \frac{c}{(1-r^q)^{1/q}} < \mathcal{K}_{p,q}(r) < \frac{\pi_{p,q}}{2 \ln c} \ln \frac{c}{(1-r^q)^{1/q}}. \tag{14}$$

**Theorem 2.** For  $r \in (0, 1)$  and  $q \in (1, \infty)$ ,

- (1) when  $p \geq 2$ , the function  $G(r) = \frac{r \mathcal{K}_{p,q}^2(r)}{\operatorname{arctanh}_q r}$  increases and maps  $(0, 1)$  onto  $\left(\frac{\pi_{p,q}^2}{4}, \infty\right)$ . Consequently, for  $r \in (0, 1)$ ,  $p \in [2, \infty)$ , and  $q \in (1, \infty)$ , we have

$$\frac{\pi_{p,q}}{2} \left(\frac{\operatorname{arctanh}_q r}{r}\right)^{1/2} < \mathcal{K}_{p,q}(r); \tag{15}$$

- (2) when  $p > 1$ , the function  $W(r) = \frac{r \mathcal{K}_{p,q}(r)}{\operatorname{arctanh}_q r}$  decreases and maps  $(0, 1)$  onto  $\left(1, \frac{\pi_{p,q}}{2}\right)$ . Consequently, for  $r \in (0, 1)$  and  $p, q \in (1, \infty)$ , we have

$$\frac{\operatorname{arctanh}_q r}{r} < \mathcal{K}_{p,q}(r) < \frac{\pi_{p,q}}{2} \frac{\operatorname{arctanh}_q r}{r}. \tag{16}$$

**Theorem 3.** For  $r \in (0, 1)$  and  $p, q \in (1, \infty)$ , the function  $H(r) = \frac{\frac{\pi_{p,q}}{2} \operatorname{arctanh}_q r - r \mathcal{K}_{p,q}(r)}{r^q \operatorname{arctanh}_q r}$  increases and maps  $(0, 1)$  onto  $\left(\frac{\pi_{p,q}}{2} \frac{1}{pq(q+1)}, \frac{\pi_{p,q}}{2} - 1\right)$ . Consequently, for  $r \in (0, 1)$  and  $p, q \in (1, \infty)$ , we have

$$\frac{\pi_{p,q}}{2} \frac{\operatorname{arctanh}_q r}{r} (1 - \alpha_2 r^q) < \mathcal{K}_{p,q}(r) < \frac{\pi_{p,q}}{2} \frac{\operatorname{arctanh}_q r}{r} (1 - \beta_2 r^q), \tag{17}$$

where  $\alpha_2 = 1 - \frac{2}{\pi_{p,q}}$  and  $\beta_2 = \frac{1}{pq(q+1)}$  are the best possible constants in the sense that they can not be replaced by any bigger and smaller constants respectively.

## 2. Lemmas and their proofs

For proving our main results, we need the following known results and lemmas.

In [22], the following two derivatives were given:

$$\frac{d\mathcal{K}_{p,q}(r)}{dr} = \frac{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)}{r(1-r^q)} \tag{18}$$

and

$$\frac{d\mathcal{E}_{p,q}(r)}{dr} = -\frac{q}{p} \frac{\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)}{r}. \tag{19}$$

**Lemma 4 (cf. [4]).** For  $a, b \in \mathbb{R}$  with  $a < b$ , let  $f$  and  $g$  be continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $g' \neq 0$  on  $(a, b)$ . If the ratio  $\frac{f'}{g'}$  is increasing on  $(a, b)$ , then both of the functions  $\frac{f(x)-f(a)}{g(x)-g(a)}$  and  $\frac{f(x)-f(b)}{g(x)-g(b)}$  are increasing with respect to  $x \in (a, b)$ .

**Lemma 5 (cf. [1]).** Suppose that the power series

$$R(x) = \sum_{n=0}^{\infty} r(n)x^n \quad \text{and} \quad S(x) = \sum_{n=0}^{\infty} s(n)x^n$$

converge for  $|x| < 1$ . If  $s(n) > 0$  for  $n \geq 0$  and the sequence  $\frac{r(n)}{s(n)}$  increases with respect to  $n \geq 0$ , then the ratio  $\frac{R(x)}{S(x)}$  increases with respect to  $x \in (0, 1)$ .

**Lemma 6 (cf. [11, Theorem 1]).** For  $r \in (0, 1)$  and  $p, q \in (1, \infty)$ , the following conclusions are valid:

- (1) the function  $\frac{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)}{r^q \mathcal{K}_{p,q}(r)}$  is decreasing and maps from  $(0, 1)$  onto  $\left(1, \frac{(p-1)q}{pq+p-q}\right)$ ;
- (2) the function  $\frac{\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)}{r^q \mathcal{K}_{p,q}(r)}$  is increasing and maps  $(0, 1)$  onto  $\left(\frac{p}{pq+p-q}, 1\right)$ . Consequently,

$$1 - r^q < \frac{\mathcal{E}_{p,q}(r)}{\mathcal{K}_{p,q}(r)} < 1 - \frac{p}{(p-1)q+p} r^q. \tag{20}$$

**Lemma 7.** For  $r \in (0, 1)$  and  $p, q \in (1, \infty)$ , the function  $f(r) = \frac{1}{q} \ln(1-r^q) + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)}$  decreases and maps  $(0, 1)$  onto  $\left(\frac{R(1-1/p, 1/q)}{q}, \frac{q(p-1)+p}{q(p-1)}\right)$ . Consequently,

$$\frac{R(1-1/p, 1/q)}{q} < \frac{1}{q} \ln(1-r^q) + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} < \frac{q(p-1)+p}{q(p-1)}. \tag{21}$$

**Proof.** By virtue of (18) and (19), differentiating  $f(r)$  gives

$$\begin{aligned} f'(r) &= -\frac{r^q}{r(1-r^q)} + \frac{r^q}{r(1-r^q)} \frac{\begin{pmatrix} q(1-r^q)\mathcal{K}_{p,q}[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)] \\ + [\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]^2 \\ -(1-q/p)(1-r^q)\mathcal{K}_{p,q}[\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)] \\ -(q-1)r^q(1-r^q)\mathcal{K}_{p,q}^2(r) \end{pmatrix}}{[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]^2} \\ &= \frac{r^q \mathcal{K}_{p,q}(r)}{r[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]^2} \begin{pmatrix} q[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)] \\ -(1-q/p)[\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)] \\ -(q-1)r^q \mathcal{K}_{p,q}(r) \end{pmatrix} \\ &= \frac{r^q \mathcal{K}_{p,q}(r)}{r[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]^2} \left( r^q \mathcal{K}_{p,q} - \frac{q(p-1)+p}{p} [\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)] \right) \\ &= \frac{(r^q \mathcal{K}_{p,q}(r))^2}{r[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]^2} \left[ 1 - \frac{q(p-1)+p}{p} \frac{\mathcal{K}_{p,q} - \mathcal{E}_{p,q}}{r^q \mathcal{K}_{p,q}} \right]. \end{aligned}$$

From the second item of Lemma 6, it follows that  $f'(r) < 0$  which means that  $f(r)$  decreases.

By the first item in Lemma 6 and [11, Theorem 3], we acquire the limits  $f(0^+) = \frac{q(p-1)+p}{q(p-1)}$  and

$$\begin{aligned} f(1^-) &= \lim_{r \rightarrow 1^-} \left[ \frac{\ln(1-r^q)}{q} + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right] \\ &= \lim_{r \rightarrow 1^-} \frac{r^q}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \left[ \mathcal{K}_{p,q} + \frac{\ln(1-r^q)}{q} \right] \\ &\quad + \lim_{r \rightarrow 1^-} \left[ 1 - \frac{r^q}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right] \frac{\ln(1-r^q)}{q} \\ &= \frac{R(1-1/p, 1/q)}{q} + \lim_{r \rightarrow 1^-} \left[ \mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q} - r^q \right] \frac{\ln(1-r^q)}{q} \\ &= \frac{R(1-1/p, 1/q)}{q} + \lim_{r \rightarrow 1^-} \frac{\mathcal{E}_{p,q}(r) - r^q}{(1-r^q)^{1/q}} (1-r^q)^{1/q} \frac{\ln(1-r^q)}{q} \\ &= \frac{R(1-1/p, 1/q)}{q}. \end{aligned}$$

The double inequality (21) follows from monotonicity of  $f(r)$ . The proof of Lemma 7 is complete. □

**Lemma 8.** For  $r \in (0, 1)$  and  $q \in (1, \infty)$ , the function  $\Phi(r) = \frac{(1-r^q)\operatorname{arctanh}_q r}{r}$  decreases and maps  $(0, 1)$  onto  $(0, 1)$ .

**Proof.** In [36], it was obtained that  $(\operatorname{arctanh}_q r)' = \frac{1}{1-r^q}$ . Employing this result and the formula (13) yields

$$r\Phi'(r) = \frac{-q(r^{q-1}\operatorname{arctanh}_q r + 1)r - (1-r^q)\operatorname{arctanh}_q r}{r} = 1 - [(q-1)r^q + 1] \frac{\operatorname{arctanh}_q r}{r} < 0.$$

Therefore, the function  $\Phi(r)$  decreases.

It is straightforward to derive  $\Phi(0^+) = 1$  and  $\Phi(1^-) = 0$ . The proof of Lemma 8 is complete. □

**Lemma 9.** For  $r \in (0, 1)$ ,  $q \in (1, \infty)$ , and  $p \in [2, \infty)$ , the function  $\phi(r) = 2\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)$  increases and maps  $(0, 1)$  onto  $(\frac{\pi_{p,q}}{2}, 2)$ . Consequently,

$$\frac{\pi_{p,q}}{4} < \mathcal{E}_{p,q}(r) - \frac{1-r^q}{2}\mathcal{K}_{p,q}(r) < 1.$$

**Proof.** Utilizing the derivative formulas (18) and (19) and differentiating give

$$\phi'(r) = \left(1 - \frac{2q}{p}\right) \frac{\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)}{r} + (q-1) \frac{r^q \mathcal{K}_{p,q}(r)}{r} = \frac{r^q \mathcal{K}_{p,q}(r)}{r} \varphi(r),$$

where

$$\varphi(r) = \left(1 - \frac{2q}{p}\right) \frac{\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)}{r^q \mathcal{K}_{p,q}(r)} + q - 1.$$

When  $p \geq 2q$ , from the second item of Lemma 6, it follows readily that  $\varphi(r) > 0$ .

When  $2 \leq p < 2q$ , by the second item of Lemma 6, it follows that

$$\varphi(r) > 0 \iff \inf_{0 < r < 1} \left[ \left(1 - \frac{2q}{p}\right) \frac{\mathcal{K}_{p,q}(r) - \mathcal{E}_{p,q}(r)}{r^q \mathcal{K}_{p,q}(r)} + q - 1 \right] > 0 \iff q \left(1 - \frac{2}{p}\right) \geq 0.$$

Accordingly, when  $p \geq 2$  and  $q > 1$ , the function  $\phi(r)$  is increasing.

By virtue of (2) and (3), it is easy to deduce the limits  $\phi(0^+) = \frac{\pi_{p,q}}{2}$  and  $\phi(1^-) = 2$ . The proof of Lemma 9 is complete. □

### 3. Proofs of main results

Now we start out to prove our main results.

**Proof of Theorem 1.** By (18), direct differentiating  $F(r)$  gives

$$\begin{aligned} \left[ \ln \frac{c}{(1-r^q)^{1/q}} \right]^2 F'(r) &= \frac{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)}{r(1-r^q)} \ln \frac{c}{(1-r^q)^{1/q}} - \frac{r^q \mathcal{K}_{p,q}(r)}{r(1-r^q)} \\ &= \frac{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)}{r(1-r^q)} \left[ \ln c - \left( \frac{1}{q} \ln(1-r^q) + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right) \right] \end{aligned}$$

By Lemma 7, we have

$$\begin{aligned} F'(r) < 0 &\iff \inf_{0 < r < 1} \left[ \ln c - \left( \frac{\ln(1-r^q)}{q} + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right) \right] < 0 \\ &\iff c \leq \inf_{0 < r < 1} \exp \left( \frac{\ln(1-r^q)}{q} + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right) \iff c \leq \exp \left( \frac{R(1-1/p, 1/q)}{q} \right) \end{aligned}$$

and

$$\begin{aligned} F'(r) > 0 &\iff \sup_{0 < r < 1} \left[ \ln c - \left( \frac{\ln(1-r^q)}{q} + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right) \right] > 0 \\ &\iff c \geq \sup_{0 < r < 1} \exp \left( \frac{\ln(1-r^q)}{q} + \frac{r^q \mathcal{K}_{p,q}(r)}{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)} \right) \iff c \geq \exp \left( \frac{q(p-1) + p}{q(p-1)} \right) \end{aligned}$$

The double inequality (14) follows from monotonicity of  $F(r)$ . The proof of Theorem 1 is complete.  $\square$

**Proof of Theorem 2.** Let  $g_1(r) = r \mathcal{K}_{p,q}^2(r)$  and  $g_2(r) = \operatorname{arctanh}_q r$ . Then  $G(r) = \frac{g_1(r)}{g_2(r)}$  and  $g_1(0) = g_2(0) = 0$ . Making use of [36] and (18), we have

$$\begin{aligned} \frac{g_1'(r)}{g_2'(r)} &= (1-r^q)\mathcal{K}_{p,q}(r) \left[ \mathcal{K}_{p,q}(r) + 2 \frac{\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)}{(1-r^q)} \right] \\ &= \mathcal{K}_{p,q}(r) [2\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]. \end{aligned}$$

From Lemma 9, it follows that the function  $G(r)$  is increasing on  $(0, 1)$ . By the L'Hôpital rule, it follows that  $G(0^+) = \frac{\pi_{p,q}^2}{4}$  and  $G(1^-) = \infty$ .

It is obvious that inequality (15) follows from monotonicity of  $G(r)$ .

Let  $g_3(r) = r \mathcal{K}_{p,q}(r)$  and  $g_4(r) = \operatorname{arctanh}_q r$ . Then  $W(r) = \frac{g_3(r)}{g_4(r)}$  and  $g_3(0) = g_4(0) = 0$ . Since

$$\frac{g_3'(r)}{g_4'(r)} = \frac{\mathcal{K}_{p,q}(r) + [\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)] / (1-r^q)}{1/(1-r^q)} = \mathcal{E}_{p,q}(r),$$

by Lemma 4, the function  $W(r)$  is decreasing on  $(0, 1)$ . By Lemma 4, those formulas in (3), and the L'Hôpital rule, we can obtain readily that  $W(0^+) = \frac{\pi_{p,q}}{2}$  and  $W(1^-) = 1$ .

The double inequality (16) follows from monotonicity of  $W(r)$  directly.  $\square$

**Proof of Theorem 3.** Let the functions  $h_1(r) = \frac{\pi_{p,q}}{2} \operatorname{arctanh}_q r - r \mathcal{K}_{p,q}(r)$  and  $h_2(r) = r^q \operatorname{arctanh}_q r$ . Then  $H(r) = \frac{h_1(r)}{h_2(r)}$  and  $h_1(0) = h_2(0) = 0$ . Using (18) and differentiating yield

$$\frac{h_1'(r)}{h_2'(r)} = \frac{\left( \frac{\pi_{p,q}}{2} - (1-r^q)\mathcal{K}_{p,q}(r) \right)}{-[\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]} \frac{1}{1 + q \frac{(1-r^q) \operatorname{arctanh}_q r}{r}} = \frac{h_3(r)}{h_4(r)},$$



where

$$h_3(r) = \frac{\frac{\pi_{p,q}}{2} - (1-r^q)\mathcal{K}_{p,q}(r) - [\mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r)]}{r^q}$$

and  $h_4(r) = 1 + q \frac{(1-r^q)\operatorname{arctanh}_q r}{r}$ . By Lemma 8, the function  $h_4(r)$  is decreasing on  $(0, 1)$ .

Let  $a = 1 - \frac{1}{p}$ ,  $b = 1 - \frac{1}{p} + \frac{1}{q}$ ,  $a_n = \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!}$ , and  $b_n = \frac{a_n}{n+b}$ . By virtue of (1) to (3), we have

$$\begin{aligned} \mathcal{E}_{p,q}(r) - (1-r^q)\mathcal{K}_{p,q}(r) &= \frac{\pi_{p,q}}{2} \left[ \sum_{n=0}^{\infty} \frac{(-\frac{1}{p})_n(\frac{1}{q})_n}{(b)_n n!} r^{qn} - \sum_{n=0}^{\infty} \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!} r^{qn} + \sum_{n=0}^{\infty} \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!} r^{q(n+1)} \right] \\ &= \frac{\pi_{p,q}}{2} \sum_{n=1}^{\infty} \left[ \frac{(a)_{n-1}(\frac{1}{q})_{n-1}}{(b)_{n-1}(n-1)!} - \frac{(a)_{n-1}(\frac{1}{q})_n}{(b)_n n!} \right] r^{qn} \\ &= \frac{a\pi_{p,q}}{2} \sum_{n=1}^{\infty} \frac{n(a)_{n-1}(\frac{1}{q})_{n-1}}{(b)_n n!} r^{qn} = \frac{a\pi_{p,q}}{2} \sum_{n=0}^{\infty} b_n r^{q(n+1)} \end{aligned}$$

and

$$\begin{aligned} \frac{\pi_{p,q}}{2} - (1-r^q)\mathcal{K}_{p,q}(r) &= \frac{\pi_{p,q}}{2} - (1-r^q) \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!} r^{qn} \\ &= \frac{\pi_{p,q}}{2} \left[ 1 - \sum_{n=0}^{\infty} \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!} r^{qn} + \sum_{n=0}^{\infty} \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!} r^{q(n+1)} \right] \\ &= \frac{\pi_{p,q}}{2} \left[ - \sum_{n=1}^{\infty} \frac{(a)_n(\frac{1}{q})_n}{(b)_n n!} r^{qn} + \sum_{n=0}^{\infty} b_n r^{q(n+1)} \right] \\ &= \frac{\pi_{p,q}}{2} \left[ - \sum_{n=0}^{\infty} \frac{(a)_{n+1}(\frac{1}{q})_{n+1}}{(b)_{n+1}(n+1)!} r^{q(n+1)} + \sum_{n=0}^{\infty} b_n r^{q(n+1)} \right] \\ &= \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} c_n r^{q(n+1)}, \end{aligned}$$

where  $c_n = \left[ 1 - \frac{(n+a)(n+\frac{1}{q})}{(n+b)(n+1)} \right] b_n$ . Furthermore, by (1), we have

$$h_3(r) = \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} (b_n + c_n) r^{qn} = \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{1}{p} \left( n + \frac{1}{q} \right) r^{qn} > 0.$$

By Lemma 7, the function  $h_3(r)$  is increasing on  $(0, 1)$ . By Lemmas 4 to 6, the function  $H(r)$  is increasing.

By the L'Hôpital rule, it follows that

$$H(0^+) = \lim_{r \rightarrow 0^+} H(r) = \lim_{r \rightarrow 0^+} \frac{h'_1(r)}{h'_2(r)} = \lim_{r \rightarrow 0^+} \frac{h_3(r)}{h_4(r)} = \frac{\pi_{p,q}}{2} \frac{b_0 + c_0}{1 + q} = \frac{\pi_{p,q}}{2pq(q+1)}$$

and

$$H(1^-) = \lim_{r \rightarrow 1^-} H(r) = \lim_{r \rightarrow 0^+} \frac{h'_1(r)}{h'_2(r)} = \lim_{r \rightarrow 0^+} \frac{h_3(r)}{h_4(r)} = \frac{\pi_{p,q}}{2} - 1.$$

Thus, the double inequality (17) holds. The proof of Theorem 3 is complete. □

#### 4. Remarks

**Remark 10.** When  $p = q = 2$ , Theorem 1 reduces to [2, Theorem 2.2 (5)].

**Remark 11.** When  $p = q = 2$ , inequalities (15) and (16) in Theorem 2 become (4). When  $p = q$ , inequality (16) becomes (12).

**Remark 12.** When  $p = q$ , the double inequality (17) improves the double inequality (12).

If setting  $p = q = 2$  in the double inequality (17) in Theorem 3, then

$$\frac{\pi}{2} \frac{\operatorname{arctanh} r}{r} \left[ 1 - \left( 1 - \frac{2}{\pi} \right) r^2 \right] < \mathcal{K}(r) < \frac{\pi}{2} \frac{\operatorname{arctanh} r}{r} \left( 1 - \frac{1}{12} r^2 \right)$$

for  $r \in (0, 1)$ . This double inequality improves the double inequalities (4) and (5).

**Remark 13.** When  $p = q = 2$ , the inequality (20) becomes

$$1 - r^2 < \frac{\mathcal{E}(r)}{\mathcal{K}(r)} < 1 - \frac{r^2}{2}$$

whose upper bound is worse than (7). This means that the inequality (7) is much better.

**Remark 14.** Interested readers who are curious about this paper not only just want to know the main research content of this paper, but also want to know the research background and research progress in this field. Enriching references as many as possible is very important for these readers. So we list several recently published papers [5, 8, 9, 19, 26–34], which are closely related to the topic of this paper, to the list of references of this paper.

## References

- [1] H. Alzer, S.-L. Qiu, “Monotonicity theorems and inequalities for the complete elliptic integrals”, *J. Comput. Appl. Math.* **172** (2004), no. 2, p. 289-312.
- [2] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, “Functional inequalities for complete elliptic integrals and their ratios”, *SIAM J. Math. Anal.* **21** (1990), no. 2, p. 536-549.
- [3] ———, “Functional inequalities for hypergeometric functions and complete elliptic integrals”, *SIAM J. Math. Anal.* **23** (1992), no. 2, p. 512-524.
- [4] ———, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, 1997.
- [5] B. A. Bhayo, L. Yin, “On a function involving generalized complete  $(p, q)$ -elliptic integrals”, *Arab. J. Math.* **9** (2020), no. 1, p. 73-82.
- [6] H.-H. Chu, Z.-H. Yang, W. Zhang, Y.-M. Chu, “Improvements of the bounds for Ramanujan constant function”, *J. Inequal. Appl.* **2016** (2016), article ID 196 (9 pages).
- [7] B.-N. Guo, F. Qi, “Some bounds for the complete elliptic integrals of the first and second kind”, *Math. Inequal. Appl.* **14** (2011), no. 2, p. 323-334.
- [8] Z.-Y. He, M.-K. Wang, Y.-P. Jiang, Y.-M. Chu, “Bounds for the perimeter of an ellipse in terms of power means”, *J. Math. Inequal.* **14** (2020), no. 3, p. 887-899.
- [9] T.-R. Huang, S.-Y. Tan, X.-Y. Ma, Y.-M. Chu, “Monotonicity properties and bounds for the complete  $p$ -elliptic integrals”, *J. Inequal. Appl.* **2018** (2018), article ID 239 (11 pages).
- [10] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang, F. Qi, “Convexity of the generalized sine function and the generalized hyperbolic sine function”, *J. Approx. Theory* **174** (2013), p. 1-9.
- [11] R. B. Jiao, S. L. Qiu, G. T. Ge, “Monotonicity and convexity properties of the generalized  $(p, q)$ -elliptic integrals”, *J. Zhejiang Sci-Tech Univ.* **39** (2018), no. 6, p. 765-769.
- [12] T. Kamiya, S. Takeuchi, “Complete  $(p, q)$ -complete elliptic integrals with application to a family of means”, *J. Class. Anal.* **10** (2017), no. 1, p. 15-25.
- [13] F. Qi, “Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities”, *Filomat* **27** (2013), no. 4, p. 601-604.
- [14] F. Qi, B.-N. Guo, “Sums of infinite power series whose coefficients involve products of the Catalan–Qi numbers”, *Montes Taurus J. Pure Appl. Math.* **1** (2019), no. 2, p. 1-12.
- [15] F. Qi, Z. Huang, “Inequalities of the complete elliptic integrals”, *Tamkang J. Math.* **29** (1998), no. 3, p. 165-169.
- [16] F. Qi, D.-W. Niu, B.-N. Guo, “Refinements, generalizations, and applications of Jordan’s inequality and related problems”, *J. Inequal. Appl.* **2009** (2009), article ID 271923 (52 pages).
- [17] F. Qi, D.-W. Niu, D. Lim, B.-N. Guo, “Closed formulas and identities for the Bell polynomials and falling factorials”, *Contrib. Discrete Math.* **15** (2020), no. 1, p. 163-174.
- [18] F. Qi, G. Rahman, S. M. Hussain, W.-S. Du, K. S. Nisar, “Some inequalities of Čebyšev type for conformable  $k$ -fractional integral operators”, *Symmetry* **10** (2018), no. 11, article ID 614 (8 pages).
- [19] W.-M. Qian, Z.-Y. He, Y.-M. Chu, “Approximation for the complete elliptic integral of the first kind”, *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM* **114** (2020), no. 2, article ID 57 (12 pages).

- [20] S. L. Qiu, M. K. Vamanamurthy, "Elliptic integrals and the modulus of Grötzsch ring", *PanAmer. Math. J.* **5** (1995), no. 2, p. 41-60.
- [21] Y.-Q. Song, Y.-M. Chu, B.-Y. Liu, M.-K. Wang, "A note on generalized trigonometric and hyperbolic functions", *J. Math. Inequal.* **8** (2014), no. 3, p. 635-642.
- [22] S. Takeuchi, "A new form of the generalized complete elliptic integrals", *Kodai Math. J.* **39** (2016), no. 1, p. 202-226.
- [23] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, John Wiley & Sons, 1996.
- [24] F. Wang, B.-N. Guo, F. Qi, "Monotonicity and inequalities related to complete elliptic integrals of the second kind", *AIMS Math.* **5** (2020), no. 3, p. 2732-2742.
- [25] F. Wang, J.-H. He, L. Yin, F. Qi, "Monotonicity properties and inequalities related to generalized Grötzsch ring functions", *Open Math.* **17** (2019), p. 802-812.
- [26] M.-K. Wang, H.-H. Chu, Y.-M. Chu, "Precise bounds for the weighted Hölder mean of the complete  $p$ -elliptic integrals", *J. Math. Anal. Appl.* **480** (2019), no. 2, article ID 12388 (9 pages).
- [27] M.-K. Wang, Z.-Y. He, Y.-M. Chu, "Sharp power mean inequalities for the generalized elliptic integral of the first kind", *Comput. Methods Funct. Theory* **20** (2020), no. 1, p. 111-124.
- [28] M.-K. Wang, M.-Y. Hong, Y.-F. Xu, Z.-H. Shen, Y.-M. Chu, "Inequalities for generalized trigonometric and hyperbolic functions with one parameter", *J. Math. Inequal.* **14** (2020), no. 1, p. 1-21.
- [29] M.-K. Wang, W. Zhang, Y.-M. Chu, "Monotonicity, convexity and inequalities involving the generalized elliptic integrals", *Acta Math. Sci.* **39** (2019), no. 5, p. 1440-1450.
- [30] Z.-H. Yang, Y.-M. Chu, "A monotonicity property involving the generalized elliptic integral of the first kind", *Math. Inequal. Appl.* **20** (2017), no. 3, p. 729-735.
- [31] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, "Monotonicity properties and bounds involving the complete elliptic integrals of the first kind", *Math. Inequal. Appl.* **21** (2018), no. 4, p. 1185-1199.
- [32] Z.-H. Yang, W.-M. Qian, Y.-M. Chu, W. Zhang, "On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind", *J. Math. Anal. Appl.* **462** (2018), no. 2, p. 1714-1726.
- [33] Z.-H. Yang, W.-M. Qian, W. Zhang, Y.-M. Chu, "Notes on the complete elliptic integral of the first kind", *Math. Inequal. Appl.* **23** (2020), no. 1, p. 77-93.
- [34] L. Yin, X.-L. Lin, F. Qi, "Monotonicity, convexity and inequalities related to complete  $(p, q, r)$ -elliptic integrals and generalized trigonometric functions", *Publ. Math.* **97** (2020), no. 1-2, p. 181-199.
- [35] L. Yin, F. Qi, "Some inequalities for complete elliptic integrals", *Appl. Math. E-Notes* **14** (2014), p. 193-199.
- [36] X.-H. Zhang, "Monotonicity and functional inequalities for the complete  $p$ -elliptic integrals", *J. Math. Anal. Appl.* **453** (2017), no. 2, p. 942-953.