Yunhyung Cho

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Remark on the Betti numbers for Hamiltonian circle actions

Yunhyung Cho

Department of Mathematics Education, Sungkyunkwan University, Seoul, Republic of Korea.
E-mail: yunhyung@skku.edu

Abstract. In this paper, we establish a certain inequality in terms of Betti numbers of a closed Hamiltonian $S^1$-manifold with isolated fixed points.

Résumé. Dans cet article, nous établissons une certaine inégalité en termes de nombres de Betti d’une $S^1$-variété hamiltonienne avec des points fixes isolés.

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1. Introduction

Let $(M, \omega)$ be a $2n$-dimensional closed symplectic manifold admitting a Hamiltonian torus action with only isolated fixed points. It has been a long-standing open problem whether $M$ admits a Kähler metric or not. Historically, Delzant [9] proved that if $M$ admits a Hamiltonian $T^n$-action, where the fixed point set is automatically discrete, then $M$ admits a $T^n$-invariant Kähler metric. Restricting to an $S^1$-action case, several results on the existence of a Kähler metric were provided in some special cases. For instance, Karshon [13] proved that every closed symplectic four manifold admitting a Hamiltonian circle action admits a Kähler metric. (In fact, the $S^1$-action is induced from a toric action when the fixed points are isolated.) Also if $\dim M = 6$ with $b_2(M) = 1$, then it turned out that $M$ admits a Kähler metric, which was proved by Tolman [18] and McDuff [15]. Recently, the author has shown that any 6-dimensional monotone closed semifree Hamiltonian $S^1$-manifold admits a Kähler metric, see [5–7].

As a counterpart, there were “candidates” of closed Hamiltonian $T$-manifolds (with isolated fixed points) which possibly fail to admit Kähler metrics. Tolman [17] and Woodward [19] constructed a six-dimensional closed Hamiltonian $T^2$-manifold with only isolated fixed points and with no $T^2$-invariant Kähler metric. Surprisingly Goertsches–Kostantis–Zoller [10] have recently
shown that the examples of Tolman and Woodward indeed admit Kähler metrics that are not $T^2$-invariant. Thus their result provides a positive evidence for the conjecture of the existence of Kähler metrics.

On the other hand, it seems reasonable to ask whether $(M, \omega)$ enjoys Kählerian properties, such as the hard Lefschetz property of the symplectic form $\omega$ or the unimodality of even Betti numbers. Recall that every closed Kähler manifold $(M, \omega, J)$ satisfies the hard Lefschetz property, that is,

$$[\omega]^{n-k} : H^k(M; \mathbb{R}) \to H^{2n-k}(M; \mathbb{R})$$

$$\alpha \mapsto \alpha \cup [\omega]^{n-k}$$

is an isomorphism for every $k = 0, 1, \cdots, n$. This implies that

$$[\omega] : H^k(M; \mathbb{R}) \to H^{k+2}(M; \mathbb{R})$$

is injective for every $k$ with $0 \leq k < n$, and therefore the sequence of even (as well as odd) Betti numbers of $M$ is unimodal. In other words,

$$b_k \leq b_{k+2}, \quad k = 0, 1, \cdots, n - 1$$

where $b_i$ denotes the $i^{th}$ Betti number of $M$. In this paper we deal with the following conjecture.

**Conjecture 1** ([12]). *Let $(M, \omega)$ be a $2n$-dimensional closed symplectic manifold equipped with a Hamiltonian $S^1$-action with only isolated fixed points. Then the sequence of even Betti numbers is unimodal, i.e.,

$$b_{2i} \leq b_{2i+2} \quad \text{for every} \quad 0 \leq i < \left\lfloor \frac{n}{2} \right\rfloor.$$*

It is worth mentioning that every odd Betti number of $M$ vanishes by Frankel’s theorem which states that a moment map is a Morse function whose critical points are of even indices. (See [2, Theorem IV.2.3].) Therefore we only need to care about even Betti numbers of $M$.

In [8], the author and Kim proved Conjecture 1 when $\dim M = 8$. The main goal of this article is to improve the result of [8] and prove the following inequality, which is automatically satisfied when Conjecture 1 is true.

**Theorem 2.** *Let $(M, \omega)$ be a closed symplectic manifold admitting a Hamiltonian circle action with only isolated fixed points where $\dim M = 8n$ or $8n + 4$. Then

$$b_2 + \cdots + b_{2+4(n-1)} \leq b_4 + \cdots + b_{4+4(n-1)}.$$*

*In particular when $\dim M = 8$ or 12, we have

$$b_2 \leq b_4.$$*

2. **Proof of the main Theorem 2**

The main technique for proving Theorem 2 is the ABBV-localization due to Atiyah–Bott and Berline–Vergne. Recall that for an $S^1$-manifold $M$, the equivariant cohomology is defined by $H^*_S(M) := H^*(M \times S^1; ES^1)$ where $ES^1$ is a contractible space on which $S^1$ acts freely. Then $H^*_S(M)$ inherits an $H^*(BS^1)$-module structure induced from the projection

$$\pi : M \times S^1 \to BS^1 := ES^1 / S^1.$$ 

Note that $H^*(BS^1; \mathbb{R}) \cong H^*(CP^\infty ; \mathbb{R}) = \mathbb{R}[u]$. Moreover, for the inclusion map $i : M^{S^1} \to M$, we have an induced ring homomorphism

$$i^* : H^*_S(M; \mathbb{R}) \to H^*_S\left( M^{S^1}; \mathbb{R} \right) \cong H^*(BS^1; \mathbb{R}) \otimes H^*(M^{S^1}; \mathbb{R}).$$
When \(M^{S^1} = \{p_1, \cdots, p_m\}\) is discrete, we may express as

\[
H^*(BS^1; \mathbb{R}) \otimes H^*(M^{S^1}; \mathbb{R}) \cong \bigoplus_{i=1}^{m} H^*(BS^1; \mathbb{R})
\]

and so

\[
i^*(\alpha) = (f_1, \cdots, f_m), \quad f_i \in \mathbb{R}[u]
\]

for \(\alpha \in H^*_{S^1}(M; \mathbb{R})\). We denote by \(\alpha|_{p_i} := f_i\) and call it the restriction of \(\alpha\) to \(p_i\). By the Kirwan’s injectivity theorem \([14]\), the map \(i^*\) is injective and hence \(H^*_{S^1}(M; \mathbb{R})\) is a free \(H^*(BS^1; \mathbb{R})\)-module.

**Theorem 3 (ABBV Localization Theorem [1, 3]).** Let \(M\) be a closed \(S^1\)-manifold with only isolated fixed points and \(\alpha \in H^*_{S^1}(M; \mathbb{R})\). Then we have

\[
\int_M \alpha = \sum_{p \in M^{S^1}} \frac{\alpha|_{p}}{(\prod_{i=1}^{n} w_i(p)) u^n},
\]

where \(w_1(p), \cdots, w_n(p)\) denote the weights of the tangential \(S^1\)-representation at \(p\).

To obtain Theorem 2, we will apply Theorem 3 to canonical classes which form a basis of \(H^*_{S^1}(M; \mathbb{R})\) as an \(H^*(BS^1; \mathbb{R})\)-module.

**Theorem 4 [16, Lemma 1.13].** Let \((M, \omega)\) be a \(2n\)-dimensional closed Hamiltonian \(S^1\)-manifold with only isolated fixed points. For each fixed point \(p \in M^{S^1}\) of index \(2k\), there exists a unique class \(\alpha_p \in H^{2k}(M; \mathbb{Z})\) such that

\[
\begin{align*}
\cdot & \alpha_p|_{q} = 0 \text{ for every } q(\neq p) \in M^{S^1} \text{ with either } H(q) \leq H(p) \text{ or } \text{ind}(q) \leq 2k, \\
\cdot & \alpha_p|_{p} = \prod_{i=1}^{k} \lambda_i u_i, \text{ where } \lambda_1, \cdots, \lambda_k \text{ are negative weights of the } S^1 \text{-action at } p.
\end{align*}
\]

Moreover, the set \(\{\alpha_p \mid p \in M^{S^1}\}\) is a basis of \(H^*_{S^1}(M; \mathbb{R})\) as an \(H^*(BS^1; \mathbb{R})\)-module.

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** We first consider the case \(\dim M = 8n\). Suppose that

\[
b_2 + \cdots + b_{2+4(n-1)} > b_4 + \cdots + b_{4+4(n-1)}.
\]

Since \(H^*_{S^1}(M)\) is a free module over \(H^*(B S^1)\), we have

\[
H^{4n-2}_{S^1}(M) \cong u^0 \otimes H^{4n-2}(M) \oplus u^1 \otimes H^{4n-4}(M) \oplus \cdots \oplus u^{(2n-1)} \otimes H^0(M)
\]

which implies that

\[
\dim_{\mathbb{R}} H^{4n-2}_{S^1}(M; \mathbb{R}) \cong b_0 + b_2 + \cdots + b_{4n-2}, \text{ and}
\]

\[
\{\alpha_p \cdot u^{2n-1-\frac{1}{2}\text{ind}(p)} \mid p \in M^{S^1}, \text{ind}(p) \leq 4n-2\}\text{ is a basis of } H^*_{S^1}(M; \mathbb{R})\text{ (as an } \mathbb{R}\text{-vector space) by Theorem 4.}
\]

Now, consider the following map

\[
\Phi: H^{4n-2}_{S^1}(M; \mathbb{R}) \rightarrow \left(\mathbb{R}^{b_0} \oplus \mathbb{R}^{b_2} \oplus \cdots \oplus \mathbb{R}^{b_{4n-2}}\right) \oplus \left(\mathbb{R}^{b_{4n}} \oplus \cdots \oplus \mathbb{R}^{b_{8n-4}}\right)
\]

\[
\alpha \rightarrow (\alpha_0, \cdots, \alpha_{4n-4}, \alpha_{4n}, \cdots, \alpha_{8n-4})
\]

with the identification

\[
\mathbb{R}^{b_{4i}} \cong \bigoplus_{\text{ind}(p)=4i} \mathbb{R} \cdot u^{2n-1} \quad \text{and} \quad \alpha_{4i} := (\alpha|_{p})_{\text{ind}(p)=4i} \in \bigoplus_{\text{ind}(p)=4i} \mathbb{R} \cdot u^{2n-1}
\]

for each \(i = 1, \cdots, n\). Since the dimension of the range of the map \(\Phi\) satisfies

\[
\dim \text{Im} \Phi \leq b_0 + \cdots + b_{4n-4} + (b_{4n} + b_{4n+4} + \cdots + b_{8n-4}) < b_0 + \cdots + b_{4n-2} + (b_{4n-2} + \cdots + b_2)
\]

\[\text{See also [18, Proposition 2.2] and [11, Lemma 2.10]}\]
by (1) and Poincaré duality, the map \( \Phi \) has a non-trivial kernel. In other words, there exists a nonzero element \( \alpha \in H^{4n}_{S^1}(M;\mathbb{R}) \) such that
\[
\alpha|_p = 0
\]
for every fixed point \( p \in M^{S^1} \) of index 0, 4, \( \cdots, 8n - 4 \).

Now fix a moment map \( H \) for the \( S^1 \)-action on \((M, \omega)\) such that \( H \) attains the maximum value 0. Denote by \( p_{\max} \) the maximal fixed point and so \( \dim(p_{\max}) = 8n \). The equivariant extension \([\omega_H] \in H^2_{S^1}(M;\mathbb{R})\) of \( \omega \) with respect to the moment map \( H \) satisfies
\[
[\omega_H]|_p = -H(p)u \in \mathbb{R}[u]
\]
for every \( p \in M^{S^1} \), see [4, Proposition 2.6]. Since \( H(p) < 0 \) for every \( p \neq p_{\max} \) by the choice of \( H \), we obtain
\[
0 = \int_M \alpha^2 \cdot |\omega_H| = \sum_{p \in M^{S^1}} \left( -\frac{\alpha^2|_p \cdot H(p)u}{\prod_{i=1}^{4n} w_i(p)} \right) u^{4n}
\]
for every \( p \in M^{S^1} \), similarly define\( \delta \) for every \( p \in M^{S^1} \) such that \( \delta|_p \neq 0 \) for \( p \neq p_{\max} \) and it violates Theorem 3
\[
0 = \int_M \alpha = \sum_{\text{ind}(p)=2 \text{ (mod 4)}} \frac{\alpha|_{p_{\max}}}{\prod_{i=1}^{4n} w_i(p)} u^{4n} \neq 0.
\]
Consequently, each summand of the rightmost equation of (3) has non-positive coefficient (of \( \frac{1}{n} \)) and at least one of those should be negative. Therefore it leads to a contradiction.

Now it remains to consider the case of \( \dim M = 8n + 4 \). Under the same assumption (1), we similarly define
\[
\Phi : H^{4n}_{S^1}(M;\mathbb{R}) \to \left( \mathbb{R}^{b_0} \oplus \mathbb{R}^{b_1} \oplus \cdots \oplus \mathbb{R}^{b_{4n}} \right) \oplus \left( \mathbb{R}^{b_{4n+2}} \oplus \cdots \oplus \mathbb{R}^{b_{8n}} \right)
\]
with the same identification as in (2). Note that \( \dim_{\mathbb{R}} H^{4n}_{S^1}(M;\mathbb{R}) = b_0 + b_2 + \cdots + b_{4n} + b_{4n+2} + b_{4n} \) and
\[
\dim \text{Im} \Phi \leq b_0 + \cdots + b_{4n} + (b_{4n+2} + \cdots + b_{8n}) = b_0 + \cdots + b_{4n} + (b_{4n} + \cdots + b_{4n+2})
\]
\[
< b_0 + \cdots + b_{4n} + (b_{4n+2} + \cdots + b_{2}) = \dim_{\mathbb{R}} H^{4n}_{S^1}(M;\mathbb{R})
\]
and so there exists a nonzero \( \alpha \in H^{4n}_{S^1}(M;\mathbb{R}) \) such that \( \alpha|_p = 0 \) for every fixed point \( p \) of index 0, 4, \( \cdots, 4n \). Therefore, we obtain
\[
0 = \int_M \alpha^2 \cdot |\omega_H| = \sum_{\text{ind}(p)=2 \text{ (mod 4)}} \frac{-\alpha^2|_p \cdot H(p)u}{\prod_{i=1}^{4n+2} w_i(p)} u^{4n+2} \neq 0
\]
which leads to a contradiction. This completes the proof of Theorem 2.  

\[
\Box
\]
References