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Dynamical systems / *Systèmes dynamiques*

# Families of polynomials of every degree with no rational preperiodic points

# *Familles de polynômes de degré arbitraire sans points prépériodiques rationnels*

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**Abstract.** Let $K$ be a number field. Given a polynomial $f(x) \in K[x]$ of degree $d \geq 2$, it is conjectured that the number of preperiodic points of $f$ is bounded by a uniform bound that depends only on $d$ and $[K:\mathbb{Q}]$. However, the only examples of parametric families of polynomials with no preperiodic points are known when $d$ is divisible by either 2 or 3 and $K = \mathbb{Q}$. In this article, given any integer $d \geq 2$, we display infinitely many parametric families of polynomials of the form $f_t(x) = x^d + c(t)$, $c(t) \in K(t)$, with no rational preperiodic points for any $t \in K$.

**Résumé.** Soit $K$ un corps de nombres. Étant donné un polynôme $f(x) \in K[x]$ de degré $d \geq 2$, il est conjecturé que le nombre de points prépériodiques de $f$ est borné par une constante ne dépendant que de $d$ et $[K:\mathbb{Q}]$. Cependant, les seuls exemples de familles paramétriques de polynômes sans points prépériodiques supposent $2|d$ ou $3|d$ et $K = \mathbb{Q}$. Dans cet article, étant donné un entier $d \geq 2$, nous démontrons qu'il existe une infinité de familles paramétriques de polynômes de la forme $f_t(x) = x^d + c(t)$, $c(t) \in K(t)$, sans points prépériodiques rationnels pour tout $t \in K$.

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## 1. Introduction

An arithmetic dynamical system over a number field $K$ consists of a rational function $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ of degree at least 2 with coefficients in $K$ where the $m^{th}$ iterate of $f$ is defined recursively by $f^1(x) = f(x)$ and $f^m(x) = f(f^{m-1}(x))$ when $m \geq 2$. A point $P \in \mathbb{P}^n(K)$ is said to be a *periodic* point for $f$ if there exists a positive integer $m$ such that $f^m(P) = P$. If $N$ is the smallest positive integer such that $f^N(P) = P$, then the periodic point $P$ is said to be of *exact period* $N$. A point $P \in \mathbb{P}^n(K)$ is said to be a *preperiodic* point for $f$ if the orbit $\{f^i(P) : i = 0, 1, 2, \ldots\}$ of $P$ is finite, i.e., if some iterate $f^i(P)$ is periodic.

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The following conjecture was proposed by Morton and Silverman in p. 4 of [6].

**Conjecture 1.** *There exists a bound $B(D,n,d)$ such that if $K/\mathbb{Q}$ is a number field of degree $D$, and $f:\mathbb{P}^n(K)\to\mathbb{P}^n(K)$ is a morphism of degree $d\geq 2$ defined over $K$, then the number of $K$-rational preperiodic points of $f$ is bounded by $B(D,n,d)$.*

When $f$ is taken to be a quadratic polynomial over $\mathbb{Q}$, the following conjecture was suggested in [2, Conjecture 2] and [9, Conjecture 2].

**Conjecture 2.** *If $N\geq 4$, then there is no quadratic polynomial $f(x)\in\mathbb{Q}[x]$ with a rational point of exact period $N$.*

The conjecture has been proved when $N=4$, see [5, Theorem 4], and $N=5$, see [2, Theorem 1]. A conditional proof for the case $N=6$ was given in [10, Theorem 7].

Although polynomials described by an equation of the form $x^d+c$, $d\geq 2$, $c\in K$, with rational preperiodic points are scarce, examples of parametric families of polynomials with no preperiodic points are very few in the literature. In Theorem 4 of [3], families of such polynomials were given when $d$ is even or when $d$ is divisible by 3, and $K=\mathbb{Q}$. The main finding of this article can be described as follows. Let $K$ be a number field. Given an arbitrary integer $d\geq 2$, we prove the existence of infinitely many parametric families of polynomials of degree $d$ with no $K$-rational preperiodic points. This is achieved using some recent results on the non existence of rational points on certain twisted superelliptic curves.

## 2. Parametric families of polynomials with no periodic points

In what follows $K$ will denote a number field with ring of integers $\mathcal{O}_K$. The following proposition is [8, Lemma 1].

**Proposition 3.** *Let $f(x)=x^d+c$, where $d\geq 2$ is an integer and $c\in K\setminus\{0\}$. If $f$ has a $K$-rational periodic point, then there exist $a,b\in\mathcal{O}_K$ such that $c=a/b^d$, and $(a\mathcal{O}_K,b^d\mathcal{O}_K)=I^d$ for some ideal $I$ in $\mathcal{O}_K$.*

Proposition 3 shows that polynomials described by an equation of the form $x^d+c$, $d\geq 2$, $c\in K$, with $K$-rational preperiodic points are rare. However, up to the knowledge of the author, the only such family is given as Theorem 4 in [3] where $K=\mathbb{Q}$. The statement of the latter theorem is as follows.

**Theorem 4.** *Let $2\mid d$ and $m\geq 4$; or $3\mid d$ and $m\geq 3$. Then for $t\in\mathbb{Q}$, the polynomial*

$$x^d+\frac{1}{1+t^m}$$

*has no $\mathbb{Q}$-rational preperiodic points.*

Now we state the main result of this work.

**Theorem 5.** *Let $K$ be a number field with ring of integers $\mathcal{O}_K$. Let $d\geq 2$ be an integer. Let $P(T)\in\mathcal{O}_K[T]$ be of degree $N$ a multiple of $d$ such that the multiplicity of each of its roots is at most $d-1$. Assume moreover that the Galois group of $P(T)$ over $K$ has an element fixing no root of $P(T)$. Then there exists $w\in\mathcal{O}_K\setminus\{0\}$ such that the polynomial*

$$f(x)=x^d+\frac{1}{w\cdot P(t)}$$

*has no $K$-rational preperiodic points for any $t\in K$.*

**Proof.** According to [4, Theorem 3.1], given that the Galois group of $P(T)$ over $K$ has an element fixing no root of $P(T)$, it follows that there exists $w \in \mathscr{O}_K \setminus \{0\}$ such that the twisted superelliptic curve defined by $y^d = w \cdot P(T)$ has no $K$-rational points. In other words, there exists no $(y,t,s) \in K^3 \setminus \{(0,0,0)\}$ such that $y^d = w \cdot Q(t,s)$, where $Q(T,S) = S^N \cdot P(T/S)$. In view of Proposition 3, $f$ has no $K$-rational periodic points of any period, hence no $K$-rational preperiodic points for any $t \in K$. □

**Remark 6.** One knows that the proportion of degree $N$ polynomials $P(T) \in \mathscr{O}_K[T]$ with height bounded by $H$ and such that the Galois group of $P(T)$ over $K$ is isomorphic to the symmetric group $S_N$ tends to 1 as $H$ tends to $\infty$, see for example [1, Theorem 2.1]. Consequently, the proportion of fixed degree polynomials $P(T)$ introduced in Theorem 5 with height bounded by $H$ tends to 1 as $H$ tends to $\infty$.

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