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Appell and Sheffer sequences: on their characterizations through functionals and examples

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Abstract. The aim of this paper is to present a new simple recurrence for Appell and Sheffer sequences in terms of the linear functional that defines them, and to explain how this is equivalent to several well-known characterizations appearing in the literature. We also give several examples, including integral representations of the inverse operators associated to Bernoulli and Euler polynomials, and a new integral representation of the re-scaled Hermite d-orthogonal polynomials generalizing the Weierstrass operator related to the Hermite polynomials.

Résumé. L'objectif de cet article est de présenter une nouvelle récurrence simple pour les suites d'Appell et de Sheffer en termes de la fonctionnelle linéaire qui les définit, et d'expliquer comment cela équivaut à plusieurs caractérisations bien connues qui apparaissent dans la littérature. Nous donnons aussi plusieurs exemples, y compris des représentations intégrales des opérateurs inverses associés aux polynômes de Bernoulli et d'Euler, et une nouvelle représentation intégrale des polynômes d'Hermite d-orthogonaux remis à l'échelle, qui généralise l'opérateur de Weierstrass associé aux polynômes d'Hermite.

Keywords. Sheffer and Appell sequences, Bernoulli, Euler and Hermite d-orthogonal polynomials.

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1. Introduction

A remarkable class of polynomials are the Appell sequences having applications in Number theory, Probability, and the theory of functions. They vastly generalize monomials arising naturally from Taylor's formula with integral rest, see [10, Chapitre VI] [11, Appendix A] and they include famous polynomial sequences. For instance, the Bernoulli polynomials useful in numerical integration and asymptotic analysis (Euler–Maclaurin formula [24]); or the Euler polynomials which lead to Euler–Boole formula [9]. On the other hand, a wide class of Appell sequences provides special values of transcendental functions, as recently proved in [20], extending the well-known case of the Bernoulli polynomials as the values at negative integers of the Hurwitz zeta function.

Appell sequences are a subclass of Sheffer sequences (*type zero* polynomials) which were treated by I. M. Sheffer as solutions of families of differential and difference equations [29]. Later on, their study, leaded by Rota and Roman, developed in what it is known today as Umbral Calculus [26–28]. In contrast, a more recent approach has been made using matrix and determinantal representations, see, e.g., [1, 2, 13, 14, 37, 38]. Also, current research has focussed on special sequences [16] and other alternative descriptions of the theory, for instance, through random variables [3, 33].

Our goal in this work is to obtain a new simple recursion for Appell sequences (Theorem 6), their expansions in terms of an arbitrary delta operator (Proposition 7) (as presented in [4] for difference operators), and several examples. Our exposition is based on Umbral Calculus, briefly recalled in Sections 2 and 3. Using this tool, we obtain in Section 3 a new recurrence for Sheffer sequences (Theorem 5) using left-inverses of a delta operator and deduce Theorem 6 as a particular case. In addition, we clarify how the characterization of Sheffer and Appell sequences through a linear functional and a linear operator are naturally equivalent (Theorem 3) (a general fact used systematically in specific cases, see, e.g., [9], [16, Theorems 2.2–2.3], [18, 34]). Our examples are presented in Section 5, in particular, we generalize the Weierstrass operator for classical Hermite polynomials by using Ecalle's accelerator functions to obtain an integral representation of the re-scaled Hermite d-orthogonal polynomials [15] (Proposition 20). We also include the integral representations of the inverse operators associated with the Bernoulli and Euler polynomials (Propositions 13 and 16). These formulas are closely related to their moment expansions [32, 33], but here we deduce them in a direct elementary way. Finally, we collect in Table 1 multiple examples of Appell sequences scattered in the literature, with their respective linear functional and characterization.

2. Preliminaries on Sheffer sequences

We briefly recall some characterizations of Sheffer sequences and set the notations used along the paper. Our summary is based on the expositions [26, 28] of Umbral Calculus whose cornerstone is the twofold identification of formal power series in one variable $\mathbb{C}[[t]]$ as the linear functionals, as well as the shift-invariant linear operators of the ring of univariate polynomials $\mathbb{C}[x]$.

Let $\partial = \partial_x$ be usual differentiation and $T_a : \mathbb{C}[x] \to \mathbb{C}[x]$, $T_a(p)(x) = p(x+a)$ the shift-operator indexed by $a \in \mathbb{C}$. A linear operator $\mathfrak{Q} : \mathbb{C}[x] \to \mathbb{C}[x]$ is *shift-invariant* if $\mathfrak{Q} \circ T_a = T_a \circ \mathfrak{Q}$, for all $a \in \mathbb{C}$. The set Σ of these operators acquires a commutative ring structure via the isomorphism

$$\iota_\partial : \mathbb{C}[[t]] \to \Sigma, \qquad \overline{B}(t) = \sum_{n=0}^{\infty} \frac{\widehat{b}_n}{n!} t^n \longmapsto \mathfrak{Q} = \overline{B}(\partial) = \sum_{n=0}^{\infty} \frac{\widehat{b}_n}{n!} \partial^n, \tag{1}$$

where the product of formal power series corresponds to composition of operators. In fact, $\widehat{b}_n$ is found by evaluating the polynomial $\mathfrak{Q}(x^n)$ at $x = 0$, i.e., $\widehat{b}_n = \mathfrak{Q}(x^n)(0)$. Now, we can extend $\mathfrak{Q}$ to $\mathbb{C}[x][[t]]$ by the rule $P(x,t) = \sum_{n=0}^{\infty} p_n(x) t^n \mapsto \mathfrak{Q}(P)(x,t) = \sum_{n=0}^{\infty} \mathfrak{Q}(p_n)(x) t^n$ (we denote this extension also by $\mathfrak{Q}$, and despite the abuse of notation, its meaning should be clear from the context). In this way, we can recover $\overline{B}(t)$ using the value of $\mathfrak{Q}$ at the exponential: $\mathfrak{Q}(e^{xt})(x,t) = \overline{B}(t)e^{xt}$ and therefore $\iota_\partial^{-1}(\mathfrak{Q})(t) = \mathfrak{Q}(e^{xt})(0,t)$, see Remark 8 for an example.

An operator $\mathfrak{Q} \in \Sigma$ is called a *delta operator* if the polynomial $\mathfrak{Q}(x)$ is a non-zero constant. In this case $\mathfrak{Q}(a) = 0$, for all $a \in \mathbb{C}$ and $\deg(\mathfrak{Q}(p)) = \deg(p) - 1$. Thus the series $\overline{B}(t)$ starts at $n = 1$ with $\widehat{b}_1 = \mathfrak{Q}(x) \neq 0$ and it admits a compositional inverse $B(t) = \sum_{n=1}^{\infty} \frac{b_n}{n!} t^n$. The role of the series

$B(t)$ is to induce the sequence $\{q_n(x)\}_{n\geq 0}$ of *basic polynomials* of $\mathfrak{Q}$. These are defined through the expansion

$$e^{xB(t)}=\sum_{n=0}^{\infty}\frac{q_n(x)}{n!}t^n,\quad \text{and they satisfy } q_n(x+x_0)=\sum_{j=0}^{n}\binom{n}{j}q_j(x)q_{n-j}(x_0). \tag{2}$$

Thus, they are of *binomial type*. Moreover, they are characterized by

$$q_0(x)=1,\quad q_n(0)=0,\quad \text{and}\quad \mathfrak{Q}(q_n)(x)=nq_{n-1}(x),\ \text{for all } n\geq 1. \tag{3}$$

The previous properties are easily established. For instance, (2) follows from equating coefficients in powers of t in $e^{(x+x_0)B(t)}=e^{xB(t)}e^{x_0B(t)}$. Likewise, (1) shows that $\mathfrak{Q}(e^{xB(t)})(x,t)=\sum_{n=1}^{\infty}\frac{\hat{b}_n}{n!}B(t)^n e^{xB(t)}=\overline{B}(B(t))e^{xB(t)}=te^{xB(t)}$, equality that proves the last equation in (3).

A delta operator $\mathfrak{Q}$ also induces a ring isomorphism $\iota_{\mathfrak{Q}}:\mathbb{C}[[t]]\to\Sigma$ by $A(t)\mapsto A(\mathfrak{Q})$, and if $\mathfrak{S}=A(\mathfrak{Q})=\sum_{n=0}^{\infty}\frac{a_n}{n!}\mathfrak{Q}^n$, then $a_n=\mathfrak{S}(q_n)(0)$. Therefore $\mathfrak{S}(e^{xB(t)})(x,t)=A(t)e^{xB(t)}$ and we recover $\iota_{\mathfrak{Q}}^{-1}(\mathfrak{S})(t)=\mathfrak{S}(e^{xB(t)})(0,t)$. Also, note that $\mathfrak{S}$ is invertible if and only if $a_0=\mathfrak{S}(1)\neq 0$, and $\mathfrak{S}^{-1}=1/A(\mathfrak{Q})$, where $1/A(t)$ is the reciprocal of $A(t)$.

We say $\{s_n(x)\}_{n\geq 0}$ is a $\mathfrak{Q}$-*Sheffer sequence* for the delta operator $\mathfrak{Q}$ if $s_0\neq 0$ is constant and

$$\mathfrak{Q}(s_n)(x)=ns_{n-1}(x),\ \text{for all } n\geq 1. \tag{4}$$

These sequences admit several characterizations. First, there is an invertible $\mathfrak{S}\in\Sigma$ satisfying

$$\mathfrak{S}(s_n)=q_n,\ \text{for all } n\geq 0. \tag{5}$$

Second, its exponential generating series has the form

$$\sum_{n=0}^{\infty}\frac{s_n(x)}{n!}t^n=\frac{e^{xB(t)}}{A(t)}, \tag{6}$$

where $A(t)\in\mathbb{C}[[t]]$ has a reciprocal, i.e., $A(0)\neq 0$. Third, the sequence satisfies

$$s_n(x+x_0)=\sum_{k=0}^{n}\binom{n}{k}s_k(x)q_{n-k}(x_0),\quad\text{and}\quad s_n(x)=\sum_{k=0}^{n}\binom{n}{k}s_k(0)q_{n-k}(x),\ \text{for all } n\geq 0. \tag{7}$$

Conditions (4), (5), (6) and both equations in (7) are equivalent to each other as can be checked. The relevant relations are $\mathfrak{S}=\iota_{\mathfrak{Q}}(A)$ and $1/A(t)=\sum_{n=0}^{\infty}s_n(0)t^n/n!$. We highlight that $\mathfrak{S}$ is uniquely associated to $\{s_n(x)\}_{n\geq 0}$ which will be referred as the $\mathfrak{Q}$-Sheffer sequence *relative* to $\mathfrak{S}$ (*$(\mathfrak{Q},\mathfrak{S})$-Sheffer* for short). Finally, after repeated application of $\mathfrak{Q}$ to (5) followed by evaluating at $x=0$, we find that

$$\mathfrak{S}(\mathfrak{Q}^m(s_n))(0)=n!\delta_{n,m},\quad \text{for all } n,m\geq 0, \tag{8}$$

where $\delta_{n,m}$ is the Kronecker delta.

3. The use of linear functionals

Another characterization of Sheffer sequences is available through functionals of $\mathbb{C}[x]$. It is based on the identification of the dual space $\mathbb{C}[x]^*$ with Σ (and thus with $\mathbb{C}[[t]]$ via (1)).

Lemma 1. *We have the linear isomorphism* $\jmath:\mathbb{C}[x]^*\to\Sigma$ *given by*

$$L\longmapsto \mathfrak{L}(p)(x):=L(T_x(p)),\quad \textit{and having as inverse}\quad \mathfrak{L}\longmapsto L(p)=\mathfrak{L}(p)(0),\quad p\in\mathbb{C}[x]. \tag{9}$$

Proof. Given $L\in\mathbb{C}[x]^*$ the map $\mathfrak{L}=\jmath(L)$ is clearly linear. It is also shift-invariant since

$$\mathfrak{L}(T_a(p))(x)=L(T_x(T_a(p)))=L(T_{x+a}(p))=\mathfrak{L}(p)(x+a)=T_a(\mathfrak{L}(p))(x),\quad \text{for all } a\in\mathbb{C}.$$

Conversely, if $\mathfrak{L}\in\Sigma$ and $L(p)=\mathfrak{L}(p)(0)$, then $L\in\mathbb{C}[x]^*$ and $L(T_{x_0}(p))=\mathfrak{L}(T_{x_0}(p))(0)=(T_{x_0}\circ\mathfrak{L})(p)(0)=\mathfrak{L}(p)(x_0)$, for all $x_0\in\mathbb{C}$. Thus the maps in (9) are inverses one of each other. □

Remark 2. We can codify $L \in \mathbb{C}[x]^*$ by the values $L_n := L(x^n)$ known as the *moments* of L. Extending L to $\mathbb{C}[x][[t]]$ by $L(\sum_{n=0}^{\infty} p_n(x)t^n) = \sum_{n=0}^{\infty} L(p_n)t^n$ we have the relation $\mathfrak{L}(e^{xB(t)})(x_0,t) = L(T_{x_0}(e^{xB(t)})) = L(e^{(x+x_0)B(t)}) = e^{x_0B(t)}L(e^{xB(t)})$. In particular, the *indicator series* of L

$$L(e^{xt}) = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!}, \quad \text{satisfies} \quad \mathfrak{L}(e^{xt})(x,t) = L(e^{xt})e^{xt}. \tag{10}$$

Now we are in position to give two equivalent ways to characterize Sheffer sequences using functionals.

Theorem 3. *Given $S \in \mathbb{C}[x]^*$, there is a unique $\mathfrak{Q}$-Sheffer sequence $\{s_n\}_{n\geq 0}$ satisfying*

$$S(s_0) = 1 \quad \textit{and} \quad S(s_n) = 0, \textit{ for all } n \geq 1, \tag{11}$$

or equivalently,

$$S(T_{x_0}(s_n)) = q_n(x_0), \quad \textit{for all } n \geq 0 \textit{ and } x_0 \in \mathbb{C}. \tag{12}$$

Indeed, $\{s_n\}_{n\geq 0}$ is the $(\mathfrak{Q},\mathfrak{S})$-Sheffer sequence, $\mathfrak{S} = \jmath(S)$, having generating series $e^{xB(t)}/S(e^{xB(t)})$. Conversely, given a $\mathfrak{Q}$-Sheffer sequence $\{s_n\}_{n\geq 0}$, there is a unique $S \in \mathbb{C}[x]^*$ such that (11) *holds.*

Proof. First note that (11) is simply (12) for $x_0 = 0$ as (3) shows. Conversely, if (11) holds, then (12) follows by applying S to the first equation in (7). Furthermore, if $\mathfrak{S} = \jmath(S)$, then equation (5) is equivalent to (12) since $S(T_{x_0}(s_n)) = \mathfrak{S}(s_n)(x_0)$, thanks to Lemma 1. Also, applying S to equation (6) we find $A(t) = S(e^{xB(t)})$.

Finally, given the $(\mathfrak{Q},\mathfrak{S})$-Sheffer sequence $\{s_n\}_{n\geq 0}$, if $S' \in \mathbb{C}[x]^*$ satisfies (11), then $\mathfrak{S}' = \jmath(S') \in \Sigma$ is invertible and satisfies $\mathfrak{S}'(s_n) = q_n$ for all n. Since $\mathfrak{S}$ is characterized by this condition, then $\mathfrak{S}' = \mathfrak{S}$ and $S' = \jmath^{-1}(\mathfrak{S}) = S$ is also uniquely determined. □

The previous theorem shows that it is equivalent to have a $\mathfrak{Q}$-Sheffer sequence $\{s_n\}_{n\geq 0}$, an invertible operator $\mathfrak{S} \in \Sigma$ or a functional $S \in \mathbb{C}[x]^*$ such that $S(1) \neq 0$. Thus we can refer to $\{s_n\}_{n\geq 0}$ as the $(\mathfrak{Q},\mathfrak{S},S)$-*Sheffer sequence*, where $\mathfrak{S} = \jmath(S)$.

Remark 4. The k-fold iteration $\mathfrak{S}^k$ of an invertible operator $\mathfrak{S} \in \Sigma$ produces the $(\mathfrak{Q},\mathfrak{S}^k,S^k)$-Sheffer sequence $\{s_n^{(k)}\}_{n\geq 0}$, $S^k := \jmath^{-1}(\mathfrak{S}^k)$, having exponential generating series $e^{xB(t)}/S(e^{xB(t)})^k$. According to (5) we find $s_n^{(k)} = \mathfrak{S}^{\circ(-k)}(q_n) = \mathfrak{S} \circ \mathfrak{S}^{\circ(-k-1)}(q_n) = \mathfrak{S}(s_n^{(k+1)})$ holding for all $k,n \in \mathbb{N}$. We also highlight that if S admits the representation

$$S(p) = \int_I p(s)w(s)\,\mathrm{d}s, \quad \text{and} \quad \mathfrak{S}(p)(x) = \int_I p(x+s)w(s)\,\mathrm{d}s,$$

where $I \subseteq \mathbb{R}$ is an interval and $w : I \to \mathbb{C}$ is such that S_n are all finite, then

$$S^k(p) = \int_{I^k} p(s_1+\cdots+s_k)w(s_1)\cdots w(s_k)\,\mathrm{d}\boldsymbol{s}, \quad \mathfrak{S}^k(p)(x) = \int_{I^k} p(x+s_1+\cdots+s_k)w(s_1)\cdots w(s_k)\,\mathrm{d}\boldsymbol{s},$$

where $\mathrm{d}\boldsymbol{s} = \mathrm{d}s_1\cdots\mathrm{d}s_k$ and the integration is taken over the kth Cartesian product $I^k \subseteq \mathbb{R}^k$.

A $\mathfrak{Q}$-Sheffer sequence can be calculate though its generating series, or using determinants, see [13, 14, 37]. Here we present a new recursion formula using left-inverses for $\mathfrak{Q}$.

Theorem 5. *Let $\mathfrak{Q}$ be a delta operator and $\overline{B} = \iota_\partial^{-1}(\mathfrak{Q})$. Then each $x_0 \in \mathbb{C}$ defines*

$$\mathfrak{Q}_{x_0}^{-1}(p) := \frac{\partial}{\overline{B}(\partial)}\left(\int_{x_0}^{x} p(s)\,\mathrm{d}s\right), \textit{ which is a left-inverse for } \mathfrak{Q}.$$

Here the integral is taken over the line segment from x_0 to x. Through it, the $(\mathfrak{Q},\mathfrak{S},S)$-Sheffer sequence $\{s_n\}_{n\geq 0}$ can be calculated recursively by $s_0 = 1/S(1)$ and

$$s_n = n\mathfrak{Q}_{x_0}^{-1}(s_{n-1}) - \frac{n}{s_0}S\left(\mathfrak{Q}_{x_0}^{-1}(s_{n-1})\right), \quad n \geq 1. \tag{13}$$

Proof. By writing $\mathfrak{Q} = \partial \circ \mathfrak{P}$, where $\mathfrak{P} \in \Sigma$ is invertible and $\mathfrak{P}^{-1} = \partial / \overline{B}(\partial)$, we find

$$\mathfrak{Q} \circ \mathfrak{Q}_{x_0}^{-1}(p)(x) = (\partial \circ \mathfrak{P}) \left(\mathfrak{P}^{-1} \left(\int_{x_0}^{x} p(s)\, \mathrm{d}s \right) \right) = \partial \left(\int_{x_0}^{x} p(s)\, \mathrm{d}s \right) = p(x),$$

as required. Now, setting $s_0' = s_0 = 1/S(1)$ and $s_n'(x)$ for the left-side of (13) we have

$$\mathfrak{Q}(s_n') = n\mathfrak{Q}(\mathfrak{Q}_{x_0}^{-1}(s_{n-1}')) = ns_{n-1}', \quad \text{and} \quad S(s_n') = nS(\mathfrak{Q}_{x_0}^{-1}(s_{n-1}')) - \frac{n}{s_0} S\left(\mathfrak{Q}_{x_0}^{-1}(s_{n-1}')\right) S(1) = 0,$$

for all $n \geq 1$. Thus $\{s_n'\}_{n\geq 0}$ is the $(\mathfrak{Q}, \mathrm{j}(S), S)$-sequence, so $s_n = s_n'$, for all n. □

4. The case of Appell sequences

The main example of Sheffer sequences are the *Appell sequences* (in honor of P. E. Appell (1880) [6]) corresponding to

$$\mathfrak{Q} = \partial, \quad \text{for which } B(t) = \overline{B}(t) = t, \quad \text{and } q_n(x) = x^n. \tag{14}$$

In this case we see $\{p_n(x)\}_{n\geq 0}$ is an Appell sequence if $p_0 \neq 0$ is a constant and

$$\frac{\mathrm{d}p_n}{\mathrm{d}x}(x) = np_{n-1}(x), \qquad n \geq 1, \tag{15}$$

Equivalently, there is a unique invertible $\mathfrak{L} \in \Sigma$ satisfying $\mathfrak{L}(p_n) = x^n$, for all $n \geq 0$, the sequence has a exponential series of the form $e^{xt}/L(e^{xt})$, where $L = \mathrm{j}^{-1}(\mathfrak{L}) \in \mathbb{C}[x]^*$, or they satisfy the corresponding equations to (7). We can refer to $\{p_n\}_{n\geq 0}$ as the $(\mathfrak{L}, L)$-*Appell* sequence. In this setting, Theorems 3 and 5 take the following form.

Theorem 6. *The $(\mathfrak{L}, L)$-Appell sequence $\{p_n\}_{n\geq 0}$ is the unique Appell sequence satisfying*

$$L(T_{x_0}(p_n)) = x_0^n, \quad \text{for all } n \geq 0 \text{ and } x_0 \in \mathbb{C}. \tag{16}$$

Furthermore, it can be calculated recursively by $p_0 = 1/L(1)$ and

$$p_n(x) = n\int_{x_0}^{x} p_{n-1}(s)\, \mathrm{d}s - \frac{n}{p_0} L\left(\int_{x_0}^{x} p_{n-1}(s)\, \mathrm{d}s \right). \tag{17}$$

Although we have deduced the previous theorem from the more general case of Sheffer sequences, it can obtained by directly means. We also remark that (16) is referred as the *mean value property* for Appell sequences connected to random variables, see [33, Proposition 2.7] and the references therein.

Additionally, we can also express an Appell sequence in terms of a delta operator as follows.

Proposition 7. *Let $\{p_n\}_{n\geq 0}$ be the $(\mathfrak{L}, L)$-Appell sequence with generating series $e^{xt}/L(e^{xt}) = C(t)e^{xt}$. If $\mathfrak{Q}$ is a delta operator with $B(t) \in \mathbb{C}[[t]]$ as in equation (2), and $(C \circ B)(t) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k$, then*

$$\mathfrak{L}^{-1} = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} \mathfrak{Q}^k, \quad \text{and thus} \quad p_n(x) = \mathfrak{L}^{-1}(x^n) = \sum_{k=0}^{n} \frac{\alpha_k}{k!} \mathfrak{Q}^k(x^n).$$

Proof. The operator $\mathfrak{Q}_1 = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} \mathfrak{Q}^k \in \Sigma$ is invertible since $\alpha_0 = C(0) \neq 0$. Recalling that $\mathfrak{Q}(e^{xt}) = \overline{B}(t)e^{xt}$, we find

$$\sum_{n=0}^{\infty} \mathfrak{Q}_1(x^n) \frac{t^n}{n!} = \mathfrak{Q}_1(e^{xt}) = (C \circ B)(\overline{B}(t))e^{xt} = C(t)e^{xt} = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} t^n.$$

Therefore, $\mathfrak{L}^{-1}(x^n) = p_n(x) = \mathfrak{Q}_1(x^n)$, for all n, and $\mathfrak{Q}_1 = \mathfrak{L}^{-1}$ as required. Finally, $\mathfrak{Q}_1(x^n) = \sum_{k=0}^{n} \frac{\alpha_k}{k!} \mathfrak{Q}^k(x^n)$, since $\mathfrak{Q}^k(x^n) = 0$ if $k > n$ as $\mathfrak{Q}^k$ lowers the degree of a polynomial by k. □

Remark 8. The previous proposition was recently studied in [4] for $\mathfrak{Q}=\Delta_1=\Delta$, the difference operator of step one. Let us recall that for each $h\in\mathbb{C}^*$, the *difference operator*

$$\Delta_h := T_h - 1, \quad \text{i.e.,} \quad \Delta_h(p)(x) = p(x+h)-p(x),$$

constitute a delta operator for which $\overline{B}(t)=\Delta_h(e^{xt})(0,t)=e^{xt}(e^{ht}-1)|_{x=0}=e^{ht}-1$ and $B(t)=\log(1+t)/h$. Thus $e^{xB(t)}=(1+t)^{x/h}=\sum_{n=0}^{\infty}(x/h)_n t^n/n!$ and Δ_h has $q_n(x)=(x/h)_n$ as basic sequence. Here $(a)_n := a(a-1)\cdots(a-n+1)$ is the falling factorial.

Remark 9. Any pair of series $B(t),\overline{B}(t)\in t\mathbb{C}[[t]]$, $B'(0)\neq 0$, $\overline{B}'(0)\neq 0$, compositional inverses one of each other, define two families of numbers $\{s_B(n,k)\}_{n\geq k}$ and $\{S_B(n,k)\}_{n\geq k}$ determined by

$$\frac{B(t)^k}{k!}=\sum_{n=k}^{\infty} s_B(n,k)\frac{t^n}{n!}, \quad \frac{\overline{B}(t)^k}{k!}=\sum_{n=k}^{\infty} S_B(n,k)\frac{t^n}{n!}, \tag{18}$$

just as $B(t)=\log(1+t)$, $\overline{B}(t)=e^t-1$ define the *Stirling numbers* of first and second kind [12, p. 50],

$$\frac{\log(1+t)^k}{k!}=\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!}, \quad \frac{(e^t-1)^k}{k!}=\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}. \tag{19}$$

If $B,\overline{B}$ are associated to the delta operator $\mathfrak{Q}$ as in (2), then (18) induces the inverse relations

$$q_n(x)=\sum_{k=0}^{n} s_B(n,k)x^k, \quad x^n=\sum_{k=0}^{n} S_B(n,k)q_k(x),$$

and

$$\alpha_k=\sum_{k=0}^{n} s_B(n,k)p_k(0), \quad p_k(0)=\sum_{k=0}^{n} S_B(n,k)\alpha_k,$$

where $\{\alpha_k\}_{k\geq 0}$ and $\{p_k(0)\}_{k\geq 0}$ are as in Proposition 7. These follows from expanding $e^{xB(t)}$, $e^{xt}=e^{xB(\bar{B}(t))}$, $C(t)$, and $C(t)=(C\circ B)(\overline{B}(t))$, respectively, as in the usual case of Stirling numbers [12, p. 144].

5. Examples

This final section is aimed to apply the previous results to concrete examples focusing on the role of the functional involved. In particular, we find integral representations for Bernoulli and Euler polynomials, and also for Hermite d-orthogonal polynomials [15]. Finally, we collect in Table 1 a list of important $(\mathfrak{L},L)$-Appell sequences and their characterization via Theorem 6, including the recent results of Kummer hypergeometric polynomials given in [16].

Remark 10. Any functional $S\in\mathbb{C}[x]^*$ admits a representation of the form

$$S(p)=\int_0^{+\infty} p(s)\,\mathrm{d}\beta(s), \qquad \mathfrak{S}(p)(x)=\int_0^{+\infty} p(x+s)\,\mathrm{d}\beta(s),$$

for some function $\beta:(0,+\infty)\to\mathbb{C}$ of bounded variation as it was proved by Boas [8] in relation to the Stieljes moment problem. Thus the characterization of Theorem 3 can be written as

$$\int_0^{+\infty} s_n(x_0+s)\,\mathrm{d}\beta(s)=q_n(x_0),$$

and equation (8) takes the form $\int_0^{+\infty}\mathfrak{Q}^{(m)}(s_n)(s)\mathrm{d}\beta(s)=n!\delta_{n,m}$. This reasoning contains the early characterization of Appell sequences of Thorne [35], soon after generalized by Sheffer [30].

Example 11. Let $\{p_n(x)\}_{n\geq 0}$ be the $(\mathfrak{L},L)$-Appell sequence, $C(t)=1/L(e^{xt})$, and $\alpha,\beta\in\mathbb{C}$ with $\beta\neq 0$. Then $\{p_n(x-\alpha)\}_{n\geq 0}$ is the $(\mathfrak{L}\circ T_\alpha, L\circ T_\alpha)$-Appell sequence and $\{\beta^{-n}p_n(\beta x)\}_{n\geq 0}$ is the $(\mathfrak{L}\circ\mathscr{H}_\alpha, L\circ\mathscr{H}_\alpha)$-Appell sequences, where $\mathscr{H}_\beta:\mathbb{C}[x]\to\mathbb{C}[x]$ is the homothecy $\mathscr{H}_\beta(p)(x)=p(x/\beta)$. Indeed, these sequences have as generating series $C(t)e^{(x-\alpha)t}=e^{xt}/L(e^{(x+\alpha)t})$, and $C(t/\beta)e^{xt}=e^{xt}/L(e^{xt/\beta})$, respectively.

Example 12 (Bernoulli polynomials). They are defined by the expansion

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \qquad B_n(x) = \sum_{j=0}^{n}\binom{n}{j}B_j(0)x^{n-j}.$$

The $B_j = B_j(0)$ are the Bernoulli numbers that satisfy $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2j+1} = 0$, $j \geq 1$. We see the Bernoulli polynomials conform the Appell sequence relative to

$$I(p) = \int_0^1 p(s)\,\mathrm{d}s \quad \text{and} \quad \mathfrak{I}(p)(x) := \int_0^1 p(x+s)\,\mathrm{d}s, \quad \text{since } I(e^{xt}) = \frac{e^t-1}{t}.$$

Theorem 6 asserts they are characterized by the condition $\mathfrak{I}(B_n)(x) = \int_0^1 B_n(x+s)ds = x^n$, or equivalently after differentiation, by the equation $\Delta(B_n)(x) = B_n(x+1) - B_n(x) = nx^{n-1}$, which is a well-known result. Furthermore, we can compute them recursively by $B_0(x) = 1$ and

$$B_n(x) = n\int_0^x B_{n-1}(t)\,\mathrm{d}t - n\int_0^1\int_0^u B_{n-1}(t)\,\mathrm{d}t\mathrm{d}u.$$

Following Remark 4 we find the k-fold iteration of I is $I^k(p) = \int_{[0,1]^k} p(s_1+\cdots+s_k)d\boldsymbol{s}$ which produces the *kth order Bernoulli polynomials* $B_n^{(k)}(x)$ having $t^k e^{xt}/(e^t-1)^k$ as generating exponential series. These are also known as *Nørlund polynomials* due to N. E. Nørlund who introduced them in 1922 [22]. Moreover, we can write these polynomials in terms of Δ as

$$B_n(x) = \sum_{j=0}^{n}\frac{(-1)^j}{j+1}\Delta^j(x^n), \quad \text{and} \quad B_n^{(k)}(x) = \sum_{j=0}^{n}\frac{k!}{(k+j)!}s(k+j,k)\Delta^j(x^n),$$

by using Proposition 7. In fact, in this case $B(t) = \log(1+t)$ and the previous formulas follow from (19) since $((e^t-1)/t)^k \circ B(t) = \log(1+t)^k/t^k$, c.f., [4, Theorem 5].

An interesting question is to determine an analytic representation for the inverse operator of $\mathfrak{I}$ which in turn gives left-inverses for Δ and an analytic representation of Bernoulli polynomials. We remark that the formula below to invert Δ is familiar in the theory of difference equations and it has been used to justify Ramanujan summation, see [11, Theorem 1].

Proposition 13. *The map* $L(p) = p(0) - \frac{p'(0)}{2} - i\int_0^{+\infty}\frac{p'(is)-p'(-is)}{e^{2\pi s}-1}\,\mathrm{d}s$ *verifies* $L(e^{xt}) = t/(e^t-1)$. *Therefore, the inverse operator of* $\mathfrak{I}(p)(x) = \int_0^1 p(x+s)\,\mathrm{d}s$ *is*

$$\mathfrak{I}^{-1}(p)(x) = p(x) - \frac{p'(x)}{2} - i\int_0^{+\infty}\frac{p'(x+is)-p'(x-is)}{e^{2\pi s}-1}\,\mathrm{d}s.$$

Moreover, the difference operator Δ *admits the left-inverses*

$$\Delta_{x_0}^{-1}(p) = \int_{x_0}^{x} p(s)\,\mathrm{d}s - \frac{p(x)}{2} - i\int_0^{+\infty}\frac{p(x+is)-p(x-is)}{e^{2\pi s}-1}\,\mathrm{d}s.$$

Furthermore, the Bernoulli polynomials admit the integral representation

$$B_n(x) = x^n - \frac{n}{2}x^{n-1} - in\int_0^{+\infty}\frac{(x+is)^{n-1}-(x-is)^{n-1}}{e^{2\pi s}-1}\,\mathrm{d}s.$$

Proof. For the first statement note $L(1) = 1 = B_0$, $L(x) = -1/2 = B_1$ and $L(x^{2j+1}) = 0$, $j \geq 1$ since the derivative of x^{2j+1} is an even function. For the even powers we find

$$L(x^{2j}) = 2j(-1)^{j+1}\int_0^{+\infty}\frac{2s^{2j-1}}{e^{2\pi s}-1}\,\mathrm{d}s = B_{2j},$$

values that are familiar in the study of Abel–Plana formula [24, p. 291], [25, 24.7.2]. Now, the operator $\mathfrak{L} = \mathrm{j}(L)$ is the inverse of $\mathfrak{I}$ since $\mathfrak{I}(e^{xt}) = (e^t-1)/t$. Finally, Proposition 5 shows $\Delta_{x_0}^{-1}(p) = \frac{\partial}{e^{h\partial}-1}\left(\int_{x_0}^x p(s)\,\mathrm{d}s\right)$ and the previous example proves $B_n(x) = \mathfrak{I}^{-1}(x^n)$ as required. □

Example 14 (Euler polynomials). We consider the Apostol–Euler polynomials determined by

$$\frac{e^{xt}}{1+\beta(e^t-1)}=\sum_{n=0}^{\infty}E_n(\beta;x)\frac{t^n}{n!},\quad \text{for a fixed } \beta\neq 0. \tag{20}$$

They are the Appell sequence relative to

$$L(p)=(1-\beta)p(0)+\beta p(1),\quad \mathfrak{L}(p)(x)=(1-\beta)p(x)+\beta p(x+1),\quad \text{since } L(e^{xt})=1+\beta(e^t-1).$$

The case $\beta=1/2$ recovers the classical Euler polynomials $E_n(1/2;x)=E_n(x)$. Theorem 6 shows the Apostol–Euler are characterized by $(1-\beta)E_n(\beta;x)+\beta E_n(\beta;x+1)=x^n$. Moreover, they are given recursively by $E_0(\beta;x)=1$ and

$$E_n(\beta;x)=n\int_0^x E_{n-1}(\beta;t)\,\mathrm{d}t-\beta n\int_0^1\int_0^u E_{n-1}(\beta;t)\,\mathrm{d}t\mathrm{d}u.$$

The *kth Apostol–Euler polynomials* $E_n^{(k)}(\beta;x)$ are the Appell sequence relative to $L^k(p)=\sum_{j=0}^k\binom{k}{j}\beta^j(1-\beta)^{k-j}p(j)$ and with exponential generating series $e^{xt}/\left(1+\beta(e^t-1)\right)^k$. The case $\beta=2$ corresponds to the *kth Euler polynomials* $E_n^{(k)}(x)$. Finally, Proposition 7 proves that

$$E_n^{(k)}(\beta;x)=\sum_{j=0}^{n}\binom{j+k-1}{j}(-1)^j\beta^j\Delta^j(x^n),\quad\text{and}\quad E_n(x)=\sum_{j=0}^{n}\frac{(-1)^j}{2^j}\Delta^j(x^n),$$

since $(1+\beta(e^t-1))^{-k}\circ\log(1+t)=(1+\beta t)^{-k}=\sum_{j=0}^{\infty}\binom{j+k-1}{j}(-1)^j\beta^jt^j$, c.f., [4, Theorem 7].

Remark 15. It is worth recalling that Apostol–Euler polynomials are usually defined through the expansion $2e^{xt}/(\lambda e^t+1)$, for a parameter $\lambda\in\mathbb{C}^*$, which up to a constant is equivalent to (20). On the other hand, these can be expressed in terms of Apostol–Bernoulli polynomials that were introduced by T. Apostol in relation with the Lerch zeta function [5], see, e.g., [21] for more details on these connections.

In analogy with Proposition 13, we can write an analytic expression for the inverse of the operator inducing the Euler polynomials. More specifically, we have.

Proposition 16. *The functional* $L(p)=\int_0^{+\infty}\frac{p\left(-\frac{1}{2}+\frac{is}{2}\right)+p\left(-\frac{1}{2}-\frac{is}{2}\right)}{e^{\pi s/2}+e^{-\pi s/2}}\,\mathrm{d}s$ *satisfies* $L(e^{xt})=2/(e^t+1)$. *In consequence,*

$$\mathfrak{J}^{-1}(p)(x)=\int_0^{+\infty}\frac{p\left(x-\frac{1}{2}+\frac{is}{2}\right)+p\left(x-\frac{1}{2}-\frac{is}{2}\right)}{e^{\pi s/2}+e^{-\pi s/2}}\,\mathrm{d}s$$

is the inverse of $\mathfrak{J}(p)(x)=(p(x+1)+p(x))/2$. *Furthermore, the Euler polynomials admit the integral representation*

$$E_n(x)=\int_0^{+\infty}\frac{\left(x-\frac{1}{2}+\frac{is}{2}\right)^n+\left(x-\frac{1}{2}-\frac{is}{2}\right)^n}{e^{\pi s/2}+e^{-\pi s/2}}\,\mathrm{d}s.$$

Proof. It is sufficient to establish the first formula, the remaining ones follow as in the previous proposition. For this purpose we write the Euler polynomials in terms of the *Euler numbers* E_n as

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{i^k}{2^k}E_k\left(x-\frac{1}{2}\right)^{n-k}\quad\text{where}\quad\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}i^nE_n\frac{t^n}{n!}.$$

In fact, $\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=2e^{xt}/(e^t+1)=2e^{(x-1/2)t}/(e^{t/2}+e^{-t/2})$. Also note that $E_{2j+1}=0$, $j\geq 0$. Now, by Example 11, it is enough to show that the operator

$$L'(p)=\int_0^{+\infty}\frac{p\left(\frac{is}{2}\right)+p\left(-\frac{is}{2}\right)}{e^{\pi s/2}+e^{-\pi s/2}}\,\mathrm{d}s,$$

satisfies $L'(e^{xt}) = 2/(e^{t/2}+e^{-t/2})$. In fact, $L = L'\circ T_{-1/2}$ and therefore $L(e^{xt}) = e^{-t/2}L'(e^{xt}) = 2/(e^t+1)$ as required. It is clear that $L'(x^{2j+1}) = 0$ since these are odd functions. For the even powers we also find [25, 24.7.6]

$$L'(x^{2j}) = \frac{(-1)^j}{2^{2j-1}}\int_0^{+\infty}\frac{s^{2j}}{e^{\pi s/2}+e^{-\pi s/2}}\,\mathrm{d}s = (-1)^j\frac{E_{2j}}{2^{2j}},$$

as required. □

Remark 17. Although the integral representations of Euler numbers

$$E_{2j} = 2\int_0^{+\infty}\frac{s^{2j}}{e^{\pi s/2}+e^{-\pi s/2}}\,\mathrm{d}s = \left(\frac{2}{\pi}\right)^{2j+1}\int_0^{+\infty}\frac{\ln(u)^{2j}}{u^2+1}\mathrm{d}u,$$

are known (here $u = e^{\pi s/2}$), we find instructive to include a simple proof using calculus of residues. Indeed, we can find $A_n = (2/\pi)^{n+1}\int_0^{+\infty}\frac{\ln(u)^n}{1+u^2}\,\mathrm{d}u$, recursively: using the branch of the logarithm $\log(z) = \ln|z| + i\arg(z)$ with $-\pi/2 < \arg(z) < 3\pi/2$, the Residue Theorem shows that $\int_{\gamma_{\epsilon,R}}\frac{\log(z)^n}{z^2+1}\,\mathrm{d}z = 2\pi i\,\mathrm{Res}\big(\frac{\log(z)^n}{z^2+1}, i\big) = i^n\pi^{n+1}/2^n$. Here $0<\epsilon<1$, $R>1$ and $\gamma_{\epsilon,R}$ is the path formed by the segments from $-R$ to $-\epsilon$ and ϵ to R, and the corresponding semicircles centered at 0 of radius ϵ and R, oriented positively. Letting $\epsilon\to 0^+$ and $R\to+\infty$ the integral over the arcs tends to 0 and we obtain

$$\int_0^{+\infty}\frac{\ln(u)^n}{1+u^2}\,\mathrm{d}u + \int_0^{+\infty}\frac{(\ln(u)+i\pi)^n}{1+u^2}\,\mathrm{d}u = \frac{i^n\pi^{n+1}}{2^n}.$$

Thus the sequence A_n satisfies $A_n + \sum_{k=0}^{n-1}\binom{n}{k}i^k2^kA_{n-k} = 2i^n$. If we set $A(t) = \sum_{n=0}^{\infty}A_nt^n/n!$, this recursion is equivalent to the equation $A(t)+A(t)e^{2it} = 2e^{it}$. Consequently, $A(t) = 2/(e^{it}+e^{-it}) = \sum_{n=0}^{\infty}(-1)^nE_nt^n/n!$ and $A_n = (-1)^nE_n$ as needed.

Now we proceed to extend the integral representation and characterization of Hermite polynomials that are essentially the only Appell orthogonal sequence [31]. But first we need a remark.

Remark 18. Given $L\in\mathbb{C}[x]^*$ and an integer $m\geq 1$, we can construct a functional recording only the moments of L indexed by multiples of m. Indeed, recalling Example 11 and fixing the m-th root of unity $\omega_m := e^{2\pi i/m}$, we see that the functional

$$L_m = \frac{1}{m}(L + L\circ\mathscr{H}_{\omega_m^{-1}} + \cdots + L\circ\mathscr{H}_{\omega_m^{-(m-1)}})$$

has moments $L_m(x^{nm}) = L_{nm}$ and equal to zero otherwise, i.e., $L_m(e^{xt}) = \sum_{n=0}^{\infty}L_{nm}\frac{t^{nm}}{(nm)!}$. This can be checked using the identity $1+\omega_m^j+\cdots+\omega_m^{j(m-1)} = 0$, valid for $j = 1,\dots,m-1$.

Example 19 (Hermite polynomials). Fix an integer $d\geq 1$. We shall describe an analytic expression for the functional defining the Appell sequence determined by the expansion

$$\exp\left(xt - t^{d+1}\right) = \sum_{n=0}^{\infty}H_n^{(d)}(x)\frac{t^n}{n!},$$

which correspond to a particular case of Gould–Hopper polynomials [17]. Our approach is based on Ecalle's accelerator operators familiar in the theory of multisummability of power series, see [7, Chapter 11]. To this end, we recall the *accelerator function*

$$C_\alpha(z) := \frac{1}{\pi}\sum_{n=0}^{\infty}\sin\left(\frac{(n+1)\pi}{\beta}\right)\Gamma\left(\frac{n+1}{\alpha}\right)\frac{z^n}{n!},\quad \text{where } \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta} = 1, \tag{21}$$

and Γ is the Gamma function. The map C_α is entire and satisfies $|C_\alpha(z)|\leq c_1\exp(-c_2|z|^\beta)$ on each sector $|\arg(z)|\leq\theta/2<\pi/(2\beta)$, for certain constants $c_j = c_j(\alpha,\theta)>0$, $j = 1,2$. Then, given $k'>k>0$, the *acceleator operator* of index (k',k), $\mathscr{A}_{k',k}(p)(z) := z^{-k}\int_0^{+\infty}p(s)C_{k'/k}((s/z)^k)\,\mathrm{d}s^k$

is well-defined for all polynomials $p \in \mathbb{C}[z]$. Its importance relies on the fact that $\mathscr{A}_{k',k}(z^n) = \frac{\Gamma(1+n/k)}{\Gamma(1+n/k')} z^n$, for all $n \geq 0$. Choosing $k = 1$, $k' = d+1$, and $z = 1$, we find that

$$\mathscr{A}_{d+1,1}(p) := \int_0^{+\infty} p(s) C_{d+1}(s)\, \mathrm{d}s \quad \text{has moments} \quad \mathscr{A}_{d+1,1}(x^n) = \frac{n!}{\Gamma\left(1+\frac{n}{d+1}\right)}.$$

Therefore, if $\omega_{d+1} = e^{2\pi i/(d+1)}$, Remark 18 proves that the functional

$$\mathscr{A}^{d+1}(p) := \frac{1}{d+1} \int_0^{+\infty} \left(p(s) + p(\omega_{d+1}s) + \cdots + p(\omega_{d+1}^d s)\right) C_{d+1}(s)\, \mathrm{d}s \tag{22}$$

satisfies $\mathscr{A}^{d+1}(e^{xt}) = \exp(t^{d+1})$. Moreover, for $\lambda \neq 0$, Example 11 shows that $\mathscr{A}^{d+1} \circ \mathscr{H}_\lambda$ produces the sequence $\{\lambda^{-n} H_n^{(d)}(\lambda x)\}_{n\geq 0}$ having $\exp(xt - (t/\lambda)^{d+1})$ as generating series. In particular, if we choose $\lambda^{d+1} = -1$, say $\lambda = e^{i\pi/(d+1)}$, the generating series would be $\exp(xt + t^{d+1})$. These considerations establish the following.

Proposition 20. *The sequence $\{H_n^{(d)}(x)\}_{n\geq 0}$ having $\exp(xt - t^{d+1})$ as exponential generating series is characterized as the Appell sequence relative to $\mathscr{A}^{d+1}$ given by* (22) *and satisfying*

$$\int_0^{+\infty} \sum_{j=0}^{d} H_n^{(d)}\left(x + e^{\frac{2\pi i j}{d+1}} s\right) C_{d+1}(s)\, \mathrm{d}s = (d+1)x^n, \tag{23}$$

where C_{d+1} is Ecalle's accelerator function (21). *Moreover, $H_n^{(d)}$ admits the integral representation*

$$H_n^{(d)}(x) = \frac{1}{d+1} \int_0^{+\infty} \sum_{j=0}^{d} \left(x + e^{\frac{2\pi i (j-1/2)}{d+1}} s\right)^n C_{d+1}(s)\, \mathrm{d}s. \tag{24}$$

On the other hand, if in the previous paragraph we take $\lambda = \lambda_d := (d!(d+1)^2)^{1/(d+1)}$ we recover the family $\widehat{H}_n(x;d) := \lambda_d^{-n} H_n^{(d)}(\lambda_d x)$ of *Hermite-type-d-orthogonal* polynomials. We briefly recall that a sequence of polynomials $\{p_n(x)\}_{n\geq 0}$ with $\deg(p_n) = n$, is *d-orthogonal* with respect to a d functionals $L_1, \ldots, L_d \in \mathbb{C}[x]^*$ if

$$L_j(p_m p_n) = 0, \text{ for } n \geq md + j, \quad \text{and} \quad L_j(p_m p_{md+j-1}) \neq 0, \quad \text{for all } m \geq 0, j = 1, \ldots, d,$$

see [15, Definition 1.5] and the references therein. For $d = 1$ this corresponds to the classical orthogonality condition. Moreover, d-orthogonal Appell polynomial sequences have a generating series of the form $\exp(xt - \sum_{j=0}^{d-1} \gamma_{d-1-j} t^{j+2}/(j+2)!)$, for certain constants γ_l [15, Theorem 3.1].

Finally, in the case $d = 1$ and $\alpha = \beta = 2$ in (21), after a direct calculation using the values $\Gamma(n+1/2) = (2n)!/(4^n n!)\sqrt{\pi}$, $n \geq 0$, we find the familiar function $C_2(z) = \exp(-z^2/4)/\sqrt{\pi}$. Then, taking $\lambda = \sqrt{2}$ we find the functional

$$(\mathscr{A}^2 \circ \mathscr{H}_{\sqrt{2}})(p) = \frac{1}{2} \int_0^{+\infty} \left(p(s/\sqrt{2}) + p(-s/\sqrt{2})\right) C_2(s)\, \mathrm{d}s = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} p(s) e^{-s^2/2}\, \mathrm{d}s,$$

corresponding to the Weierstrass operator [16, p. 746] who induces the classical Hermite polynomials $\{He_n(x)\}_{n\geq 0}$ with generating series $\exp(xt - t^2/2)$. Finally, equations (23) and (24) take the familiar form [19, p. 254]

$$x^n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} He_n(x+s) e^{-s^2/2}\, \mathrm{d}s, \qquad He_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x+is)^n e^{-s^2/2}\, \mathrm{d}s.$$

We conclude with one final worked example in relation with functionals induced by entire functions.

Example 21. Given an entire function $F(z) = \sum_{n=0}^{\infty} f_n z^n$ and $\lambda \in \mathbb{C}$ such that $F(\lambda) \neq 0$, let

$$L_{F,\lambda}(p) = \frac{1}{F(\lambda)} \sum_{k=0}^{\infty} p(k) f_k \lambda^k, \quad \text{and} \quad \mathfrak{L}_{F,\lambda}(p)(x) = \frac{1}{F(\lambda)} \sum_{k=0}^{\infty} p(x+k) f_k \lambda^k.$$

To see these are well-defined we use the differential operator $\delta = z\frac{\partial}{\partial z}$ to note that $\delta^j(F)(z) = \sum_{k=1}^{\infty} k^j f_k z^k$, $j \geq 1$ are again entire and thus

$$L_{F,\lambda}(x^n) = \frac{\delta^n(F)(\lambda)}{F(\lambda)}, \quad \text{and} \quad \mathfrak{L}_{F,\lambda}(x^n)(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{\delta^j(F)(\lambda)}{F(\lambda)} x^{n-j}.$$

Moreover, $L_{F,\lambda}(e^{xt}) = \sum_{k=0}^{\infty} e^{kt} f_k \lambda^k / F(\lambda) = F(\lambda e^t)/F(\lambda)$. In this way we obtain an Appell sequence $\{P_{F,\lambda,n}(x)\}_{n\geq 0}$ characterized by the equation

$$\sum_{k=0}^{\infty} P_{F,\lambda,n}(x+k) f_k \lambda^k = F(\lambda) x^n.$$

If F has no zeros in the complex plane, $1/F(z) = \sum_{k=0}^{\infty} f'_k z^k$ is again entire and $L_{1/F,\lambda}(e^{xt}) = F(\lambda)/F(\lambda e^t)$. Therefore, $\mathfrak{L}_{F,\lambda}^{-1} = \mathfrak{L}_{1/F,\lambda}$ and we can invert the previous equation to obtain

$$P_{F,\lambda,n}(x) = F(\lambda) \sum_{k=0}^{\infty} (x+k)^n f'_k \lambda^k = \sum_{j=0}^{n} \binom{n}{j} \frac{\delta^j(1/F)(\lambda)}{(1/F)(\lambda)} x^{n-j}. \tag{25}$$

Examples of this situation are given by $F(z) = \exp(P(z))$, where P is a non-constant polynomial as considered by Touchard [36]. The best-known case corresponds to $F(z) = e^z$, for which

$$L_{e^z,\lambda}(e^{xt}) = \exp(\lambda(e^t - 1)) = \sum_{k=0}^{\infty} T_k(\lambda) \frac{t^k}{k!}, \quad \text{where } T_k(\lambda) = e^{-\lambda} \cdot \delta^n(e^z)(\lambda) = \sum_{k=0}^{n} S(n,k) \frac{\lambda^k}{k!}$$

are the *exponential* or *Touchard* polynomials [26, p. 63] (recall equation (19)). In particular, (25) takes the form $P_{e^z,\lambda,n}(x) = \sum_{j=0}^{n} \binom{n}{j} \sum_{l=0}^{j} S(j,l)(-\lambda)^l x^{n-j}/l!$, see [9, Example 4.4].

Table 1 contains more examples of Appell sequences as Nørlund [23], Laguerre [26, p. 108], Strodt [9], and the Bernoulli-type polynomials [34] (see Lemma 1 where a should be only equal to 1). Moreover, the Bernoulli hypergeometric [18] and Kummer hypergeometric Bernoulli polynomials [16].

Table 1. Some families of Appell polynomials

Polynomials	Functional $L(p)$	Indicator series $L(e^{xt})$ / Characterization
Monomials $(x-a)^n$	$p(a)$	e^{at}
Bernoulli $B_n(x)$	$\int_0^1 p(t)\,\mathrm{d}t$	$(e^t-1)/t$
		$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n$
kth Bernoulli $B_n^{(k)}(x)$	$\int_{[0,1]^k} p(s_1+\cdots+s_k)\,\mathrm{d}\boldsymbol{s}$	$(e^t-1)^k/t^k$
		$\int_{[0,1]^k} B_n^{(k)}(x+s_1+\cdots+s_k)\,\mathrm{d}\boldsymbol{s} = x^n$
Bernoulli Nørlund	$\int_{[0,1]^k} p(\omega_1 s_1+\cdots+\omega_k s_k)\,\mathrm{d}\boldsymbol{s}$	$\prod_{j=1}^{k} \frac{e^{\omega_j t}-1}{\omega_j t}$
$B_n^{(k)}(x\vert\omega)$	$\omega = (\omega_1,\ldots,\omega_k) \in \mathbb{C}^k$	$\int_{[0,1]^k} B_n^{(k)}(x+\sum_{j=1}^{k}\omega_j s_j \vert \omega)\,\mathrm{d}\boldsymbol{s} = x^n$
Apostol Euler $E_n(\beta;x)$	$(1-\beta)p(0)+\beta p(1)$	$1+\beta(e^t-1)$
		$(1-\beta)E_n(\beta;x)+\beta E_n(\beta;x+1) = x^n$

Euler $E_n(x)$	$\dfrac{p(0)+p(1)}{2}$	$(1+e^t)/2$
		$E_n(x)+E_n(x+1)=2x^n$
kth Apostol Euler $E_n^{(k)}(\beta;x)$	$\sum_{j=0}^{k}\binom{k}{j}\beta^j(1-\beta)^{k-j}p(j)$	$\left(1+\beta(e^t-1)\right)^k$
		$\sum_{j=0}^{k}\binom{k}{j}\beta^j(1-\beta)^{k-j}E_n^{(k)}(\beta;x+j)=x^n$
Euler Nørlund $E_n^{(k)}(x\|\omega)$	$2^{-k}\sum_{n_j\in\{0,1\}}p\left(\sum_{j=1}^{k}n_j\omega_j\right)$	$2^{-k}\prod_{j=1}^{k}(e^{\omega_j t}+1)$
	$\omega=(\omega_1,\dots,\omega_k)\in\mathbb{C}^k$	$\sum_{n_j\in\{0,1\}}E_n^{(k)}\left(x+\sum_{j=1}^{k}n_j\omega_j \mid \omega\right)=2^kx^n$
w-Strodt $S_{n,w}(x)$	$\sum_{j=1}^{N}w_j\,p\left(x_j\right)$	$\sum_{j=1}^{N}w_je^{x_jt}$
	$x_j\in\mathbb{R},\ 0<w_j<1,\ \sum_{j=1}^{N}w_j=1$	$\sum_{j=0}^{n-1}S_{n,w}\left(x+w_j\right)=x^n$
Bernoulli-type $B_{n,1}^{(m-l)}(x)$	$\sum_{j=l}^{m}a_j\int_0^j p(s)\,\mathrm{d}s$	$\sum_{j=l}^{m}a_je^{jt}/t$
	$l,m\in\mathbb{Z},\ \sum_{j=l}^{m}a_j=0,\ \sum_{j=l}^{m}ja_j=1$	$\sum_{j=l}^{m}a_jB_{n,1}^{(m-l)}(x+j)=nx^{n-1}$
Hermite $He_n(x)$	$\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}p(s)e^{-s^2/2}\,\mathrm{d}s$	$\exp(t^2/2)$
		$\int_{-\infty}^{+\infty}He_n(x+s)e^{-s^2/2}\,\mathrm{d}s=\sqrt{2\pi}x^n$
d-Hermite $H_n^{(d)}(x)$	$\int_0^{+\infty}\sum_{j=0}^{d}p\left(e^{\frac{2\pi ij}{d+1}}s\right)\dfrac{C_{d+1}(s)}{d+1}\,\mathrm{d}s$	$\exp(t^{d+1})$, $d\geq 0$ integer
		$\int_0^{+\infty}\sum_{j=0}^{d}H_n^{(d)}\left(x+e^{\frac{2\pi ij}{d+1}}s\right)\dfrac{C_{d+1}}{d+1}(s)\,\mathrm{d}s=x^n$
Laguerre $(-1)^n n!L_n^{(\alpha-n)}(x)$	$\int_0^{+\infty}p(s)\dfrac{s^\alpha e^{-s}}{\Gamma(1+\alpha)}\,\mathrm{d}s$, $\mathrm{Re}(\alpha)>-1$	$(1-t)^{-\alpha-1}$
		$\int_0^{+\infty}L_n^{(\alpha-n)}(x+t)\dfrac{t^\alpha e^{-t}}{\Gamma(1+\alpha)}\,\mathrm{d}t=\dfrac{(-1)^nx^n}{n!}$
Bernoulli hypergeometric $B_{N,n}(x)$	$N\int_0^1 p(s)(1-s)^{N-1}\,\mathrm{d}s,\ N\geq 1$	$\left(e^t-\sum_{j=0}^{N-1}t^j/j!\right)/(t^N/N!)$
		$N\int_0^1 B_{N,n}(x+s)(1-s)^{N-1}\mathrm{d}s=x^n$
Kummer hypergeometric $B_{a,b,n}(x)$	$\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 p(s)s^{a-1}(1-s)^{b-1}\,\mathrm{d}s$	$1+\sum_{n=1}^{\infty}\dfrac{a(a+1)\cdots(a+n-1)}{(a+b)(a+b+1)\cdots(a+b+n-1)}\dfrac{t^n}{n!}$
	$\mathrm{Re}(a),\mathrm{Re}(b)>0$	$\int_0^1 B_{a,b,n}(x+s)\dfrac{s^{a-1}}{\Gamma(a)}\dfrac{(1-s)^{b-1}}{\Gamma(b)}\,\mathrm{d}s=\dfrac{x^n}{\Gamma(a+b)}$
$P_{F,\lambda,n}(x)$	$\dfrac{1}{F(\lambda)}\sum_{k=0}^{\infty}\dfrac{p(x+k)}{k!}F^{(k)}(0)\lambda^k$	$F(\lambda e^t)/F(\lambda)$
	$F:\mathbb{C}\to\mathbb{C}$ entire, $F(\lambda)\neq 0$	$\sum_{k=0}^{\infty}P_{F,\lambda,n}(x+k)\dfrac{F^{(k)}(0)\lambda^k}{k!}=F(\lambda)x^n$

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