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Analysis / *Analyse*

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Estimées de rigidité pour les champs incompatibles comme conséquence du résultat div-rot de Bourgain et Brezis

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Abstract. In this note we show that a sharp rigidity estimate and a sharp Korn's inequality for matrix-valued fields whose incompatibility is a bounded measure can be obtained as a consequence of a Hodge decomposition with critical integrability due to Bourgain and Brezis.

Résumé. Dans cette note, nous démontrons qu'une estimée de rigidité et une inégalité de Korn pour des champs avec des valeurs matricielles dont l'incompatibilité est une mesure bornée peuvent être obtenues comme conséquence d'une décomposition de Hodge avec intégrabilité critique dû à Bourgain et Brezis.

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1. Introduction

The celebrated rigidity estimate of Friesecke, James, and Müller [8, 9] is a fundamental tool for the analysis of variational, geometrically nonlinear models in elasticity. Indeed it allows to obtain compactness of the deformation gradient (and therefore, for example, linearization or dimension-reduction results) when the elastic energy has a degeneracy due to frame indifference. The linear counterpart of this estimate is the classical Korn inequality, which permits to deal with problems in linear elasticity where the energy, due to the approximation of the frame indifference, does not control the antisymmetric part of the displacement gradient, the so called infinitesimal rotations.

In the presence of plastic deformations, and then beyond the elastic regime mentioned above, the variational models should account for plastic slips that occur at microscopic scale and for the presence of topological defects which induce elastic distortion. Therefore, the relevant variable is not the gradient of a deformation but a more general matrix-valued field, the elastic strain, which may not be a gradient and may have a non trivial curl, possibly concentrated on low-dimensional sets.

This framework calls for a version of the rigidity result, and of its linear counterpart, for fields that may be incompatible, in the sense that are not gradients, but for which one knows that the curl is a bounded measure. Such results do not seem to be available in the literature in dimension greater than two, and are stated in Theorems 1 and 2.

In the sequel, given a matrix-valued function $\beta \in L^1(\Omega; \mathbb{R}^{n \times n})$, $\operatorname{Curl} \beta$ denotes the tensor-valued distribution whose rows are the curl, in the sense of distributions, of the rows of β, in the sense that $\operatorname{Curl} \beta_{ijk} := \partial_k \beta_{ij} - \partial_j \beta_{ik}$, moreover $\mathcal{M}(\Omega; \mathbb{R}^k)$ denotes the space of Radon measures in Ω with values in $\mathbb{R}^k$. In dimension 3, $\operatorname{Curl} \beta$ can as usual be identified with a matrix-valued distribution $\alpha_{il} := \sum_{j,k} \epsilon_{ljk} \operatorname{Curl} \beta_{ijk}$. The natural space is then the one obtained by the Sobolev-conjugate exponent $1^* := n/(n-1)$.

Theorem 1. *Given an open, bounded, connected, and Lipschitz set Ω in $\mathbb{R}^n$, with $n \geq 2$, there exists a constant $C = C(n, \Omega) > 0$ such that for every $\beta \in L^1(\Omega; \mathbb{R}^{n \times n})$ with $\operatorname{Curl} \beta \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n \times n})$, there exists a rotation $R \in \operatorname{SO}(n)$ such that*

$$\left\| \beta - R \right\|_{L^{1^*}(\Omega)} \leq C \left(\left\| \operatorname{dist}(\beta, \operatorname{SO}(n)) \right\|_{L^{1^*}(\Omega)} + \left| \operatorname{Curl} \beta \right|(\Omega) \right). \tag{1}$$

Theorem 2. *Given an open, bounded, connected, and Lipschitz set Ω in $\mathbb{R}^n$, with $n \geq 2$, there exists a constant $C = C(n, \Omega) > 0$ such that for every $\beta \in L^1(\Omega; \mathbb{R}^{n \times n})$ with $\operatorname{Curl} \beta \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n \times n})$ there exists an antisymmetric matrix A such that*

$$\left\| \beta - A \right\|_{L^{1^*}(\Omega)} \leq C \left(\left\| \beta + \beta^T \right\|_{L^{1^*}(\Omega)} + \left| \operatorname{Curl} \beta \right|(\Omega) \right). \tag{2}$$

In the above results $|\operatorname{Curl} \beta|(\Omega)$ denotes the total variation of the measure $\operatorname{Curl} \beta$. In the case when $\operatorname{Curl} \beta$ is absolutely continuous with respect to the Lebesgue measure it is simply given by the L^1 norm of $\operatorname{Curl} \beta$. We remark that the main point in this estimate is that it provides an estimate with the total variation and therefore allows for concentrated incompatibilities, which correspond to concentration of crystal defects. Korn's inequalities with incompatibility involving the L^p norm of $\operatorname{Curl} \beta$, with $p > 1$, instead of the total variation, can be found in [15] and the references therein.

One can see Theorem 2 as a a linearization of Theorem 1. Indeed, it could be proven that way, using a suitable density argument to reduce to fields $\beta \in L^\infty$. This procedure is normally not used in the case that β is a gradient, as the linear result is a key ingredient in the proof of the nonlinear rigidity estimate.

In 2 dimensions we have $1^* = 2$ and Korn and rigidity estimates for incompatible field (Theorems 2 and 1 with $n = 2$) have been obtained in [10, Theorem 11] and [16, Theorem 3.3], respectively. A key ingredient in the proof is an interpolation inequality due to Bourgain and Brezis which provides an estimate of the H^{-1} norm of a field in terms of the L^1 norm and the H^{-2} norm of its divergence, [2].

In higher dimension a recent result by Lauteri and Luckhaus, [14], shows that the rigidity estimate of Theorem 1 can be obtained in the spaces L^p with $p < 1^*$ and the Lorentz space $L^{1^*,\infty}$, but their results do not cover the case L^{1^*} which has the natural scaling.

Here, we show that the result in dimension three and above, Theorems 1 and 2, can be obtained very easily via Hodge decomposition using a sharp regularity result due to Bourgain and Brezis for the div-curl system [1, 2] (see also [13] and [18]) in the case that Ω is a cube, and then generalized to Lipschitz domains using a Whitney-type argument.

A similar Hodge decomposition argument had been used in the subcritical case in [4]. We had used the Bourgain–Brezis regularity result for the div-curl system to show existence of plastic strain in the presence of dislocations in [6]. After this paper was submitted, a related argument has been used in [11] to obtain similar rigidity estimates with zero boundary data or on cubes for other differential operators, including the deviatoric strain.

2. Proof of the results for $n \geq 3$

Lemma 3. *Let $n \geq 3$ and $Y \in L^1(Q; \mathbb{R}^n)$ be such that $\operatorname{div} Y = 0$ distributionally and $Y n = 0$ on ∂Q. Then*

$$\|Y\|_{L^{1^*}(Q)} \leq c(n) \|\operatorname{curl} Y\|_{L^1(Q)}. \tag{3}$$

Here and below $Q := (0,1)^n$; the assumption on Y means that extending Y by zero to $\mathbb{R}^n$ we obtain a divergence-free field.

We recall that this assertion does not hold for $n = 2$, as one can see with $\operatorname{curl} Y = \delta_{x_0}$ for some $x_0 \in Q$, see for example [1, Remark 2]. Indeed, the proof of Theorem 1 and Theorem 2 for $n = 2$ given in [10, Theorem 11] and [16, Theorem 3.3] is different, although it also uses in a crucial way the construction by Bourgain and Brezis.

We remark that for $n = 3$ this implies that for every $f \in L^1(\Omega; \mathbb{R}^3)$, with $\operatorname{div} f = 0$, there exists a unique solution $Z \in L^{3/2}(Q; \mathbb{R}^3)$ of the system

$$\begin{cases} \operatorname{div} Z = 0 & \text{in } Q, \\ \operatorname{curl} Z = f & \text{in } Q, \\ Z \cdot n = 0 & \text{on } \partial Q \end{cases} \tag{4}$$

and it satisfies

$$\|Z\|_{L^{3/2}(Q)} \leq C\|f\|_{L^1(Q)} \tag{5}$$

for some constant independent on f. By an approximation argument it follows that the same result holds for measures, i.e., $f \in \mathcal{M}(Q; \mathbb{R}^3)$, with $\operatorname{div} f = 0$ in the sense of distributions. By [3, Theorem 4.1] this result can be extended to smooth bounded sets $\Omega \subset \mathbb{R}^n$. This argument was used for the study of geometrically linear dislocations in [6].

Proof. By density, it suffices to prove the bound for regular fields Y (one can reflect Y across all boundaries and then mollify). Let $X \in L^n(Q; \mathbb{R}^n)$. By [2, Corollary 18'] there are $\varphi \in W^{1,n}(Q)$ and $\gamma \in W_0^{1,n} \cap L^\infty(Q; \mathbb{R}^{n \times n})$ such that $\gamma + \gamma^T = 0$, $X = D\varphi + \operatorname{div}\gamma$, and

$$\|\varphi\|_{W^{1,n}(Q)} + \|\gamma\|_{L^\infty(Q)} + \|\gamma\|_{W^{1,n}(Q)} \leq c(n)\|X\|_{L^n(Q)}. \tag{6}$$

We estimate

$$
\begin{aligned}
\int_Q Y \cdot X \, \mathrm{d}x &= \int_Q Y \cdot D\varphi \, \mathrm{d}x + \int_Q Y \cdot \operatorname{div}\gamma \, \mathrm{d}x = \int_{\partial Q} Y \cdot \gamma n \, \mathrm{d}\mathcal{H}^{n-1} - \int_Q DY \cdot \gamma \, \mathrm{d}x \\
&= -\frac{1}{2} \int_Q \left(DY - DY^T \right) \cdot \gamma \, dx = -\frac{1}{2} \int_Q \operatorname{curl} Y \cdot \gamma \, \mathrm{d}x \\
&\leq \frac{1}{2} \left\| \operatorname{curl} Y \right\|_{L^1(Q)} \left\| \gamma \right\|_{L^\infty(Q)} \leq c(n) \left\| \operatorname{curl} Y \right\|_{L^1(Q)} \|X\|_{L^n(Q)}.
\end{aligned}
\tag{7}
$$

$\square$

Proof of Theorem 1. To simplify notation, we write $s := 1^* = n/(n-1)$ and denote by c a constant that depends only on the spatial dimension n, and that may change from line to line. It suffices to prove the assertion if the right-hand side is finite.

First we prove the result in the case that $\Omega = Q = (0,1)^n$.

By density it suffices to prove the assertion for β smooth. Indeed, it is sufficient to extend β by reflection and then to mollify. We define $u \in W^{2,1}(Q;\mathbb{R}^n)$ as the solution of $\Delta u = \operatorname{div}\beta$ in Q, $\partial_n u = \beta n$ on ∂Q, and set $Y := \beta - Du$. Since $\operatorname{div} Y = 0$, $Y n = 0$ on ∂Q, and $\operatorname{curl} Y = \operatorname{curl}\beta$, application of Lemma 3 to each row of Y yields

$$
\|Y\|_{L^s(Q)} \leq c \left\| \operatorname{curl}\beta \right\|_{L^1(Q)}.
\tag{8}
$$

Therefore, since $Du = \beta - Y$,

$$
\begin{aligned}
\|\operatorname{dist}(Du, \mathrm{SO}(n))\|_{L^s(Q)} &\leq \left\| \operatorname{dist}\left(\beta, \mathrm{SO}(n)\right) \right\|_{L^s(Q)} + \|Y\|_{L^s(Q)} \\
&\leq \left\| \operatorname{dist}\left(\beta, \mathrm{SO}(n)\right) \right\|_{L^s(Q)} + c \left\| \operatorname{curl}\beta \right\|_{L^1(Q)}.
\end{aligned}
\tag{9}
$$

We then use the geometric rigidity estimate by Friesecke, James, and Müller [8, 9] (see also [5] for the version in L^p) which states that for any bounded Lipschitz connected domain Ω and any $p \in (1, \infty)$ there is a constant $c = c(\Omega, n, p)$ such that for any $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ there is $R \in \mathrm{SO}(n)$ such that

$$
\|Du - R\|_{L^p(\Omega)} \leq c \|\operatorname{dist}(Du, \mathrm{SO}(n))\|_{L^p(\Omega)}
\tag{10}
$$

and obtain the assertion. By scaling invariance, we have shown that for any cube $Q_r := x_* + (-r, r)^n$ and any $\beta : Q_r \to \mathbb{R}^{n \times n}$ there is $R \in \mathrm{SO}(n)$ such that

$$
\left\| \beta - R \right\|_{L^s(Q_r)} \leq c \left(\|\operatorname{dist}\left(Du, \mathrm{SO}(n)\right)\|_{L^s(Q_r)} + \left\| \operatorname{curl}\beta \right\|_{L^1(Q_r)} \right).
\tag{11}
$$

We now deal with a generic bounded Lipschitz connected set Ω. We consider a Whitney covering of Ω by countably many cubes $Q^j := x_j + (-r_j, r_j)^n$ with finite overlap, so that

$$
\chi_\Omega \leq \sum_j \chi_{\hat{Q}^j} \leq \sum_j \chi_{Q^j} \leq c \chi_\Omega,
\tag{12}
$$

where

$$
\widehat{Q}^j := x_j + \left(-\frac{1}{2} r_j, \frac{1}{2} r_j \right)^n,
$$

and such that $r_j \leq \operatorname{dist}(Q_j, \partial\Omega) \leq c r_j$, see for example [17, Section VI.1] or [7, Section 6.5]. Condition (12) states that the smaller cubes cover the domain Ω and the bigger ones still have finite overlap. This permits to construct a partition of unity $\varphi_j \in C_c^\infty(Q^j)$ subordinated to the cubes Q^j, with $\sum_j \varphi_j = 1$ in Ω and such that $|D\varphi_j| \leq c/r_j$.

We apply the estimate (11) on each cube Q^j and obtain rotations $R_j \in SO(n)$. If $Q_j \cap Q_k \neq \emptyset$, then by a triangular inequality we obtain

$$
\int_{Q_j \cap Q_k} \left| R_j - R_k \right|^s \, \mathrm{d}x \leq c \int_{Q_j \cap Q_k} \left| \beta - R_j \right|^s \, \mathrm{d}x + c \int_{Q_j \cap Q_k} \left| \beta - R_k \right|^s \, \mathrm{d}x.
\tag{13}
$$

We define $R : \Omega \to \mathbb{R}^{n \times n}$ by $R := \sum_j \varphi_j R_j$, and observe that

$$
\left\| \beta - R \right\|_{L^s(\Omega)}^s \leq c \sum_j \left\| \beta - R_j \right\|_{L^s(Q_j)}^s.
\tag{14}
$$

From $\sum_j \varphi_j = 1$ we obtain $\sum_j D\varphi_j = 0$ on Ω, so that

$$DR = \sum_j D\varphi_j R_j = \sum_j D\varphi_j (R_j - \beta). \tag{15}$$

Recalling that $\text{dist}(Q_j, \partial\Omega) \le cr_j$ and (12),

$$\int_\Omega \text{dist}^s(x, \partial\Omega)|DR|^s \, dx \le c \sum_j \int_{Q_j} r_j^s |D\varphi_j|^s |\beta - R_j|^s \, dx \le c \sum_j \int_{Q_j} |\beta - R_j|^s \, dx. \tag{16}$$

Since (11) holds in each cube Q^j,

$$\begin{aligned}
\int_\Omega \text{dist}^s(x, \partial\Omega)|DR|^s \, dx &\le c \sum_j \left[\left\| \text{dist}(\beta, \text{SO}(n)) \right\|_{L^s(Q^j)}^s + \left(|\text{curl}\,\beta|(Q_j) \right)^s \right] \\
&\le c \left\| \text{dist}(\beta, \text{SO}(n)) \right\|_{L^s(\Omega)}^s + c \left(|\text{curl}\,\beta|(\Omega) \right)^{s-1} \sum_j |\text{curl}\,\beta|(Q_j) \\
&\le c \left\| \text{dist}(\beta, \text{SO}(n)) \right\|_{L^s(\Omega)}^s + c \left(|\text{curl}\,\beta|(\Omega) \right)^s.
\end{aligned} \tag{17}$$

By the weighted Poincaré inequality [12, Theorem 8.8] there is a matrix $R_* \in \mathbb{R}^{n \times n}$ such that

$$\|R - R_*\|_{L^s(\Omega)}^s \le c \int_\Omega \text{dist}^s(x, \partial\Omega)|DR|^s \, dx. \tag{18}$$

Let now $\widehat{R} \in \text{SO}(n)$ be a matrix such that $|\widehat{R} - R_*| = \text{dist}(R_*, \text{SO}(n))$. Then, recalling (14),

$$\begin{aligned}
|\Omega|^{1/s} |\widehat{R} - R_*| &\le \|R_* - R\|_{L^s} + \|R - \beta\|_{L^s} + \|\text{dist}(\beta, \text{SO}(n))\|_{L^s} \\
&\le c \|\text{dist}(\beta, \text{SO}(n))\|_{L^s(\Omega)} + c |\text{curl}\,\beta|(\Omega).
\end{aligned} \tag{19}$$

$\square$

Proof of Theorem 2. This follows by the same proof, replacing geometric rigidity in L^s, Equation (10), by Korn's inequality in L^s, which states that for every bounded Lipschitz connected domain Ω and any $p \in (1, \infty)$ there is a constant $c = c(\Omega, n, p)$ such that for any $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ there is $R \in \text{SO}(n)$ such that

$$\|Du - R\|_{L^p(\Omega)} \le c \left\| Du + Du^T \right\|_{L^p(\Omega)} \tag{20}$$

(see, for example, [5] and references therein). $\square$

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