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Chern characters in equivariant basic cohomology

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Abstract. The purpose of this Note is to establish a geometric realization of the cohomological isomorphism in the case of a transversely oriented Killing foliation on a compact smooth manifold through equivariant basic Chern characters.

Résumé. L'objet de cette Note est d'établir une réalisation géométrique de l'isomorphisme cohomologique dans le cas d'un feuilletage de Killing transversalement orienté sur une variété compacte à travers les caractères de Chern basiques équivariants.

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1. Introduction

Let $(M,\mathscr{F})$ be a smooth compact manifold equipped with a Riemannian foliation $\mathscr{F}$. Let $E = E^+ \oplus E^-$ be a foliated vector bundle over M and $D_b^{E+} : C^\infty_{\mathscr{F}\text{-bas}}(M,E^+) \to C^\infty_{\mathscr{F}\text{-bas}}(M,E^-)$ be a transversely elliptic basic differential operator on basic sections. Thus, we can consider its basic index $\mathrm{Index}_b(D_b^{E+}) \in \mathbb{Z}$, and a natural question emerges: how to obtain a cohomological formula as Atiyah–Singer index theorem for this invariant. Some (partial) responses have been proposed in the literature for this problem [1,3,5]. In the 1990s, El Kacimi proposed to tackle this problem by using Molino's theory. Let us recall the main lines of this proposal. To each transversely oriented Riemannian foliation $(M,\mathscr{F})$ of codimension q, Molino associated an oriented manifold W equipped with an action of the orthogonal group $SO(q)$. In this setting, the space of the leaf closures of the foliation $\mathscr{F}$ exhibits a natural identification with the quotient $W/SO(q)$. In [3], El Kacimi constructed a $SO(q)$-equivariant bundle $\mathscr{E}$ over W. Brüning, Kamber and Richardson [1] studied the basic Dirac operator D_b^{E+} over a basic Clifford bundle. An $SO(q)$-transversely elliptic operator $\mathscr{D}^{\mathscr{E}}$ is associated such that $\mathrm{Index}_b(\mathscr{D}_b^{E+})$ is equal to the index of $\mathscr{D}^{\mathscr{E}}$ when restricted to invariant sections. Leveraging this valuable identity, we investigate the acquisition of the $\mathrm{Index}_b(\mathscr{D}_b^{E+})$ information from that of $\mathscr{D}^{\mathscr{E}}$. And in turn, to obtain an index formula by results in [9].

Let $H^\bullet(M,\mathscr{F})$ be the basic de Rham cohomology of the foliated manifold $(M,\mathscr{F})$. If $(M,\mathscr{F})$ is a Killing foliation (a priori Riemannian), there is an abelian Lie algebra $\mathfrak{a}$, which acts transversely on $(M,\mathscr{F})$. The orbit of the leaves of the foliation $\mathscr{F}$ under the action of $\mathfrak{a}$ is exactly the leaf closure. Let $H^\bullet_{\mathfrak{a}}(M,\mathscr{F})$ be the $\mathfrak{a}$-equivariant basic de Rham cohomology with polynomial coefficients. Goertsches and Töben [4] proved a cohomological isomorphism $H^\bullet_{\mathfrak{a}}(M,\mathscr{F}) \simeq H^\bullet_{so(q)}(W)$. Duflo and Vergne extended the equivariant cohomology to the case of C^∞-coefficients in [2]. In this Note, we first present an assumption for the existence of a basic connection and proceed to introduce the definition of the equivariant basic Chern character. Then, we establish a geometric realization of the cohomological isomorphism through equivariant basic Chern characters.

2. Equivariant basic Chern character

We use the notation presented in Section 1 in the following definitions.

Definition 1. *A foliated principal bundle $(P,\mathscr{F}_P)$ is a principal bundle P equipped with a foliation $\mathscr{F}_P$ such that*

(1) *$\mathscr{F}_P$ is invariant under the action of the structure group of P, and*
(2) *the projection of $\mathscr{F}_P$ on M is $\mathscr{F}$ and at each point of P, the projection is an isomorphism.*

Let $(E,\mathscr{F}_E)$ be a foliated hermitian bundle over $(M,\mathscr{F})$. In order to define the basic Chern character $\mathrm{Ch}(E,\mathscr{F}_E) \in H^\bullet(M,\mathscr{F})$, $(E,\mathscr{F}_E)$ must be associated with a foliated principal bundle $(P,\mathscr{F}_P)$. Let ω be a connection on P. We denote by $\mathfrak{X}(\mathscr{F}_P)$ the Lie algebra of the vector fields on M that are tangent to $\mathscr{F}_P$. If $\mathfrak{X}(\mathscr{F}_P)$ belongs to the kernel of ω, ω is said to be adapted to $\mathscr{F}_P$. By [6], an adapted connection always exists. If ω is invariant with respect to $\mathfrak{X}(\mathscr{F}_P)$ under the Lie derivative, i.e., $\mathscr{L}_Z\omega = 0, \forall\, Z \in \mathfrak{X}(\mathscr{F}_P)$, it is a *basic connection*. That is the case for which $(P,\mathscr{F}_P)$ is a principal $\mathscr{F}$-bundle, introduced in [3, 6]. However, in general, the existence of a basic connection cannot be guaranteed; a counterexample is presented in [7, Section 4.1.1].

Example 2. Let $M = S^1 \times S^1$ and $P = M \times S^1$. The coordinates are denoted by $(x, y, z) \in M \times S^1$, and the foliation $\mathscr{F}$ on M is given by $\partial/\partial x$. Let $f \in C^\infty(S^1)$ be a non-constant real function. The foliation $\mathscr{F}_P$ is generated by $V = \partial/\partial x + f(y)\partial/\partial z$.

Let $\mathrm{Hol}(M,\mathscr{F})$ be the holonomy groupoid of $(M,\mathscr{F})$. If a vector bundle E is equipped with a groupoid action of $\mathrm{Hol}(M,\mathscr{F})$, it is $\mathrm{Hol}(M,\mathscr{F})$-equivariant. Henceforth, we fix $P = U(E)$ as the unitary frame bundle of E. Naturally, P is also equipped with an action of $\mathrm{Hol}(M,\mathscr{F})$. The groupoid action generates a foliation $\mathscr{F}_P$ on P such that $(P,\mathscr{F}_P)$ is a foliated principal bundle. Unfortunately, under these conditions, we also have a counterexample [7, Section 4.2.1]. Therefore, an assumption is necessary to progress further in our work. Let $\mathrm{Hol}(M,\mathscr{F})^{\mathscr{T}}_{\mathscr{T}}$ be the étale holonomy groupoid restricted on a complete transversal section $\mathscr{T}$ of $(M,\mathscr{F})$. In [5], Gorokhovsky and Lott defined its closure $\overline{\mathrm{Hol}(M,\mathscr{F})^{\mathscr{T}}_{\mathscr{T}}}$. Respectively, these restricted bundles on $\mathscr{T}$ are denoted by $E|_{\mathscr{T}}$ and $P|_{\mathscr{T}}$. We state the assumption as follows.

Assumption 3. *The groupoid action of* $\mathrm{Hol}(M,\mathscr{F})^{\mathscr{T}}_{\mathscr{T}}$ *on* $E|_{\mathscr{T}}$ *extends to a groupoid action of* $\overline{\mathrm{Hol}(M,\mathscr{F})^{\mathscr{T}}_{\mathscr{T}}}$ *on* $E|_{\mathscr{T}}$.

Under Assumption 3, we can construct a basic connection ω_P on P and the basic Chern character is well-defined. This result is detailed in Section 4. Let $l(M,\mathscr{F})$ be the Lie algebra of the transverse fields on $(M,\mathscr{F})$. Assume that $(M,\mathscr{F})$ is a Killing foliation. In this setting, the leaf closure $\overline{\mathscr{F}}$ is given by the action of $\mathfrak{a}$, i.e., $\mathfrak{a}\cdot\mathscr{F} = \overline{\mathscr{F}}$, see [4]. We consider the Lie algebroid of $\overline{\mathrm{Hol}(M,\mathscr{F})^{\mathscr{T}}_{\mathscr{T}}}$. By [5], it is trivialisable to $\mathscr{T} \times \mathfrak{a}$. Let us consider the infinitesimal version of the action of $\overline{\mathrm{Hol}(M,\mathscr{F})^{\mathscr{T}}_{\mathscr{T}}}$ on $P|_{\mathscr{T}}$. We obtain a transverse action $\mathfrak{a} \to l(P,\mathscr{F}_P)^G$ (i.e., a Lie homomorphism) and a commutative diagram of Lie homomorphisms, where G denotes the structure group of P and $(-)^G$ denotes the G-invariant transverse fields.

$$\begin{tikzcd} & l(P,\mathscr{F}_P)^G \arrow[d] \\ \mathfrak{a} \arrow[ur] \arrow[r] & l(M,\mathscr{F}) \end{tikzcd} \tag{1}$$

The diagram (1) is essential for the definition of the equivariant basic Chern character. Additionally, the basic connection ω_P must be compatible with the transverse action $\mathfrak{a} \to l(P,\mathscr{F}_P)^G$. This compatibility is guaranteed because they all result from the action of $\overline{\mathrm{Hol}(M,\mathscr{F})}^{\mathcal{T}}_{\mathcal{T}}$. Next, the associated bundle E is considered, we have the following definition.

Definition 4. *The $\mathfrak{a}$-equivariant basic Chern character associated with the vector $\mathscr{F}$-bundle E is defined by the following relation*

$$\mathrm{Ch}_{\mathfrak{a}}(E,\mathscr{F}_E) = \left[\mathrm{Ch}_{\mathfrak{a}}(E,\mathscr{F}_E,,\nabla^E,Y)\right] \in H^{\infty}_{\mathfrak{a}}(M,\mathscr{F})$$

where, $\mathrm{Ch}_{\mathfrak{a}}(E,\mathscr{F}_E,\nabla^E,Y) = \mathrm{Tr}\left(e^{-(R^E+\mu^E(Y))}\right)$ *with R^E representing the curvature of ∇^E and μ^E representing the moment map on E and $Y \in \mathfrak{a}$.*

3. Our main result

The main result presented in [7] is as follows.

Theorem 5. *Each* $\mathrm{Hol}(M,\mathscr{F})$*-equivariant hermitian bundle satisfying Assumption 3 is a vector $\mathscr{F}$-bundle.*

Theorem 6. *Let $(M,\mathscr{F})$ be a transversely oriented Killing foliation of codimension q and $E \to (M,\mathscr{F})$ be a hermitian* $\mathrm{Hol}(M,\mathscr{F})$*-equivariant vector bundle. Under Assumption 3, we can associate a $SO(q)$-equivariant vector bundle $\mathscr{E} \to W$ to E such that we have a geometric realization of cohomological isomorphism through equivariant basic Chern characters.*

$$H_{\mathfrak{a}}(M,\mathscr{F}) \simeq H_{so(q)}(W), \quad \mathrm{Ch}_{\mathfrak{a}}(E,\mathscr{F}_E) \simeq \mathrm{Ch}_{so(q)}(\mathscr{E}). \tag{2}$$

4. Proof of Theorem 5

We use the notation presented in Section 2 in this proof. Evidently, the holonomy groupoid action on E induces a corresponding action on P. Under Assumption 3, the $\mathrm{Hol}(M,\mathscr{F})^{\mathcal{T}}_{\mathcal{T}}$-action on $P|_{\mathcal{T}}$ extends to a $\overline{\mathrm{Hol}(M,\mathscr{F})}^{\mathcal{T}}_{\mathcal{T}}$-action on $P|_{\mathcal{T}}$. It is enough to construct a basic connection on P. We take an adapted connection ω on $(P,\mathscr{F}_P)$ and restrict it on $P|_{\mathcal{T}}$, denoted by $\omega_{\mathcal{T}}$. By [5], the groupoid $\overline{\mathrm{Hol}(M,\mathscr{F})}^{\mathcal{T}}_{\mathcal{T}}$ is proper. By [10], every proper groupoid admits a Haar system and a "cut-off" function. The cross-product groupoid, denoted by $\mathscr{G}_P$, is $\overline{\mathrm{Hol}(M,\mathscr{F})}^{\mathcal{T}}_{\mathcal{T}} \ltimes P_{\mathcal{T}}$. The key step is to average $\omega_{\mathcal{T}}$ by $\mathscr{G}_P$. We lift a Haar system and a "cut-off" function on $\overline{\mathrm{Hol}(M,\mathscr{F})}^{\mathcal{T}}_{\mathcal{T}}$ to $\mathscr{G}_P$, denoted by $\mu^{\mathscr{G}_P}$ and by $\varphi^{\mathscr{G}_P}$. We define

$$(\widehat{\omega_{\mathcal{T}}})|_p = \int_{\gamma\in(\mathscr{G}_P)_p} \gamma^*(\omega_{\mathcal{T}}|_{r(\gamma)})\varphi^{\mathscr{G}_P}\big(r(\gamma)\big)\,\mathrm{d}\mu_p^{\mathscr{G}_P}(\gamma), \quad \forall\, p \in P_{\mathcal{T}}. \tag{3}$$

Then, by the holonomy of $\mathscr{F}_P$, we construct a connection ω_P on P from $\widehat{\omega_{\mathcal{T}}}$. This connection is invariant with respect to $\mathfrak{X}(\mathscr{F}_P)$, so it is basic.

5. Proof of Theorem 6

With the construction of the basic connection, we establish constructions and calculations on principal bundles for the case in which vector bundles are always associated. By Molino's theory [8], the transversely oriented frame bundle $\widetilde{M}$ of $(M,\mathscr{F})$ is naturally equipped with a tranversely parallelizable (T.P.) foliation $\widetilde{\mathscr{F}}$. In addition, $(\widetilde{M},\widetilde{\mathscr{F}})$ is a principal $\mathscr{F}$-bundle of structure group $SO(q)$ equipped with the Bott connection. The space of the leaf closures $W = \widetilde{M}/\overline{\widetilde{\mathscr{F}}}$ is a manifold. By [4, Proposition 4.9] [2], the cohomological isomorphism is given by $H^{\infty}_{\mathfrak{a}}(M,\mathscr{F}) \to H^{\infty}_{\mathfrak{a}\times so(q)}(\widetilde{M},\widetilde{\mathscr{F}})$ and $H^{\infty}_{\mathfrak{a}\times so(q)}(\widetilde{M},\widetilde{\mathscr{F}}) \to H^{\infty}_{so(q)}(W)$. The first isomorphism is the inclusion, and the second one is the equivariant Chern–Weil map $\mathrm{CW}_{so(q)}$. We prove Theorem 6 with two steps below:

First step. Consider the pullback bundle $\widetilde{P}$ of P over $(\widetilde{M},\widetilde{\mathscr{F}})$. We also lift the foliation $\mathscr{F}_P$ and the basic connection ω_P to $\widetilde{P}$, denoted by $\mathscr{F}_{\widetilde{P}}$ and $\omega_{\widetilde{P}}$, respectively. Under Assumption 3, the action of $\overline{\mathrm{Hol}(M,\mathscr{F})}^{\mathcal{T}}_{\mathcal{F}}$ can be lifted to $\widetilde{P}$. The lifted groupoid action induces a transverse action $\mathfrak{a} \to l(\widetilde{P},\mathscr{F}_{\widetilde{P}})^{G\times SO(q)}$. Analogous to Definition 4, the $\mathfrak{a}\times so(q)$-equivariant basic Chern character $\mathrm{Ch}_{\mathfrak{a}\times so(q)}(\widetilde{E},\mathscr{F}_{\widetilde{E}}) \in H^{\infty}_{\mathfrak{a}\times so(q)}(\widetilde{M},\widetilde{\mathscr{F}})$ is well-defined where $\widetilde{E}$ is the pullback bundle of E. Naturally, $\mathrm{Ch}_{\mathfrak{a}\times so(q)}(\widetilde{E},\mathscr{F}_{\widetilde{E}})$ is the inclusion of $\mathrm{Ch}_{\mathfrak{a}}(E,\mathscr{F}_E)$ as we do the constructions by the "pullback" approach.

Second step. The lifted foliation $(\widetilde{P},\mathscr{F}_{\widetilde{P}})$ is T.P., just as $(\widetilde{M},\widetilde{\mathscr{F}})$. Again, by Molino's theory, the space of the leaf closure $\widetilde{W} = \widetilde{P}/\overline{\mathscr{F}_{\widetilde{P}}}$ is a manifold. The key step is to prove that $\overline{\mathscr{F}_{\widetilde{P}}}$ is also given by the action of $\mathfrak{a}$, i.e., $\mathfrak{a}\cdot\mathscr{F}_{\widetilde{P}} = \overline{\mathscr{F}_{\widetilde{P}}}$. Furthermore, the foliation $\overline{\mathscr{F}_{\widetilde{P}}}$ has no torsion, see [7, Proposition 5.3.2], which infers that G acts freely on $\widetilde{W}$. Thus, $\widetilde{W} \to W$ is a principal bundle (see the diagram below).

$$
\begin{tikzcd}
(\widetilde{P},\mathscr{F}_{\widetilde{P}}) \arrow[dr, "\widetilde{\omega}", "\mathfrak{a}\times SO(q)"'] & & & \widetilde{W} = \widetilde{P}/\overline{\mathscr{F}_{\widetilde{P}}} \arrow[dl, "\overline{\omega}"', "\circlearrowleft SO(q)"] \\
& (\widetilde{M},\widetilde{\mathscr{F}}) \arrow[r] \arrow[d, "/SO(q)"] & W = \widetilde{M}/\overline{\widetilde{\mathscr{F}}} & \\
(P,\mathscr{F}_P) \arrow[r, "\omega_P", "\mathfrak{a}"'] & (M,\mathscr{F}) & &
\end{tikzcd}
$$

As can be seen in the diagram, the connection $\widetilde{\omega}$ is a modification of the lifted connection by adding a term of the action of $\mathfrak{a}$ such that it is $\overline{\mathscr{F}_{\widetilde{P}}}$-basic. By quotient, we have an $SO(q)$-invariant connection $\overline{\omega}$ on $\widetilde{W}$. Thus, $\mathscr{E}$ is the associated bundle $\widetilde{W}\times_G \mathbb{C}^N$ with $G = U(N)$ and N the rank of E. Hence, the $so(q)$-equivariant Chern character $\mathrm{Ch}_{so(q)}(\mathscr{E}) \in H^{\infty}_{so(q)}(W)$ is well-defined.

We have explicitly calculated the results to show that $\mathrm{Ch}_{so(q)}(\mathscr{E}) = \mathrm{CW}_{so(q)}\big(\mathrm{Ch}_{\mathfrak{a}\times so(q)}(\widetilde{E},\mathscr{F}_{\widetilde{E}})\big)$. We note that E and $\mathscr{E}$ have the same rank because $\mathscr{E}$ is the associated bundle. Therefore, the proof of Theorem 6 is complete.

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