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A nonlinear Korn inequality on a surface with an explicit estimate of the constant

Une inégalité de Korn non linéaire sur une surface avec une majoration explicite de la constante

Maria Malin [a] **and Cristinel Mardare** [*, b]

[a] Department of Mathematics, University of Craiova, Craiova, Romania
[b] Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong
E-mails: malinmaria@yahoo.com (M. Malin), cmardare@cityu.edu.hk (C. Mardare)

Abstract. A nonlinear Korn inequality on a surface estimates a distance between a surface $\boldsymbol{\theta}(\omega)$ and another surface $\boldsymbol{\phi}(\omega)$ in terms of distances between their fundamental forms in the space $L^p(\omega)$, $1 < p < \infty$.

We establish a new inequality of this type. The novelty is that the immersion $\boldsymbol{\theta}$ belongs to a specific set of mappings of class $\mathscr{C}^1$ from $\overline{\omega}$ into $\mathbb{R}^3$ with a unit vector field also of class $\mathscr{C}^1$ over $\overline{\omega}$.

Résumé. Une inégalité de Korn non linéaire sur une surface estime une distance entre une surface $\boldsymbol{\theta}(\omega)$ et une autre surface $\boldsymbol{\phi}(\omega)$ en fonction des distances entre leur formes fondamentales dans l'espace $L^p(\omega)$, $1 < p < \infty$.

Nous établissons une nouvelle inégalité de ce type. La nouveauté réside dans l'appartenance de l'immersion $\boldsymbol{\theta}$ à un ensemble particulier d'applications de classe $\mathscr{C}^1$ de $\overline{\omega}$ dans $\mathbb{R}^3$ avec un champ de vecteurs normaux unitaires aussi de classe $\mathscr{C}^1$ dans $\overline{\omega}$.

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1. Notation and definitions

Vector and matrix fields are denoted by boldface letters.

Given any open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, any subset $V \subset Y$ of a finite-dimensional vector space Y, and any integer $\ell \geqslant 0$, the notation $\mathscr{C}^\ell(\Omega; V)$ designates the set of all fields $\boldsymbol{v} = (v_i) : \Omega \to Y$ such that $\boldsymbol{v}(x) \in V$ for all $x \in \Omega$ and $v_i \in \mathscr{C}^\ell(\Omega)$. Likewise, given any real number $p > 1$, the notation

[*] Corresponding author.

$L^p(\Omega; V)$, resp. $W^{\ell,p}(\Omega; V)$, designates the set of all fields $\boldsymbol{v} = (v_i) : \Omega \to Y$ such that $\boldsymbol{v}(x) \in V$ for almost all $x \in \Omega$ and $v_i \in L^p(\Omega)$, resp. $v_i \in W^{\ell,p}(\Omega)$.

The space of all real matrices with k rows and ℓ columns is denoted $\mathbb{M}^{k\times\ell}$. We also let

$$\mathbb{M}^k := \mathbb{M}^{k\times k},\ \mathbb{S}^k := \left\{\boldsymbol{A} \in \mathbb{M}^k;\ \boldsymbol{A} = \boldsymbol{A}^T\right\},$$
$$\mathbb{S}^k_> := \left\{\boldsymbol{A} \in \mathbb{S}^k;\ \boldsymbol{A} \text{ is positive-definite}\right\}, \quad \text{and } \mathbb{O}^k_+ := \left\{\boldsymbol{A} \in \mathbb{M}^k;\ \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I} \text{ and } \det \boldsymbol{A} = 1\right\}.$$

A $k \times \ell$ matrix whose column vectors are the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell \in \mathbb{R}^k$ is denoted $(\boldsymbol{v}_1|\dots|\boldsymbol{v}_\ell)$. If $\boldsymbol{A} \in \mathbb{S}^k_>$, there exists a unique matrix $\boldsymbol{U} \in \mathbb{S}^k_>$ such that $\boldsymbol{U}^2 = \boldsymbol{A}$; this being the case, we let $\boldsymbol{A}^{1/2} := \boldsymbol{U}$.

The Euclidean norm in $\mathbb{R}^3$ is denoted $|\cdot|$. Spaces of matrices are equipped with the Frobenius norm, also denoted $|\cdot|$. The spaces $L^p(\Omega)$, $L^p(\Omega;\mathbb{R}^k)$, and $L^p(\Omega;\mathbb{M}^{k\times\ell})$, are respectively equipped with the norms denoted and defined by

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} := \left(\int_\Omega |\boldsymbol{v}(x)|^p \,\mathrm{d}x\right)^{1/p},$$
$$\text{and} \quad \|\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)} := \left(\int_\Omega |\boldsymbol{A}(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

A *domain* Ω in $\mathbb{R}^n$, $n \geqslant 2$, is a connected and open subset of $\mathbb{R}^n$ that is bounded and has a Lipschitz-continuous boundary, the set Ω being locally on the same side of its boundary (cf. Adams [1], Maz'ya [10], or Nečas [11]).

Given an open subset Ω of $\mathbb{R}^n$ and any integer $\ell \geqslant 0$, the notation $\mathscr{C}^\ell(\overline{\Omega})$ designates the space of all functions $u \in \mathscr{C}^\ell(\Omega)$ such that u and all their partial derivatives up to order ℓ possess continuous extensions to the closure $\overline{\Omega}$ of Ω. If $\Omega \subset \mathbb{R}^n$ is a domain, then $\mathscr{C}^\ell(\overline{\Omega}) = \{f|_{\overline{\Omega}};\ f \in \mathscr{C}^\ell(\mathbb{R}^n)\}$, where $f|_{\overline{\Omega}}$ denotes the restriction of the function f to the set $\overline{\Omega}$, thanks to Whitney's extension theorem: cf. Whitney [12]; see also Ciarlet & Mardare [5].

Given a connected open subset Ω of $\mathbb{R}^n$, the *geodesic distance* between two points $x, y \in \Omega$ is defined by

$$\operatorname{dist}_\Omega(x,y) := \inf\Big\{\ell \in \mathbb{R};\ \text{there exists a path } c \in \mathscr{C}^1\left([0,\ell];\mathbb{R}^n\right)$$
$$\text{such that } c(0) = x,\ c(\ell) = y,\ c(s) \in \Omega \text{ and } \left|c'(s)\right| = 1 \text{ for all } s \in [0,\ell]\Big\}. \quad (1)$$

If $\Omega \subset \mathbb{R}^n$ is a domain, then there exists a constant $C_\Omega \geqslant 1$ such that

$$\operatorname{dist}_\Omega(x,y) \leqslant C_\Omega \left|x - y\right| \text{ for all } x, y \in \overline{\Omega}; \quad (2)$$

see e.g. Anicic, Le Dret & Raoult [2, Proposition 5.1] .

A generic point in an open subset ω of $\mathbb{R}^2$ is denoted $y = (y_1, y_2)$ and partial derivative operators with respect to y_1 and y_2 are denoted ∂_1 and ∂_2.

2. Main result

An *immersion of class* $\mathscr{C}^1$ from a two-dimensional domain $\omega \subset \mathbb{R}^2$ into the three-dimensional Euclidean space $\mathbb{R}^3$ is a mapping $\boldsymbol{\phi} : \omega \to \mathbb{R}^3$ of class $\mathscr{C}^1$ such that the two vector fields $\partial_1\boldsymbol{\phi} : \omega \to \mathbb{R}^3$ and $\partial_2\boldsymbol{\phi} : \omega \to \mathbb{R}^3$ are linearly independent at each point of ω. This means that the image of ω by $\boldsymbol{\phi}$ is a surface in $\mathbb{R}^3$ whose tangent plane at its point $\boldsymbol{\phi}(y)$, $y \in \omega$, is spanned by the two vectors $\partial_1\boldsymbol{\phi}(y)$ and $\partial_2\boldsymbol{\phi}(y)$. Consequently,

$$\boldsymbol{v}(\boldsymbol{\phi}) := \frac{\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}}{\left|\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}\right|} \quad (3)$$

is a continuous unit vector field from ω into $\mathbb{R}^3$ that is normal to the surface $\boldsymbol{\phi}(\omega)$.

Given an immersion $\boldsymbol{\phi} : \omega \to \mathbb{R}^3$ of class $\mathscr{C}^1$, we let

$$\nabla\boldsymbol{\phi} := \big(\partial_1\boldsymbol{\phi} \,\big|\, \partial_2\boldsymbol{\phi}\big) \text{ and } \boldsymbol{A}(\boldsymbol{\phi}) := \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\phi}. \tag{4}$$

Note that $\nabla\boldsymbol{\phi}$ is field of 3×2 matrices whose column vectors are the partial derivatives of $\boldsymbol{\phi}$ and that $\boldsymbol{A}(\boldsymbol{\phi})$ is a field of 2×2 positive-definite symmetric matrices whose components are the covariant components of the *first fundamental form* associated with the immersion $\boldsymbol{\phi}$.

Given an immersion $\boldsymbol{\phi} : \omega \to \mathbb{R}^3$ of class $\mathscr{C}^1$ such that the unit vector field $\boldsymbol{\nu}(\boldsymbol{\phi}) : \omega \to \mathbb{R}^3$ is also of class $\mathscr{C}^1$, we let

$$\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) = \big(\partial_1\boldsymbol{\nu}(\boldsymbol{\phi}) \,\big|\, \partial_2\boldsymbol{\nu}(\boldsymbol{\phi})\big) \text{ and } \boldsymbol{B}(\boldsymbol{\phi}) := \nabla\boldsymbol{\phi}^T\nabla\boldsymbol{\nu}(\boldsymbol{\phi}). \tag{5}$$

Note that $\nabla\boldsymbol{\nu}(\boldsymbol{\phi})$ is field of 3×2 matrices whose column vectors are the partial derivatives of the vector field $\boldsymbol{\nu}(\boldsymbol{\phi})$ and that $\boldsymbol{B}(\boldsymbol{\phi})$ is a field of 2×2 symmetric matrices whose components are the covariant components of the *second fundamental form* associated with the immersion $\boldsymbol{\phi}$.

The above definitions and notations apply as well to immersions $\boldsymbol{\phi} : \omega \to \mathbb{R}^3$ and their associated unit vector fields $\boldsymbol{\nu}(\boldsymbol{\phi}) : \omega \to \mathbb{R}^3$ that are both of class $\mathscr{C}^1$ up to the boundary of ω, or of class $W^{1,p}$ in ω, $1 < p < \infty$. This being the case, we let

$$\mathscr{C}^1_+\left(\overline{\omega};\mathbb{R}^3\right) := \left\{\boldsymbol{\phi} \in \mathscr{C}^1\left(\overline{\omega};\mathbb{R}^3\right);\ \left|\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}\right| > 0 \text{ in } \overline{\omega},\ \boldsymbol{\nu}(\boldsymbol{\phi}) \in \mathscr{C}^1\left(\overline{\omega};\mathbb{R}^3\right)\right\} \tag{6}$$

and

$$W^{1,p}_+\left(\omega;\mathbb{R}^3\right) := \left\{\boldsymbol{\phi} \in W^{1,p}\left(\omega;\mathbb{R}^3\right); \left|\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}\right| > 0 \text{ a.e. in } \omega,\ \boldsymbol{\nu}(\boldsymbol{\phi}) \in W^{1,p}\left(\omega;\mathbb{R}^3\right)\right\}. \tag{7}$$

The objective of this Note is to indicate how to establish a *nonlinear Korn inequality with an explicit estimate of the constant* that appears in it for mappings $\boldsymbol{\phi} \in W^{1,p}_+(\omega;\mathbb{R}^3)$ and $\boldsymbol{\theta} \in \mathscr{C}^1_+(\overline{\omega};\mathbb{R}^3)$; see Theorem 1 below. We will show in particular that the estimate for the constant depends on $\boldsymbol{\theta}$ only via two scalar parameters, denoted ρ and δ in what follows, which are related to the assumption that $\boldsymbol{\theta}$ is an immersion such that $\boldsymbol{\theta}$ and $\boldsymbol{\nu}(\boldsymbol{\theta})$ are continuously differentiable vector fields over $\overline{\omega}$.

Note that the nonlinear Korn inequality of Theorem 1 constitutes an improvement, when $n = 3$, over two previous results by the authors about hypersurfaces in $\mathbb{R}^n$, $n \geqslant 3$: see [7, Theorem 3.1 and Lemma 3.2], or [6, Lemma 2].

The definition of the constant C_ω in the next statement is justified by relations (1)-(2) in Section 1.

Theorem 1. *Given any domain $\omega \subset \mathbb{R}^2$ and any real numbers $p > 1$, $1 \geqslant \rho > 0$ and $\delta > 0$, there exists a constant $C = C(\omega, p, \rho, \delta)$ such that*

$$\begin{aligned}
&\inf_{\boldsymbol{R}\in\mathbb{O}^3_+}\Big(\left\|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right\|_{\boldsymbol{L}^p(\omega)} + \left\|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\right\|_{\mathbb{L}^p(\omega)} + \left\|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right\|_{\mathbb{L}^p(\omega)}\Big)\\
&\qquad\leqslant C\left\|\inf_{\boldsymbol{R}\in\mathbb{O}^3_+}\Big(\left|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right| + \left|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\right| + \left|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right|\Big)\right\|_{L^p(\omega)}\\
&\qquad\leqslant C\sqrt{3}\Big(\left\|\boldsymbol{A}(\boldsymbol{\phi})^{1/2} - \boldsymbol{A}(\boldsymbol{\theta})^{1/2}\right\|_{\mathbb{L}^p(\omega)} + \left\|\boldsymbol{A}(\boldsymbol{\phi})^{-1/2}\boldsymbol{B}(\boldsymbol{\phi}) - \boldsymbol{A}(\boldsymbol{\theta})^{-1/2}\boldsymbol{B}(\boldsymbol{\theta})\right\|_{\mathbb{L}^p(\omega)}\Big)
\end{aligned}$$

for all mappings $\boldsymbol{\phi} \in W^{1,p}_+(\omega;\mathbb{R}^3)$ and $\boldsymbol{\theta} \in \mathscr{C}^1_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$, where

$$\mu > 0 \textit{ is any real number such that } \mu \leqslant \frac{\rho^{11}}{468\,(1 + C_\omega)},$$
$$C_\omega \geqslant 1 \textit{ is any constant such that } \operatorname{dist}_\omega\left(y, \widetilde{y}\right) \leqslant C_\omega\left|y - \widetilde{y}\right| \text{ for all } y, \widetilde{y} \in \omega,$$

and

$$\mathscr{C}^1_{\rho,\delta,\mu}\left(\overline{\omega};\mathbb{R}^3\right) := \Big\{\boldsymbol{\theta}\in\mathscr{C}^1_+\left(\overline{\omega};\mathbb{R}^3\right);\ \inf_{y\in\overline{\omega}}\left|\partial_1\boldsymbol{\theta}(y)\wedge\partial_2\boldsymbol{\theta}(y)\right|\geqslant\rho,\ \sup_{y\in\overline{\omega}}|\nabla\boldsymbol{\theta}(y)|\leqslant\frac{1}{\rho},\ \sup_{y\in\overline{\omega}}|\nabla\boldsymbol{\nu}(\boldsymbol{\theta})(y)|\leqslant\frac{1}{\rho},$$
$$\sup_{\begin{cases} y,\tilde{y}\in\overline{\omega},\\ |y-\tilde{y}|\leqslant\delta\end{cases}}\left|\nabla\boldsymbol{\theta}(y)-\nabla\boldsymbol{\theta}(\tilde{y})\right|\leqslant\mu,\quad \sup_{\begin{cases} y,\tilde{y}\in\overline{\omega},\\ |y-\tilde{y}|\leqslant\delta\end{cases}}\left|\nabla\boldsymbol{\nu}(\boldsymbol{\theta})(y)-\nabla\boldsymbol{\nu}(\boldsymbol{\theta})(\tilde{y})\right|\leqslant\mu\Big\}.$$

The proof of the Theorem 1 is sketched in Section 3 below; the details are given in [9].

The restriction in Theorem 1 that $\boldsymbol{\theta}$ belongs to the subset $\mathscr{C}^1_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ of the set $\mathscr{C}^1_+(\overline{\omega};\mathbb{R}^3)$, rather than to the set $\mathscr{C}^1_+(\overline{\omega};\mathbb{R}^3)$ itself, is essential (i.e., not merely an artefact of the proof). However, this inconvenient is alleviated by the fact that, as $\rho\to 0^+$ and $\delta\to 0^+$, the subset $\mathscr{C}^1_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ becomes as large in $\mathscr{C}^1_+(\overline{\omega};\mathbb{R}^3)$ as one wants. More specifically, for each $\mu>0$,

$$\mathscr{C}^1_+\left(\overline{\omega};\mathbb{R}^3\right) = \lim_{\rho\to 0^+}\left(\lim_{\delta\to 0^+}\mathscr{C}^1_{\rho,\delta,\mu}\left(\overline{\omega};\mathbb{R}^3\right)\right),$$

where the limits above are defined as the union of an increasing sequence of sets.

3. Sketch of the proof of Theorem 1

The proof is broken for clarity into six steps, numbered (i) to (vi).

Proof. As in the statement of the Theorem 1, let there be given a domain $\omega\subset\mathbb{R}^2$, a constant C_ω such that

$$\mathrm{dist}_\omega(y,\tilde{y})\leqslant C_\omega\left|y-\tilde{y}\right| \text{ for all } y,\tilde{y}\in\omega,$$

four real numbers $p>1$, $1\geqslant\rho>0$, $\delta>0$, $\mu>0$, and two mappings

$$\boldsymbol{\theta}\in\mathscr{C}^1_{\rho,\delta,\mu}\left(\overline{\omega};\mathbb{R}^3\right) \text{ and } \boldsymbol{\phi}\in W^{1,p}_+\left(\omega;\mathbb{R}^3\right).$$

Then let $\lambda:=\rho/3$, $\eta:=13\mu\rho^{-8}/3$, $\varepsilon:=\rho^6/3$, let $\Omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)$, and let $\boldsymbol{\Theta}:\Omega^\varepsilon\to\mathbb{R}^3$ and $\boldsymbol{\Phi}:\Omega^\varepsilon\to\mathbb{R}^3$ be the mappings defined by

$$\boldsymbol{\Theta}(x):=\boldsymbol{\theta}(y)+x_3\boldsymbol{\nu}(\boldsymbol{\theta})(y)\quad\text{for all}\quad x=(y,x_3)\in\Omega^\varepsilon$$

and

$$\boldsymbol{\Phi}(x):=\boldsymbol{\phi}(y)+x_3\boldsymbol{\nu}(\boldsymbol{\phi})(y)\quad\text{for almost all}\quad x=(y,x_3)\in\Omega^\varepsilon.$$

Note that the above definition of the constants λ, η and ε is justified by the estimates established in Step (iv) below.

Step (i). There exists a constant $C_1(p,\rho)>0$ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^3_+}\|\nabla\boldsymbol{\Phi}-\boldsymbol{R}\nabla\boldsymbol{\Theta}\|_{\mathbb{L}^p(\Omega^\varepsilon)} \geqslant C_1(p,\rho)\inf_{\boldsymbol{R}\in\mathbb{O}^3_+}\left(\left\|\boldsymbol{\nu}(\boldsymbol{\phi})-\boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right\|_{\boldsymbol{L}^p(\omega)}+\left\|\nabla\boldsymbol{\phi}-\boldsymbol{R}\nabla\boldsymbol{\theta}\right\|_{\mathbb{L}^p(\omega)}+\left\|\nabla\boldsymbol{\nu}(\boldsymbol{\phi})-\boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right\|_{\mathbb{L}^p(\omega)}\right).$$

The proof of this inequality relies on Clarkson's inequalities in the space $L^p(\Omega^\varepsilon)$ (see, e.g., Adams [1, Theorem 2.28]) and follows an argument previously used in Ciarlet, Malin & Mardare [4, Proof of Theorem 4.2].

Step (ii). There exists a constant $C_2(p,\rho) > 0$ such that

$$\left\| \inf_{\boldsymbol{R}\in\mathbb{O}^3_+} |\nabla\boldsymbol{\Phi} - \boldsymbol{R}\nabla\boldsymbol{\Theta}| \right\|_{L^p(\Omega^\varepsilon)} \leqslant C_2(p,\rho) \left\| \inf_{\boldsymbol{R}\in\mathbb{O}^3_+} \Big(\big|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\big| + \big|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\big| + \big|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\big| \Big) \right\|_{L^p(\omega)}.$$

The proof of this inequality uses either Jensen's inequality if $p > 2$, or the inequality $(a+b+c)^{p/2} \leqslant a^{p/2} + b^{p/2} + c^{p/2}$ if $p \leqslant 2$ for some appropriate nonnegative real numbers a, b and c, followed by an appropriate application of Fubini's theorem.

Step (iii). The following assertions hold:

$$\boldsymbol{A}(\boldsymbol{\theta}) \in \mathscr{C}^0\left(\overline{\omega};\mathbb{S}^2_>\right), \quad \boldsymbol{A}(\boldsymbol{\theta})^{-1} \in \mathscr{C}^0\left(\overline{\omega};\mathbb{S}^2_>\right), \quad \boldsymbol{B}(\boldsymbol{\theta}) \in \mathscr{C}^0\left(\overline{\omega};\mathbb{S}^2\right), \quad \boldsymbol{\Theta} \in \mathscr{C}^1\left(\overline{\Omega^\varepsilon};\mathbb{R}^3\right),$$

and

$$\boldsymbol{A}(\boldsymbol{\phi})^{1/2} \in L^p\left(\omega;\mathbb{S}^2_>\right), \quad \boldsymbol{A}(\boldsymbol{\phi})^{-1/2}\boldsymbol{B}(\boldsymbol{\phi}) \in L^p\left(\omega;\mathbb{M}^2\right), \quad \boldsymbol{\Phi} \in W^{1,p}\left(\Omega^\varepsilon;\mathbb{R}^3\right).$$

These assertions are straightforward generalisations of similar ones established in Ciarlet, Gratie & Mardare [3] for $p = 2$, and for this reason their proof is omitted.

Step (iv). The mapping $\boldsymbol{\Theta}$ satisfies the following properties:

$$\det\nabla\boldsymbol{\Theta}(x) \geqslant \lambda \text{ and } |\nabla\boldsymbol{\Theta}(x)| \leqslant \frac{1}{\lambda} \quad \text{for all } x \in \overline{\Omega^\varepsilon},$$

and

$$|\nabla\boldsymbol{\Theta}(x) - \nabla\boldsymbol{\Theta}(\widetilde{x})| \leqslant \eta \quad \text{for all } x, \widetilde{x} \in \overline{\Omega^\varepsilon} \text{ such that } |x - \widetilde{x}| \leqslant \delta.$$

Using the estimates in terms of ρ of the partial derivatives of $\boldsymbol{\theta}$ and $\boldsymbol{\nu}(\boldsymbol{\theta})$ appearing in the definition of the set $\mathscr{C}^1_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ (see the statement of Theorem 1), we first deduce from Weingarten's equations that

$$|\nabla\boldsymbol{\Theta}(x)| \leqslant \frac{7}{3\rho} \text{ and } \det\nabla\boldsymbol{\Theta}(x) \geqslant \frac{11\rho}{18} \quad \text{for all } x = (y, x_3) \in \overline{\Omega^\varepsilon},$$

then we deduce from the definition of the vector field $\boldsymbol{\nu}(\boldsymbol{\theta})$ in terms of the partial derivatives of $\boldsymbol{\theta}$ (see relation (3)) that

$$\big|\boldsymbol{\nu}(y) - \boldsymbol{\nu}(\widetilde{y})\big| \leqslant \frac{3}{\rho^8}\big|\nabla\boldsymbol{\theta}(y) - \nabla\boldsymbol{\theta}(\widetilde{y})\big| \quad \text{for all } y, \widetilde{y} \in \overline{\omega}.$$

Combined with the definition of the mapping $\boldsymbol{\Theta}$ in terms of $\boldsymbol{\theta}$ and the definition of the parameter ε in terms of ρ, the last inequality implies that, for each $x = (y, x_3) \in \overline{\Omega^\varepsilon}$ and each $\widetilde{x} = (\widetilde{y}, \widetilde{x}_3) \in \overline{\Omega^\varepsilon}$,

$$\begin{aligned}|\nabla\boldsymbol{\Theta}(x) - \nabla\boldsymbol{\Theta}(\widetilde{x})| &\leqslant \big|\nabla\boldsymbol{\theta}(y) - \nabla\boldsymbol{\theta}(\widetilde{y})\big| + \varepsilon\big|\nabla\boldsymbol{\nu}(y) - \nabla\boldsymbol{\nu}(\widetilde{y})\big| + \big|\boldsymbol{\nu}(y) - \boldsymbol{\nu}(\widetilde{y})\big| \\ &\leqslant \left(1 + \frac{3}{\rho^8}\right)\big|\nabla\boldsymbol{\theta}(y) - \nabla\boldsymbol{\theta}(\widetilde{y})\big| + \frac{\rho^6}{3}\big|\nabla\boldsymbol{\nu}(y) - \nabla\boldsymbol{\nu}(\widetilde{y})\big|.\end{aligned}$$

Assume next that x and $\widetilde{x}$ satisfy $|x - \widetilde{x}| \leqslant \delta$, so that, in particular, $|y - \widetilde{y}| \leqslant \delta$. Then we infer from the definition of the space $\mathscr{C}^1_{\rho,\delta,\mu}(\overline{\omega};\mathbb{R}^3)$ and from the previous estimate that

$$|\nabla\boldsymbol{\Theta}(x) - \nabla\boldsymbol{\Theta}(\widetilde{x})| \leqslant \left(1 + \frac{3}{\rho^8} + \frac{\rho^6}{3}\right)\mu \leqslant \frac{13\mu}{3\rho^8}.$$

Step (v). Assume that the given constant $\mu > 0$ satisfies $\mu \leqslant \frac{\rho^{11}}{468(1+C_\omega)}$. Then there exists a constant $C_3(\omega, p, \rho, \delta)$ depending only on ω, p, ρ, δ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^3_+} \|\nabla\boldsymbol{\Phi} - \boldsymbol{R}\nabla\boldsymbol{\Theta}\|_{\mathbb{L}^p(\Omega^\varepsilon)} \leqslant C_3(\omega, p, \rho, \delta) \left\| \inf_{\boldsymbol{R}\in\mathbb{O}^3_+} |\nabla\boldsymbol{\Phi} - \boldsymbol{R}\nabla\boldsymbol{\Theta}| \right\|_{L^p(\Omega^\varepsilon)}. \tag{8}$$

First, the inequalities established in (iv) imply that the mapping $\boldsymbol{\Theta}$ belong to the set:

$$\mathscr{C}^1_{\lambda,\delta,\eta}\left(\overline{\Omega^\varepsilon};\mathbb{R}^3\right) := \left\{ \boldsymbol{\Theta} \in \mathscr{C}^1\left(\overline{\Omega^\varepsilon};\mathbb{R}^3\right);\ \inf_{x\in\overline{\Omega^\varepsilon}} \det\nabla\boldsymbol{\Theta}(x) \geqslant \lambda,\ \sup_{x\in\overline{\Omega^\varepsilon}} |\nabla\boldsymbol{\Theta}(x)| \leqslant \frac{1}{\lambda},\ \sup_{\begin{cases} x,\tilde{x}\in\overline{\Omega^\varepsilon}, \\ |x-\tilde{x}|\leqslant\delta \end{cases}} |\nabla\boldsymbol{\Theta}(x) - \nabla\boldsymbol{\Theta}(\tilde{x})| \leqslant \eta \right\}.$$

Secondly, by (iii),

$$\boldsymbol{\Phi} \in W^{1,p}\left(\Omega^\varepsilon;\mathbb{R}^3\right).$$

Thirdly, the definition of the set Ω^ε in terms of ω, the definition of the geodesic distance in Ω^ε (see relation (1)), and the definition of the constant C_ω (see the statement of Theorem 1), together show that, for each $x = (y, x_3) \in \overline{\Omega^\varepsilon}$ and each $\tilde{x} = (\tilde{y}, \tilde{x}_3) \in \overline{\Omega^\varepsilon}$,

$$\operatorname{dist}_{\Omega^\varepsilon}(x, \tilde{x}) \leq \operatorname{dist}_\omega(y, \tilde{y}) + |x_3 - \tilde{x}_3| \leqslant C_\omega \left|y - \tilde{y}\right| + |x_3 - \tilde{x}_3| \leqslant (1 + C_\omega)\,|x - \tilde{x}|.$$

Fourthly, the assumption on μ made in (v) implies that

$$\eta \leqslant \frac{\lambda^3}{4(1+C_\omega)}.$$

The four observations above together imply that the assumptions of [8, Theorem 1 (a)] are satisfied by the domain Ω^ε and by the mappings $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ from Ω^ε into $\mathbb{R}^3$. Thus inequality (8) holds as a consequence of this theorem.

Step (vi). Combining the three inequalities established in steps (i), (ii) and (v) above yields the inequality

$$\inf_{\boldsymbol{R}\in\mathbb{O}^3_+} \left(\left\|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right\|_{\boldsymbol{L}^p(\omega)} + \left\|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\right\|_{\mathbb{L}^p(\omega)} + \left\|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right\|_{\mathbb{L}^p(\omega)} \right)$$
$$\leqslant C \left\| \inf_{\boldsymbol{R}\in\mathbb{O}^3_+} \left(\left|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\right| + \left|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right| + \left|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right| \right) \right\|_{L^p(\omega)},$$

where $C := (C_1(p,\rho))^{-1} C_2(p,\rho) C_3(\omega, p, \rho, \delta)$. This establishes the first inequality of Theorem 1.

Using the polar decomposition of the 3×3 matrix fields

$$(\nabla\boldsymbol{\theta}|\boldsymbol{\nu}(\boldsymbol{\theta})) \text{ and } \left(\nabla\boldsymbol{\phi}\middle|\boldsymbol{\nu}(\boldsymbol{\phi})\right)$$

and a method similar to one used in the proof of [7, Theorem 3.1], we next show that the following inequality holds almost everywhere in ω:

$$\inf_{\boldsymbol{R}\in\mathbb{O}^3_+} \left(\left|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right|^2 + \left|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\right|^2 + \left|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right|^2 \right)$$
$$\leqslant \left|\boldsymbol{A}(\boldsymbol{\phi})^{1/2} - \boldsymbol{A}(\boldsymbol{\theta})^{1/2}\right|^2 + \left|\boldsymbol{A}(\boldsymbol{\phi})^{-1/2}\boldsymbol{B}(\boldsymbol{\phi}) - \boldsymbol{A}(\boldsymbol{\theta})^{1/2}\boldsymbol{B}(\boldsymbol{\theta})\right|^2.$$

Consequently,

$$\left\| \inf_{\boldsymbol{R}\in\mathbb{O}^3_+} \left(\left|\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\boldsymbol{\nu}(\boldsymbol{\theta})\right| + \left|\nabla\boldsymbol{\phi} - \boldsymbol{R}\nabla\boldsymbol{\theta}\right| + \left|\nabla\boldsymbol{\nu}(\boldsymbol{\phi}) - \boldsymbol{R}\nabla\boldsymbol{\nu}(\boldsymbol{\theta})\right|\right)\right\|_{L^p(\omega)}$$
$$\leqslant \sqrt{3}\left(\left\|\boldsymbol{A}(\boldsymbol{\phi})^{1/2} - \boldsymbol{A}(\boldsymbol{\theta})^{1/2}\right\|_{\mathbb{L}^p(\omega)} + \left\|\boldsymbol{A}(\boldsymbol{\phi})^{-1/2}\boldsymbol{B}(\boldsymbol{\phi}) - \boldsymbol{A}(\boldsymbol{\theta})^{-1/2}\boldsymbol{B}(\boldsymbol{\theta})\right\|_{\mathbb{L}^p(\omega)}\right).$$

This establishes the second inequality of Theorem 1. □

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