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<https://doi.org/10.5802/crmath.148>
Partial differential equations / Équations aux dérivées partielles

Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production

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Abstract. This paper deals with the chemotaxis system with nonlinear signal secretion

\[
\begin{align*}
&u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v), \quad x \in \Omega, \quad t > 0, \\
&v_t = \Delta v - v + g(u), \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). The diffusion function $D(s) \in C^2([0,\infty))$ and the chemotactic sensitivity function $S(s) \in C^2((0,\infty))$ are given by $D(s) \geq C_d (1 + s)^{-\alpha}$ and $0 < S(s) \leq C_s s(1 + s)^{\beta - 1}$ for all $s \geq 0$ with $C_d, C_s > 0$ and $\alpha, \beta \in \mathbb{R}$. The nonlinear signal secretion function $g(s) \in C^1([0,\infty))$ is supposed to satisfy $g(s) \leq C_g s^\gamma$ for all $s \geq 0$ with $C_g, \gamma > 0$. Global boundedness of solution is established under the specific conditions:

$$0 < \gamma \leq 1 \quad \text{and} \quad \alpha + \beta < \min \left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma \right\}.$$ 

The purpose of this work is to remove the upper bound of the diffusion condition assumed in [9], and we also give the necessary constraint $\alpha + \beta < 1 + \frac{1}{n}$, which is ignored in [9, Theorem 1.1].


Funding. This work is supported by the Chongqing Research and Innovation Project of Graduate Students (No. CYS20271) and Chongqing Basic Science and Advanced Technology Research Program (No. cstc2017jcyjXb0037).

Manuscript received 1st September 2020, revised 9th October 2020 and 9th December 2020, accepted 10th December 2020.

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1. Introduction

In the present work, we consider the following system, which describes the fully parabolic chemotaxis system with nonlinear diffusion, sensitivity and signal secretion

\[
\begin{align*}
  u_t &= \nabla \cdot (D(u)\nabla u - S(u)\nabla v), \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v - v + g(u), \quad x \in \Omega, \quad t > 0, \\
  \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  (u, v)(x, 0) &= (u_0(x), v_0(x)), \quad x \in \Omega,
\end{align*}
\]

(1)

with homogeneous Neumann boundary conditions, where \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a bounded domain, and \( \frac{\partial}{\partial \nu} \) is the derivative of the normal with respect to \( \partial \Omega \). In system (1), \( u = u(x, t) \) and \( v = v(x, t) \) represent the density of population and the concentration of chemicals, respectively. In this article, the diffusion function \( D \in C^2([0, \infty)) \) and the chemotactic sensitivity function \( S \in C^2([0, \infty)) \) with \( S(0) = 0 \) are given by

\[
D(s) \geq C_d(1 + s)^{-\alpha} \quad \text{and} \quad 0 \leq S(s) \leq C_s(1 + s)^{\beta-1} \quad \text{for all} \quad s \geq 0
\]

(2)

with \( C_d, C_s > 0 \) and \( \alpha, \beta \in \mathbb{R} \). The signal secretion function \( g \in C^1([0, \infty)) \) is nonnegative and satisfies

\[
g(s) \leq C_g s^\gamma \quad \text{for all} \quad s \geq 0 \quad \text{with} \quad C_g, \gamma > 0.
\]

(3)

The well-known chemotaxis model for the chemotactic movement of one specie [4] proposed by Keller and Segel, which describes the aggregation phenomenon of the Dictyostelium discoideum, there are many results about this system [1, 3, 9, 12, 13, 15, 16]. For instance, in case \( g(u) = u \), the asymptotics of \( \frac{S(u)}{D(u)} \approx u^\frac{1}{n} \) is critical to distinguish the blow-up and global boundedness: under the condition \( \frac{S(u)}{D(u)} \leq cu^{\frac{2}{n}} - \epsilon \) for all \( u > 1 \) with \( \epsilon > 0 \), Tao and Winkler [12] obtained the global boundedness of solution; while if \( \frac{S(u)}{D(u)} \leq cu^{\frac{2}{n} + \epsilon} \) for all \( u > 1 \) [16], the solution of (1) blow-up either in infinite time or finite time. We note that in [9], global boundedness of solution is established under the conditions that \( \alpha + \beta + \gamma < 1 + \frac{1}{n} \) and \( d_0(1 + u)^{\alpha} \leq D(u) \leq d_1(1 + u)^{\alpha_1} \) with \( d_0, d_1 > 0 \) and \( \alpha, \alpha_1 \in \mathbb{R} \). The purpose of this work is to remove the upper bound of the diffusion condition and give the necessary constraint \( \alpha + \beta < 1 + \frac{1}{n} \) that is ignored in [9, Theorem 1.1]. The main result of this article is described below.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) be a smooth bounded domain. The nonnegative initial data \( (u_0, v_0) \in C^0(\Omega) \times C^1(\Omega) \). Assume that (2) and (3) hold. If \( 0 < \gamma \leq 1 \) and

\[
\alpha + \beta < \min \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma \right\},
\]

then system (1) possesses a unique global bounded classical solution \( (u, v) \) in the sense that there exists some constant \( C > 0 \) satisfying

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} < C \quad \text{for all} \quad t > 0.
\]

**Remark 2.** Compared with the previous study in [9, Theorem 1.1], we give the necessary constraint \( \alpha + \beta < 1 + \frac{1}{n} \) that is ignored in it, and we also remove the restriction on the upper bound of the diffusion function \( D(s) \).

2. Boundedness

Let us state the local existence result, which has been established in [1, 3, 8, 10, 17, 18].

C. R. Mathématique, 2021, 359, nº 2, 161-168
Lemma 3. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that (2) and (3) hold, then there exists $t \in (0, T_{\text{max}})$ such that system (1) has a unique non-negative solution and satisfies
\[ u, v \in C(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^2,1(\overline{\Omega} \times (0, T_{\text{max}})), \]
where $T_{\text{max}}$ denotes the maximal existence time. Moreover, if $T_{\text{max}} < \infty$, then
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \text{ as } t \to T_{\text{max}}. \]

In order to obtain the global boundedness of solution to system (1), we first establish a series of prior estimates; then we treat the dissipative terms on the right hand side of the inequality by using the Gagliardo–Nirenberg inequality; last, we get our final results by controlling the parameter range in the inequality. The ideas come from [9, 12–14].

Lemma 4. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$. Assume that (2) and (3) hold, then the first term of the solution to system (1) satisfies
\[ \|u(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \text{ for all } t \in (0, T_{\text{max}}). \]

Furthermore, assume that $0 < \gamma \leq 1$, if $s \in \left[1, \frac{n}{(n-1)\gamma}\right]$, then there exits $C > 0$ such that
\[ \|v(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}}). \]

Proof. Integrating the first equation of (1) over $\Omega$, (4) can be easily obtained. From the Neumann semigroup estimates method in [5, Lemma 1], (5) can be obtained. $\square$

Before we give the result of main part, we first select the appropriate parameters.

Lemma 5. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain, the nonnegative initial data $(u_0, v_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega})$, assume that (2) and (3) hold. In case $0 < \gamma \leq 1$, if
\[ \alpha + \beta < \min \left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma \right\}, \]
then there exists $s \in \left[1, \frac{n}{(n-1)\gamma}\right]$ such that
\[ \gamma - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha - \beta. \]

Moreover, let $1 < a < \min \left\{\frac{n-2}{n-2-s}, \frac{s}{2} \frac{n}{2} \frac{1}{a} \right\}$ and $b > \max \left\{\frac{n}{2}, \frac{1}{2}\right\}$, we choose some $p_* > 1 + \frac{n+a}{2}$ and $q_* > 1 + \frac{s}{2}$ such that for all $p > p_*$ and $q > q_*$, then we have
\[ \frac{n-2}{n} \frac{p}{p-\alpha} < \frac{1}{a} < \frac{p}{p-\alpha} + 2\beta - 2, \quad \frac{n-2}{n} \frac{2\gamma}{p-\alpha} < \frac{1}{b} < \frac{2\gamma}{p-\alpha} + \frac{1}{q} \left(1 - \frac{2}{n}\right) \quad \text{and} \quad \frac{2b(q-1)}{b-1} > s. \]

Proof. The proof is similar to [9] (also see [12]), so we omitted it here. $\square$

In the following lemma, we obtain the uniform boundedness of $\|u\|_{L^p(\Omega)}$ by establishing a priori estimates and taking appropriate parameters.
Lemma 6. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain. The nonnegative initial data $(u_0, v_0) \in C^0(\Omega) \times C^1(\Omega)$. Assume that (2) - (3) and Lemma 5 hold. If

$$0 < \gamma \leq 1 \quad \text{and} \quad \alpha + \beta < \min \left\{1 + \frac{1}{n}, 1 + \frac{2}{n} - \frac{1}{\gamma}\right\},$$

then there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} + \|\nabla v(\cdot, t)\|_{L^p(\Omega)} \leq C$$

for all $t \in (0, T_{\text{max}})$ with all $p \in [1, \infty) > p_*$ and $q \in \left(\frac{3}{2}, \infty\right) > q_*$. 

Proof. Multiplying both sides the first equation of (1) by $p(u + 1)^{p-1}$ and integrating, then using Young’s inequality, we have

$$\frac{d}{dt} \int_{\Omega} (1 + u)^p \leq -C_d p(p-1) \int_{\Omega} (1 + u)^{p-a-2} |\nabla u|^2 + C_s p(p-1) \int_{\Omega} (1 + u)^{p+a-2} |\nabla u||\nabla v|$$

$$\leq -\frac{C_d p(p-1)}{2} \int_{\Omega} (1 + u)^{p-a-2} |\nabla u|^2 + \frac{C_s^2 p(p-1)}{2C_d} \int_{\Omega} (1 + u)^{p+a+2\beta-2} |\nabla v|^2.$$ (11)

The first term on the right-hand side of the inequality (11) can be expressed as

$$\frac{C_d p(p-1)}{2} \int_{\Omega} (1 + u)^{p-a-2} |\nabla u|^2 = \frac{2C_d p(p-1)}{(p-a)^2} \int_{\Omega} |\nabla (1 + u)^{\frac{p-a}{a}}|^2,$$

this together with (11) which implies

$$\frac{d}{dt} \int_{\Omega} (1 + u)^p + \frac{2C_d p(p-1)}{(p-a)^2} \int_{\Omega} |\nabla (1 + u)^{\frac{p-a}{a}}|^2 \leq \frac{C_s^2 p(p-1)}{2C_d} (1 + u)^{p+a+2\beta-2} |\nabla v|^2$$

for all $t \in (0, T_{\text{max}})$. For a prior estimate of $v$, one can see [9,12,13], for completeness, a brief proof is given here. Applying the second equation of (1), the point-wise identity $\Delta |\nabla v|^2 = 2 |D^2 v|^2 + 2 \nabla v \cdot \nabla \Delta v$ and the fact $|\Delta v|^2 \leq n |D^2 v|^2$, we derive

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{2}{n} \int_{\Omega} |\nabla v|^{2(q-1)} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^{2q}$$

$$\leq \int_{\partial \Omega} |\nabla v|^{2(q-1)} |\Delta v|^2 + 2 \int_{\partial \Omega} |\nabla v|^{2(q-1)} |\nabla v \cdot \nabla g(u)$$

$$-(q-1) \int_{\Omega} |\nabla v|^{2(q-2)} |\nabla v|^{2} \cdot |\nabla v|^{2} + \int_{\Omega} |\nabla v|^{2(q-1)} \frac{\partial |\nabla v|^2}{\partial v} dS$$

$$-2(q-1) \int_{\Omega} |\nabla v|^{2(q-2)} |\nabla v|^{2} \cdot |\nabla v \cdot g(u) - 2 \int_{\Omega} |\nabla v|^{2(q-1)} \Delta v \cdot g(u)$$

for all $t \in (0, T_{\text{max}})$. Using the property of boundary integral without the convexity of domain [6, Lemma 4.2] and the trace inequality [2, Proposition 2.2, 2.4] we have

$$\int_{\partial \Omega} |\nabla v|^{2(q-1)} \frac{\partial |\nabla v|^2}{\partial v} dS \leq 2k_\Omega \int_{\Omega} |\nabla v|^{2q} dS \leq \frac{q-1}{q^2} \int_{\Omega} |\nabla v|^{q} \int_{\Omega} |\nabla v|^{q} + C_1 \int_{\Omega} |\nabla v|^{2q}$$

with some $k_\Omega, C_1 > 0$. Combining (13) with (14) and using Young’s inequality yield

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{2}{n} \int_{\Omega} |\nabla v|^{2(q-1)} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^{2q}$$

$$\leq -\frac{q-1}{2} \int_{\Omega} |\nabla v|^{2(q-2)} |\nabla v|^{2} |\nabla v|^{2} + \frac{q-1}{q^2} \int_{\Omega} |\nabla v|^{q} \int_{\Omega} |\nabla v|^{2q}$$

$$+ \frac{2}{n} \int_{\Omega} |\nabla v|^{2(q-1)} |\Delta v|^2 + \frac{2(q-1)}{q^2} \int_{\Omega} |\nabla v|^{2(q-2)} |\nabla v|^2$$

$$= -\frac{q-1}{2} \int_{\Omega} |\nabla v|^{q} \int_{\Omega} |\nabla v|^{q} + C_1 \int_{\Omega} |\nabla v|^{2q} + \frac{2}{n} \int_{\Omega} |\nabla v|^{2(q-1)} |\Delta v|^2$$

$$+ \left(2(q-1) + \frac{n}{2}\right) \int_{\Omega} |\nabla v|^{2(q-1)} |\Delta v|^2.$$
thus, this together with (3) which implies
\[
\frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} + \frac{q - 1}{q^2} \int_\Omega |\nabla|\nabla v|^{q}|^2 \leq C_8^2 \left(2(q - 1) + \frac{n}{2}\right) \int_\Omega u^{2q} |\nabla v|^{2(q - 1)} + (C_1 - 2) \int_\Omega |\nabla v|^{2q}
\] (15)
for all \( t \in (0, T_{\max}) \). Combining (12) and (15) we have
\[
\frac{d}{dt} \int_\Omega \left( (1 + u)^p + \frac{1}{q} |\nabla v|^{2q} \right) + \frac{q - 1}{q^2} \int_\Omega |\nabla|\nabla v|^{q}|^2 + \frac{2C_4 p(p - 1)}{(p - \alpha)^2} \int_\Omega |\nabla (1 + u)|^{p-\alpha} \leq C_2 \int_\Omega (1 + u)^{p+\alpha+2\beta-2} |\nabla v|^2 + C_2 \int_\Omega (1 + u)^{2\gamma} |\nabla v|^{2(q - 1)} + C_2 \int_\Omega |\nabla v|^{2q}
\] (16)
for all \( t \in (0, T_{\max}) \) with \( C_2 := \max \left\{ C_1, C_2 \left(2(q - 1) + \frac{n}{2}\right) \right\} > 0 \). According to Lemma 5, \( a, b > 1 \), let \( a' := \frac{a}{a - 1} > 1 \) and \( b' := \frac{b}{b - 1} > 1 \), applying Hölder’s inequality to the first two terms on the right-hand side of the inequality (16), we infer
\[
\int_\Omega (1 + u)^{p+\alpha+2\beta-2} |\nabla v|^2 \leq \left( \int_\Omega (1 + u)^{(p+\alpha+2\beta-2)\theta} \right)^{\frac{1}{\theta}} \left( \int_\Omega |\nabla v|^{2\theta} \right)^{\frac{1}{\theta}}
\] (17)
and
\[
\int_\Omega (1 + u)^{2\gamma} |\nabla v|^{2(q - 1)} \leq \left( \int_\Omega (1 + u)^{(2\gamma)\theta} \right)^{\frac{1}{\theta}} \left( \int_\Omega |\nabla v|^{2\theta} \right)^{\frac{1}{\theta}}.
\] (18)
In view of (4) and Gagliardo–Nirenberg inequality \([7, 11] \) we have
\[
\left( \int_\Omega (1 + u)^{(p+\alpha+2\beta-2)\theta} \right)^{\frac{1}{\theta}} = \left\| (1 + u)^{p-\alpha} \right\|_{L^{2(p+\alpha+2\beta-2)\theta}(\Omega)}^{\frac{2(p+\alpha+2\beta-2)\theta}{p-\alpha}} \leq C_3 \left\| (1 + u) \right\|_{L^{2\theta}(\Omega)}^{p-\alpha} \left\| u \right\|_{L^{2\theta}(\Omega)}^{2 \theta (1 - \theta)} + C_3 \left\| (1 + u) \right\|_{L^{2\theta}(\Omega)}^{p-a} \left\| u \right\|_{L^{2\theta}(\Omega)}^{2 \theta (1 - \theta)}
\] (19)
\[
\leq C_4 \left( \int_\Omega |\nabla v|^{2\theta} \right)^{\frac{1}{\theta}} + C_4
\]
with \( C_3, C_4 > 0 \), and \( \theta = \frac{p-\alpha}{2 - \frac{2(a+2\beta-2)\theta}{p-\alpha}} \in (0, 1) \) is guaranteed by (7). Similarly, according to (5) and Gagliardo–Nirenberg inequality again we have
\[
\left( \int_\Omega |\nabla v|^{2\theta} \right)^{\frac{1}{\theta}} = \left\| (1 + u)^{p-\alpha} \right\|_{L^{2\theta}(\Omega)}^{\frac{2\theta}{p-\alpha}} \leq C_5 \left\| (1 + u)^{p-\alpha} \right\|_{L^{2\theta}(\Omega)}^{\frac{2\theta}{p-\alpha}} \left\| u \right\|_{L^{2\theta}(\Omega)}^{2 \theta (1 - \theta)} + C_5 \left\| u \right\|_{L^{2\theta}(\Omega)}^{\frac{2\theta}{p-\alpha}}
\] (20)
\[
\leq C_6 \left( \int_\Omega |\nabla v|^{2\theta} \right)^{\frac{1}{\theta}} + C_6
\]
with \( C_5, C_6 > 0 \), and \( \delta = \frac{3}{2} \frac{\theta}{p-\alpha} \in (0, 1) \) is guaranteed by (8). Combining (19) and (20) with (17), there exists a positive constant \( C_7 > 0 \) such that
\[
C_2 \int_\Omega (1 + u)^{p+\alpha+2\beta-2} |\nabla v|^2 \leq C_7 \left( \left( \int_\Omega |\nabla (1 + u)|^{p-\alpha} \right)^{\frac{p+\alpha+2\beta-2}{p-\alpha}} \right)^{1} \left( \left( \int_\Omega |\nabla v|^{2\theta} \right)^{\frac{1}{\theta}} + 1 \right)
\] (21)
Similarly, in view of Lemma 4, (9) and Gagliardo–Nirenberg inequality again we derive
\[
\left( \int_\Omega (1 + u)^{2\gamma} \right)^{\frac{1}{\theta}} = \left\| (1 + u)^{p-\alpha} \right\|_{L^{2\theta}(\Omega)}^{\frac{2\theta}{p-\alpha}} \leq C_8 \left( \int_\Omega |\nabla (1 + u)|^{p-\alpha} \right)^{\frac{2\theta}{p-\alpha}} + C_8
\] (22)
and
\[
\left( \int \Omega |\nabla v|^{2(q-1)b'} \right)^{\frac{1}{b'}} = \| \nabla v \|^{\frac{2(q-1)}{q}}_{L^2(\Omega)^{q}} \leq C_9 \left( \int \Omega |\nabla v|^q \right)^{\frac{|q-1|\delta}{q}} + C_9
\]  \tag{23}
with some $C_8, C_9 > 0, \bar{\theta} = \frac{p-a}{2} - \frac{p-a}{q} \in (0, 1)$ and $\delta = \frac{\frac{q}{2} + \frac{n}{2} - \frac{p-a}{2} - \frac{p-a}{q}}{\frac{n}{2} + \frac{p-a}{2} - \frac{p-a}{q}} \in (0, 1)$. Then combining (22) and (23) with (18), there exists a positive constant $C_{10} > 0$ such that
\[
C_2 \int_\Omega (1 + u)^{2\gamma} |\nabla v|^{2(q-1)} \leq C_{10} \left( \int_\Omega \left| (1 + u)^{\frac{p-a}{2}} - 1 \right| \right)^{\frac{2\gamma}{p-a}} + \left( \int_\Omega \| \nabla v \|^q \right)^{\frac{|q-1|\delta}{q}} + C_{10}
\]  \tag{24}
Therefore, using (16) in conjunction with (21) and (24), we infer
\[
C_2 \int_\Omega (1 + u)^{2\gamma} |\nabla v|^{2(q-1)} \leq C_{11} \left( \int_\Omega \left| (1 + u)^{\frac{p-a}{2}} - 1 \right| \right)^{\frac{2\gamma}{p-a}} + \left( \int_\Omega \| \nabla v \|^q \right)^{\frac{|q-1|\delta}{q}} + C_{12}
\]  \tag{25}
for all $t \in (0, T_{\text{max}})$ with some $C_{11} > 0$. Thus, according to [12, Lemma 3.1] and Young's inequality, we can obtain
\[
\frac{d}{dt} \int_\Omega \left( (1 + u)^p + \frac{1}{q} |\nabla v|^{2q} \right) + \frac{C_4}{p-a} \int_\Omega \left| (1 + u)^{\frac{p-a}{2}} - 1 \right|^2 + \frac{q-1}{2q^2} \int_\Omega |\nabla v|^q \leq C_2 \int_\Omega |\nabla v|^{2q} + C_{12}
\]  \tag{26}
with $C_{12} > 0$ if the assumptions
\[
\frac{p + a + 2\beta - 2 - \frac{\delta}{q}}{p-a} \leq 1 \quad \text{and} \quad \frac{2\gamma}{p-a} + \frac{(q-1)\delta}{q} \leq 1
\]  \tag{27}
are satisfied. Therefore, in order for the assumptions in (27) to be satisfied, let
\[
h(q) := \frac{p + a + 2\beta - 2 - \frac{\delta}{q}}{p-a} = \frac{p + a + 2\beta - 2}{2} - \frac{2\gamma}{2} + \frac{1}{n} - \frac{1}{2} + \frac{p-a}{2} = \frac{1}{n} - \frac{1}{2} + \frac{p-a}{2}
\]
and
\[
\bar{h}(q) := \frac{2\gamma}{p-a} + \frac{(q-1)\delta}{q} = \frac{\gamma}{n} - \frac{1}{2} + \frac{p-a}{2} = \frac{1}{n} - \frac{1}{2} + \frac{p-a}{2}
\]
according to the condition (6) of Lemma 5, we have
\[
h(q(p)) < 1 \quad \text{and} \quad \bar{h}(q(p)) < 1
\]
with $q(p) := \frac{p-a}{2}$. Since $q(p) \to +\infty$ as $p \to -\infty$, for all $p \geq p^*$, there exists $q \geq q^*$ such that
\[
h(q) < 1 \quad \text{and} \quad \bar{h}(q) < 1,
\]
thus, the assumptions in (27) are satisfied. In order for the inequality (26) to satisfy the form of Gronwall's inequality, using Gagliardo–Nirenberg inequality and Lemma 4 imply
\[
\int_\Omega (1 + u)^p \leq \left( \int_\Omega \left| (1 + u)^{\frac{p-a}{2}} - 1 \right| \right)^{\frac{2p}{p-a}} + C_{13}
\]  \tag{28}
with some \( C_{13} > 0 \), and \( \sigma = \frac{p-a}{\frac{q}{2} + \frac{1}{2}} \in (0, 1) \) is satisfied because of the condition \( p > 1 + \frac{q}{2} \) in Lemma 5. In the same way, we obtain

\[
\left( \frac{1}{q} + C_2 \right) \int_\Omega |\nabla u|^2q = \left( \frac{1}{q} + C_2 \right) \| \nabla v \|^2_{L^1(\Omega)} \leq C_{14} \left( \int_\Omega |\nabla v|^q \right)^{\frac{q}{2}} + C_{14} \leq \frac{q-1}{2q^2} \int_\Omega |\nabla v|^q \leq C_{15}
\]

with some \( C_{14}, C_{15} > 0 \), and \( \overline{\sigma} = \frac{q-\frac{1}{2}}{\frac{q}{2} + \frac{1}{2}} \in (0, 1) \) is satisfied because of the condition \( q > 1 + \frac{q}{2} \) in Lemma 5. Therefore, combining (28) and (29) with (26), which implies

\[
\frac{d}{dt} \int_\Omega \left( (1 + u)^p + \frac{1}{q} |\nabla v|^2q \right) + C_{16} \left( \int_\Omega (1 + u)^p \right)^{\frac{p-a}{mp}} + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \leq C_{17}
\]

for all \( t \in (0, T_{\max}) \) with some \( C_{16}, C_{17} > 0 \), therefore, according to the ODI comparison principle with (30), which implies (10). \( \square \)

Now, we can easily prove Theorem 1.

**Proof of Theorem 1.** In view of [12, Lemmas 3.3 and A.1], we obtain the desired results. \( \square \)

**Acknowledgments**

The authors thank the anonymous referee for their positive and useful comments, which helped him improve further the exposition of the paper.

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