Chandrashekhar B. Khare and Michael Larsen

**Abelian varieties with isogenous reductions**


<https://doi.org/10.5802/crmath.129>
Number Theory / Théorie des nombres

Abelian varieties with isogenous reductions

Chandrashekhar B. Khare⁎,a and Michael Larsenb

a UCLA Department of Mathematics, Box 951555, Los Angeles, CA 90095, USA
b Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
E-mails: shekhar@math.ucla.edu (C. B. Khare), mjlarsen@indiana.edu (M. Larsen)

Abstract. Let A1 and A2 be abelian varieties over a number field K. We prove that if there exists a non-trivial morphism of abelian varieties between reductions of A1 and A2 at a sufficiently high percentage of primes, then there exists a non-trivial morphism A1 → A2 over ¯K. Along the way, we give an upper bound for the number of components of a reductive subgroup of GLn whose intersection with the union of Q-rational conjugacy classes of GLn is Zariski-dense. This can be regarded as a generalization of the Minkowski–Schur theorem on faithful representations of finite groups with rational characters.

Résumé. Soient A1 et A2 deux variétés abéliennes sur un corps de nombres K. Nous montrons que, s’il existe un morphisme non trivial de variétés abéliennes entre réductions de A1 et A2 pour une proportion suffisamment grande d’idéaux premiers, il existe un morphisme non trivial A1 → A2 sur ¯K. Nous donnons également une majoration du nombre des composantes d’un sous-groupe réductif de GLn dont l’intersection avec l’union des classes de conjugaison Q-rationnelles de GLn est dense pour la topologie de Zariski; c’est une généralisation d’un théorème de Minkowski–Schur sur les représentations fidèles des groupes finis à caractère rationnel.

Funding. Michael Larsen was partially supported by NSF grant DMS-2001349.

In this note, we answer a recent question of Dipendra Prasad and Ravi Raghunathan [6, Remark 1]. We are grateful to Dipendra Prasad and Jean-Pierre Serre for helpful correspondence. We would also like to thank the referee for several improvements and corrections.

Let K be a number field and A1 and A2 abelian varieties over K. If ϕ is a prime of K, we denote by kϕ the residue field of ϕ. If ϕ is a prime of good reduction for A1, we denote by A1ϕ the reduction and by Frobϕ the Frobenius element regarded as an automorphism, well defined up to conjugacy, of the ℓ-adic Tate module of A1 or, dually, of H1(A1, Z/ℓ).

Theorem 1. Let A1 and A2 be abelian varieties over a number field K. Suppose that for a density one set of primes ϕ of K, there exists a non-trivial morphism of abelian varieties over ¯kϕ from A1ϕ to A2ϕ. Then there exists a non-trivial morphism of abelian varieties from A1 to A2 defined over ¯K.

* Corresponding author.
Let $G$ be a connected reductive algebraic group over an algebraically closed field $F$ of characteristic 0, and let $V$ be a finite dimensional representation of $G$. Let $T$ be a maximal torus of $G$ and $W$ the Weyl group of $G$ with respect to $T$. If $V$ is irreducible, we say it is minuscule if $W$ acts transitively on the weights of $V$ with respect to $T$. The highest weight of $V$ with respect to any choice of Weyl chamber has multiplicity 1, so every element of the Weyl orbit has multiplicity one.

For general finite dimensional representations $V$, we say $V$ is minuscule if each of its irreducible factors is so. Regarding the character of a representation $V$ as a function $f_V$ from $W$-orbits in $X^* (T)$ to non-negative integers, when $V$ is minuscule, for any dominant weight $\lambda$, the multiplicity in $V$ of the irreducible $G$-representation $V_\lambda$ with highest weight $\lambda$ is the value of $f_V$ on the $W$-orbit containing $\lambda$.

**Proposition 2.** Let $V_1$ and $V_2$ be minuscule representations of $G$. If $\dim \text{Hom}_T(V_1, V_2) > 0$, then $\dim \text{Hom}_G(V_1, V_2) > 0$.

**Proof.** If $\dim \text{Hom}_T(V_1, V_2) > 0$, then $V_1$ and $V_2$ must have a common $T$-irreducible factor, and that means they have a common weight $\chi$ with respect to $T$. If $\lambda$ is the dominant weight in the orbit of $\chi$, then $V_1$ and $V_2$ each contain $V_\lambda$ as a subrepresentation, so $\dim \text{Hom}_G(V_1, V_2) > 0$. □

Now let $A_1$ and $A_2$ denote abelian varieties over a number field $K$ with absolute Galois group $G_K := \text{Gal}(\overline{K}, K)$. Let $\ell$ be a fixed rational prime, and let $F = \overline{\mathbb{Q}}_\ell$. Let $V_i = H^1(A_i, F)$, regarded as $G_K$-modules. Let $V_{12} := V_1 \oplus V_2$ as $G_K$-module and $G_{12}$ the Zariski closure of $G_K$ in $\text{Aut}_F(V_{12})$. By the semisimplicity of Galois representations defined by abelian varieties [3], $G_{12}$ is reductive. Let $G$ denote the identity component $G_{12}$.

**Proposition 3.** There exists a positive density set of primes $\wp$ of $K$ such that $A_1 \times A_2$ has good reduction at $\wp$, and $\text{Frob}_\wp$ generates a Zariski dense subgroup of a maximal torus of $G$.

**Proof.** The condition that $\text{Frob}_\wp$ lies in the identity component $G$ has density $[G_{12} : G]^{-1} > 0$. By a theorem of Serre [4, Theorem 1.2], there exists a proper closed, conjugation-stable subvariety $X$ of $G$ such that $\text{Frob}_\wp \in G \setminus X$ implies that $\text{Frob}_\wp$ generates a Zariski-dense subgroup of a maximal torus of $G$. However, by a second theorem of Serre [8, Théorème 10], the set of $\wp$ such that $\text{Frob}_\wp \in X$ has density 0. □

We can now prove the main Theorem 1.

**Proof.** A well-known theorem of Tate [11] asserts that the existence of a non-trivial $\mathbb{F}_q$-morphism between abelian varieties over $\mathbb{F}_q$ is equivalent to the existence of a $\text{Frob}_q$-stable morphism of their $\ell$-adic Tate modules. By the easy direction of this result, the existence of a non-trivial morphism defined over $\mathbb{F}_q$ implies the existence of a $\text{Frob}_q^m$-stable morphism of their Tate modules for some positive integer $m$.

By Proposition 3, the hypothesis of the Theorem 1 therefore implies that

$$\dim \text{Hom}(V_1, V_2)_{\text{Frob}_\wp^m} > 0$$

for some prime $\wp$ for which $\text{Frob}_\wp$ generates a Zariski-dense subgroup of a maximal torus $T$ of $G$ and some positive integer $m$. As $T$ is connected, $\text{Frob}_\wp^m$ likewise generates a Zariski-dense subgroup of $T$. Thus $\dim \text{Hom}_T(V_1, V_2) > 0$. By a theorem of Pink [5, Corollary 5.11], the $G$-representations $V_1$ and $V_2$ are minuscule. Thus Proposition 2 implies that $\dim \text{Hom}_G(V_1, V_2)$ > 0. Finally, Faltings’ proof of Tate’s Conjecture [3] implies $\text{Hom}_K(A_1, A_2)$ is non-zero. □

**Remark 4.** One might ask whether there exists a non-trivial homomorphism $A_1 \to A_2$ defined over $K$ itself if for a density one set of $\wp$ there exists a non-trivial $k_\wp$-homomorphism $A_{1, \wp} \to A_{2, \wp}$. D. Prasad pointed out the following counterexample to us. Let $E$ be an elliptic curve over $\mathbb{Q}$ which does not have complex multiplication. Let $E_n$ denote the quadratic twist of $E$ by $n \in \mathbb{Q}^\times$. Let
$A_1 = E$, $A_2 = E_2 \times E_3 \times E_6$. For every rational prime $p > 3$, either 2, 3, or 6 lies in $\mathbb{F}_p^\times$, so if $E$ has good reduction at $p$, the same is true for both $A_1$ and $A_2$, and there exists an $\mathbb{F}_p$-isomorphism from $(A_1)_p$ to at least one of $(E_2)_p$, $(E_3)_p$, and $(E_6)_p$, and therefore a non-trivial $\mathbb{F}_p$-homomorphism to $(A_2)_p$. On the other hand, there is no $\mathbb{Q}$-isogeny from $A_1$ to any one of $E_2$, $E_3$, or $E_6$, and therefore no non-trivial $\mathbb{Q}$-homomorphism to $A_2$.

We can prove a stronger version of Theorem 1 in analogy with the theorem of C. S. Rajan [7].

**Theorem 5.** Let $n$ be a positive integer. If $A_1$ and $A_2$ are abelian varieties of dimension $\leq n$ over a number field $K$ and the set of primes $\mathfrak{p}$ of $K$ for which there exists a non-trivial $\mathbb{F}_\mathfrak{p}$-morphism of abelian varieties from $A_1|_{\mathbb{F}_\mathfrak{p}}$ to $A_2|_{\mathbb{F}_\mathfrak{p}}$ has upper density $> 1 - e^{-6n^2}$, then there exists a non-trivial $\bar{K}$-morphism of abelian varieties from $A_1$ to $A_2$.

The only additional ingredient necessary to prove Theorem 5 is an upper bound, depending only on $n$, on the number of components of $G_{12}$. This is an immediate consequence of the following theorem.

**Theorem 6.** Let $n$ be a positive integer, $F$ a field of characteristic 0, and $G \subset \text{GL}_n$ a reductive $F$-subgroup. If the set of $\bar{F}$-points of $G$ consisting of matrices whose characteristic polynomials lie in $\mathbb{Q}[x]$ is Zariski-dense, then $|G/G^0| < e^{6n^2} n!2^{2n}$.

We remark that without the rationality assumption, this statement fails even for $n = 1$, where $G$ could be an arbitrarily large cyclic group.

**Proof.** The locus of $\bar{F}$-points of $G$ whose characteristic polynomials lie in $\mathbb{Q}[x]$ is $G_{\bar{F}}$-stable, so the Zariski-closure does not change when the base field is changed from $F$ to $\bar{F}$. This justifies assuming that $F$ is algebraically closed.

We can write $G^0 = DZ^0$, where $D$ and $Z := Z(G^0)$ are the derived group and the center of $G^0$ respectively. By [10, Corollary 2.14], the outer automorphism group of $D$ is contained in the automorphism group of the Dynkin diagram $\Delta$ of $D$. Every automorphism of $\Delta$ preserves the set of isomorphic components. We claim that $|\text{Aut}(\Delta)| \leq n!$. It suffices to prove this when $\Delta$ consists of $m$ mutually isomorphic connected diagrams $\Delta_0$ of rank $r = n/m$. The claim obviously holds when $r = 1$. It is easily verified for $n \leq 4$. For $n \geq 5$, the classification of connected Dynkin diagrams gives $|\text{Aut}(\Delta_0)|^{2/r} \leq \sqrt{6} < n/2$, so if $r \geq 2$,

$$|\text{Aut}(\Delta)| = |\text{Aut}(\Delta_0)|^{n/r} (n/r)! < (n/2)^{n/2} [n/2]! < n!.$$

Any automorphism of $G^0$ is determined by its restrictions to the characteristic subgroups $D$ and $Z^0$. An automorphism which is inner on $D$ and trivial on $Z^0$ is inner. Thus, the homomorphism $\text{Aut}(G^0) \to \text{Aut}(D) \times \text{Aut}(Z^0)$ gives an injective homomorphism

$$\text{Out}(G^0) \to \text{Out}(D) \times \text{Aut}(Z^0) = \text{Out}(D) \times \text{GL}_k(Z),$$

where $k = \dim Z^0 \leq n$. By Minkowski’s theorem [9, Theorem 9.1], every finite subgroup of $\text{GL}_k(Z)$ has order at most

$$M(k) := \prod_p \left\lfloor \frac{k}{(p-1)p^2} \right\rfloor.$$

We have

$$\log M(k) \leq \sum_{p=2}^{k+1} \frac{kp\log p}{(p-1)^2} = k \sum_{i=1}^k \frac{(i+1)\log(i+1)}{i^2} \leq 2k^2,$$

since $(i+1)\log(i+1) \leq 2i^2$ for all $i \geq 1$. Thus, any finite subgroup of $\text{Out}(G^0)$ has order $\leq n!e^{2kn^2}$. 

C. R. Mathématique. 2020, 358, no 9-10, 1085-1099
The conjugation action on \( G^o \) defines a homomorphism \( G/G^o \to \text{Out}(G^o) \). Let \( \Gamma_0 \) denote the kernel of this homomorphism and \( G_0 \) the inverse image of \( \Gamma_0 \) in \( G \). Thus, the index of \( \Gamma_0 \) in the component group \( G/G^o \) is \( \leq n!e^{2n^2} \leq e^{3n^2} \). Arguing by contradiction, we may assume the order of \( \Gamma_0 \) is at least 
\[
e^{-3n^2} \left| \frac{G}{G^o} \right| \geq e^{3n^2} n^{2n}.
\]

Let \( \Gamma := Z_{G_0}(G^o)/Z^o \), so \( \Gamma_0 \cong Z_{G_0}(G^o)/Z \) is a quotient group of \( \Gamma \). Consider the short exact sequence
\[
0 \to Z^o \to Z_{G_0}(G^o) \to \Gamma \to 0.
\]
The extension class \( \alpha \in H^2(\Gamma, Z^o) \) is annihilated by \( N := |\Gamma| \). As \( Z^o \cong (F^o)^k \) is a divisible group, it follows that the extension class \( \alpha \) lies in the image of \( H^2(\Gamma, Z^o[N]) \), where \( Z^o[N] \) denotes the kernel of the \( N \)th power map on \( Z^o \). We can therefore represent \( \alpha \) by a 2-cocycle with values in \( Z^o[N] \). This means that there exists a set-theoretic section \( i: \Gamma \to Z_{G_0}(G^o) \) such that the associated 2-cocycle takes values in \( Z^o[N] \), and it follows that \( \Gamma_0 := Z^o[N]i(\Gamma) \) is a finite subgroup of \( Z_{G_0}(G^o) \subset G \) which maps onto \( \Gamma \) and therefore onto \( \Gamma_0 \).

By Jordan’s theorem, \( \Gamma_0 \) contains an abelian normal subgroup \( \tilde{A}_0 \) of index \( \leq J(n) \), a constant depending only on \( n \). The optimal Jordan constant has been computed by Michael Collins [2], and for all \( n \), we have \( J(n) \leq e^{2n^2} \). Indeed, for \( n \geq 71 \), the bound, \((n+1)!\), is given by Theorem A, and
\[
(n+1)! < (n+1)^n < \left( e^n \right)^n < 2^{n^2}.
\]
For \( 20 \leq n \leq 70 \) and \( n \leq 19 \), the bounds are given by Theorems B and D respectively, and they can be checked by machine to be less than \( e^{2n^2} \) in every case.

Let \( T \) be a maximal torus of \( G^o \), so \( \tilde{A}_0 T \) is a commutative subgroup of \( G_0 \). As
\[
\tilde{A}_0 \cap T < \tilde{A}_0 \cap G^o = \ker \tilde{A}_0 \to \Gamma_0,
\]
we have
\[
|\tilde{A}_0 T/T| = |\tilde{A}_0/\tilde{A}_0 \cap T| \geq |\text{Im} \tilde{A}_0 \to \Gamma_0| \geq \frac{|\Gamma_0|}{e^{2n^2}} \geq e^{n^2} n^{2n}.
\]
Therefore, if \( M := e^n n^{2n} \), then \( \tilde{A}_0 T \) has at least \( M^n \) components. Since \( \tilde{A}_0 T/T \) is a quotient group of \( \tilde{A}_0 \subset \text{GL}_n(F) \), it contains no elementary \( p \)-group of rank \( > n \), so it must have an element of order \( \geq M \). Let \( g \in \tilde{A}_0 \) map to such an element.

By hypothesis, there exists \( t \in G^o \times [g] \) such that the characteristic polynomial of \( gt \) has coefficients in \( Q \). We can further assume that \( t \) is semisimple, so we can choose our maximal torus \( T \) to contain \( t \). Let \( T' = \langle g \rangle T \). Every element of \( T' \) is the product of two commuting elements, one of which is of finite order, and one which belongs to \( T \), so both are semisimple, from which it follows that their product is semisimple. Thus \( T' \) is diagonalizable, so it is a closed subgroup of a maximal torus of \( \text{GL}_n \) [1, Proposition 8.4]. Without loss of generality, we may assume this maximal torus is the group \( \text{GL}_n^o \) of invertible diagonal matrices.

The contravariant functor taking an algebraic group to its character group gives an equivalence of categories between diagonalizable groups and finitely generated abelian groups [1, Proposition 8.12]. In particular, there is a bijective correspondence between subgroups \( \Lambda \subset \mathbb{Z}^n \) and closed subgroups \( D_\Lambda \) of the group \( \text{GL}_n^o \) of diagonal matrices in \( \text{GL}_n \), where
\[
D_\Lambda = \{(x_1, \ldots, x_n) \in \text{GL}_n^o | \Lambda(x_1, \ldots, x_n) = 1 \ \forall \ \Lambda \in \Lambda \}.
\]

Let \( \Lambda \) be the subgroup of \( \mathbb{Z}^n \) such that \( D_\Lambda = T \) and \( \Lambda' \) the subgroup such that \( D_{\Lambda'} = T' \). The inclusion \( T \to T' \) corresponds to the surjection \( \mathbb{Z}^n/\Lambda' \to \mathbb{Z}^n/\Lambda \) and thus to the inclusion \( \Lambda' \subset \Lambda \). As \( T'/T \) is cyclic, \( \Lambda/\Lambda' \) is cyclic of the same order \( k \). Let \( \hat{\lambda} \in \Lambda \) map to a generator of \( \Lambda/\Lambda' \). Then the smallest integer \( m \) such that \( \hat{\lambda}((gt)^m) = 1 \) is the smallest such that \( \hat{\lambda}(g^m) = 1 \), which is \( k \).

Writing \( gt = (x_1, \ldots, x_n) \in \text{GL}_1(F)^n \subset \text{GL}_n(F) \), the \( x_i \) are the eigenvalues of \( gt \), so they all lie in some Galois extension of \( Q \) of degree \( \leq n! \). Therefore \( \hat{\lambda}(gt) \) lies in this extension. Since it is a
primitive $k$th root of unity, this implies $\phi(k) \leq n!$. Now $\phi(q) \geq \sqrt{q}$ for all prime powers $q$ except 2, and it follows from the multiplicativity of $\phi$ that $\phi(k) \geq \sqrt{k}/2$ for all $k \geq 1$, so $M \leq k \leq 2n!^2$, which is a contradiction. □

References