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Mathematical Analysis

An example of a $C^{1,1}$ polyconvex function with no differentiable convex representative

Un exemple de fonction $C^{1,1}$ polyconvexe sans reprèsentant convexe differentiable

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Abstract

We construct a $C^{1,1}$ polyconvex function W such that there exists a fixed 2×2 matrix Y with the property that all convex representatives of W have at least two distinct subgradients (and are hence not differentiable) at the point $(Y, \det Y)$, showing in particular that a polyconvex function can be smoother than any of its convex representatives. ***To cite this article: J. Bevan, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On construit une fonction $C^{1,1}$ polyconvexe W tel qu'il existe une matrice 2×2 Y satisfaisant la propriété suivante : tous les representants convexes de W ont au moins deux sousgradients distincts (et ne sont donc pas differentiable) au point $(Y, \det Y)$. Ceci montre, en particulier, qu'une fonction polyconvexe peut être plus differentiable que tous ses representants convex. ***Pour citer cet article : J. Bevan, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction and notation

We denote the real 2×2 matrices by $\mathbf{R}^{2\times 2}$ and define the function $R : \mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2} \times \mathbf{R}$ by $R(Z) = (Z, \det Z)$. We recall that a convex representative φ of a polyconvex function W must satisfy $W(A) = \varphi(R(A))$ for all A in $\mathbf{R}^{2\times 2}$, and that there is a largest such (see [5] and [1]), φ_W, given by

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$$\varphi_W(A,\delta)=\inf\left\{\sum_{i=1}^{6}\lambda_i W(A_i),\ \sum_{i=1}^{6}\lambda_i=1,\ \lambda_i\geqslant 0,\ \sum_{i=1}^{6}\lambda_i R(A_i)=(A,\delta)\right\}.$$

If a polyconvex function W has a strictly convex representative then W is said to be strictly polyconvex. Suppose $\Omega\subset\mathbf{R}^n$ is a bounded domain and that u is a fixed mapping in the Sobolev space $W^{1,2}(\Omega,\mathbf{R}^m)$. Consider, when $n=m=2$ and Ω is the unit ball in $\mathbf{R}^2$, the following problem: find a nonnegative strictly polyconvex function $f:\mathbf{R}^{2\times 2}\to\mathbf{R}$ which satisfies

$$\{Du(x),\ x\in\Omega\}\subset\{\xi,\ f(\xi)=0\}. \tag{1}$$

For example, when $u:\Omega\to\mathbf{R}^2$ is defined in polar coordinates by $u(r,\theta)=\frac{1}{\sqrt{2}}(r,2\theta)$ it is shown in [4] that a solution f to the problem exists. It follows that u is a singular minimizer of $I(v)=\int_\Omega f(Dv(x))\,\mathrm{d}x$ among $v\in W^{1,2}(\Omega,\mathbf{R}^2)$ satisfying $v|_{\partial\Omega}=u(\cdot)$. If in addition f can be smooth and strongly quasiconvex in the sense of [6] then Evans' partial regularity theorem would be optimal in dimensions $n=m=2$. This is certainly the case when n and m are large enough, as was first shown by Nečas in [7] and later by Šverák and Yan in [10], who gave an example of a singular minimizer of a smooth strongly convex functional in dimensions $m=5,\ n=3$.

In seeking a strictly polyconvex function satisfying (1) one is naturally led to study the properties of its possible convex representatives, in particular their regularity. It is trivially true that f is at least as differentiable as its smoothest convex representative, but is it true that there is a convex representative which is as smooth as f is? We show that the answer is in general no. We do not know if the function W constructed in this paper is smoother than $C^{1,1}$, or whether there is an example with a possibly different smooth W.

The example in this Note is reminiscent of the loss of regularity which occurs when an isotropic, frame indifferent function $W:D\to\mathbf{R}$, is expressed as $W(F)=H(v_1+v_2,v_1v_2)$, where D is an SO(2)-invariant subset of $\{\xi\in\mathbf{R}^{2\times 2},\ \det\xi>0\}$ and v_1, v_2 are the principal values of F. The remarks in [2, Theorem 6.9] show that H is in general less differentiable than W.

We denote the subdifferential of a convex function $f:\mathbf{R}^n\to\mathbf{R}$ at a point x by $\partial f(x)$ and recall that this is the set

$$\{\nu\in\mathbf{R}^n,\ f(U)\geqslant f(X)+\nu\cdot(U-X)\text{ for all }U\text{ in }\mathbf{R}^n\} \tag{2}$$

where "·" represents the usual Euclidean inner product. For later use we define the inner product of two matrices E and F in $\mathbf{R}^{2\times 2}$ by $E:F=\operatorname{tr}E^{\mathrm{T}}F$. We define $|F|_2^2=F:F$. As usual we write $B(A,r)$ for the open ball with radius r and centre A in the topology induced by $|\cdot|_2$. We reserve $|\cdot|$ for the Euclidean norm on $\mathbf{R}^{2\times 2}\times\mathbf{R}$, and write $|(A,\delta)|^2=|A|_2^2+\delta^2$. By [9, Theorem 23.4] a real valued convex function f defined on all of $\mathbf{R}^n$ has a nonempty, compact and convex subdifferential $\partial f(x)$ at each x, whose elements are then referred to as subgradients of f at x. Further, by [9, Theorem 25.1], f is differentiable at the point $x\in\mathbf{R}^n$ if and only if $\partial f(x)$ consists of a singleton, written $Df(x)$. Finally, for any function $f:\mathbf{R}^{2\times 2}\to\mathbf{R}$ we use the notation f^{pc} to represent the largest polyconvex function majorised by f (identically $-\infty$ if no such function exists). For further details on polyconvexity see [1,5] and [8].

2. The counterexample

Fix $Y\in\mathbf{R}^{2\times 2}$ such that $\det Y<0$ and define the function $\omega:\mathbf{R}^{2\times 2}\to\mathbf{R}$ by

$$\omega(Z)=\max\{0,\det(Z-Y)\}.$$

It is easy to see that ω is polyconvex (for example, let

$$\phi(A,\delta)=\delta+\det Y-\operatorname{cof}Y:A, \tag{3}$$

note that $\psi(A,\delta) \stackrel{\text{def}}{=} \max\{0, \phi(A,\delta)\}$ is convex, being the maximum of two convex functions, and that $\omega(Z) = \psi(R(Z)))$. Fix $X \in \mathbf{R}^{2\times 2}$ such that $\det(X - Y) > 0$. For any positive real number r let the set of 2×2 matrices $K(r)$ be defined by

$$K(r) = B(0,r) \cup B(X,r). \tag{4}$$

By continuity of $f(Z) \stackrel{\text{def}}{=} \det(Z - Y)$ there exists $\tau > 0$ such that $\omega(Z) = 0$ if $Z \in B(0,\tau)$ and $\omega(Z) = f(Z)$ if $Z \in B(X,\tau)$, so that by taking $\varepsilon = \frac{1}{3}\min\{\tau, \operatorname{dist}(Y, \{0, X\})\}$ we ensure that ω is smooth on $K(2\varepsilon)$ and $\operatorname{dist}(Y, K(2\varepsilon)) \geqslant \varepsilon$. Let η be a smooth cut-off function whose support lies in $K(2\varepsilon)$ and which satisfies $0 \leqslant \eta \leqslant 1$, with $\eta(Z) = 1$ if $Z \in K(\varepsilon)$. Define the function $g : \mathbf{R}^{2\times 2} \to \mathbf{R}$ by

$$g(Z) = \big(1 - \eta(Z)\big)\frac{|Z - Y|^2}{2} + \eta(Z)\omega(Z).$$

Observe that since $|Z - Y|^2/2 \geqslant \omega(Z)$ for all Z we have $g(Z) \geqslant \omega(Z)$ for all Z, and that g is smooth with $|D^2 g(\cdot)| \leqslant c$ for some fixed positive constant c. By the comments in the opening paragraph of [3, Section 4], [3, Proposition 3.7] also applies to polyconvex envelopes; we can then conclude by [3, Remark 1, p. 347] that

$$W(Z) \stackrel{\text{def}}{=} g^{\mathrm{pc}}(Z)$$

is $C^{1,1}_{\mathrm{loc}}$. Using this and the following lemma we claim DW is in fact globally Lipschitz. By an abuse of notation we let $|\xi| = |\xi|_2$.

Lemma 2.1. *ξ is a point of strict convexity of g if $|\xi|$ is sufficiently large.*

Proof. Fix ξ such that $\operatorname{dist}(\xi, K(2\varepsilon)) > \varepsilon$ and let $m = \sup\{|\omega(A) - \frac{1}{2}|A - Y|^2|, A \in K(2\varepsilon)\}$. By definition of g it follows that

$$g(Z) \geqslant \frac{1}{2}|Z - Y|^2 - m\eta(Z) \tag{5}$$

for all Z. We wish to show $g(Z) - g(\xi) > Dg(\xi) : (Z - \xi)$ for all $Z \neq \xi$, provided $|\xi|$ is large enough; by (5) it is sufficient to prove

$$\tfrac{1}{2}|Z - Y|^2 - m\eta(Z) - \tfrac{1}{2}|\xi - Y|^2 > \xi : Z + \xi : Y - Y : Z - |\xi|^2,$$

which is satisfied if and only if

$$\tfrac{1}{2}|\xi - Z|^2 - m\eta(Z) > 0. \tag{6}$$

Now for large enough $|\xi|$, $\inf\{\frac{1}{2}|\xi - Z|^2, Z \in K(2\varepsilon)\} \geqslant 2m$, so that (6) holds for all $Z \neq \xi$. □

We claim that $W(\xi) = g(\xi)$ for ξ such that Lemma 2.1 holds. By [3, Section 4, Eq. (4.1)],

$$W(\xi) = \inf\left\{\sum_{i=1}^{6}\lambda_i g(\xi_i),\ \lambda_i \geqslant 0, \sum_{i=1}^{6}\lambda_i = 1, \sum_{i=1}^{6}\lambda_i R(\xi_i) = R(\xi)\right\},$$

so we can apply Lemma 2.1 with $Z = \xi_i$ to each summand in $\sum_{i=1}^{6}\lambda_i g(\xi_i)$ to deduce $W(\xi) \geqslant g(\xi)$. The reverse inequality is true by definition of the polyconvex envelope so that $W(\xi) = g(\xi)$ for all large enough ξ. It now follows easily that the derivative of W is globally Lipschitz.

For future use we note that since ω is polyconvex and bounds g below, $g \geqslant W \geqslant \omega$ on $\mathbf{R}^{2\times 2}$; and since $g = \omega$ on $K(\varepsilon)$ we have $W = \omega$ on $K(\varepsilon)$. The following lemmas are used in Theorem 2.4, where we show that W has the property stated in the title of this Note.

Lemma 2.2. *Let $\varepsilon > 0$ and let ξ be any fixed 2×2 matrix. Then $R(\xi)$ lies in the interior of the convex hull of the set $\{(U, \det U),\ U \in B(\xi, \varepsilon)\}$.*

Proof. Suppose not. Then there exist $C \in \mathbf{R}^{2\times 2}$, $c \in \mathbf{R}$ not both zero, such that $(C, c) \cdot R(\xi + \tau A) \geqslant (C, c) \cdot R(\xi)$ for all $A \in B(0, \varepsilon)$ and $|\tau| \leqslant 1$, which holds if and only if $\tau(A : C + cA : \operatorname{cof}\xi) + c\tau^2 \det A \geqslant 0$. Dividing through by $\tau^2 \neq 0$ and letting $\tau \to 0+$ it follows that $A : C + cA : \operatorname{cof}\xi \geqslant 0$. Letting $\tau \to 0-$ we have $A : C + cA : \operatorname{cof}\xi = 0$ for all $A \in B(0, \varepsilon)$, implying that $c \det A \geqslant 0$ for such A. Choose in particular A such that $\det A \neq 0$ and let $\bar{A} \in B(0, \varepsilon)$ satisfy $\det \bar{A} = -\det A$ (for example, exchange the rows of A and call the result $\bar{A}$). Then $c \det \bar{A} \geqslant 0$, $c \det A \geqslant 0$ together imply $c = 0$, from which it follows that $C : A = 0$ for all $A \in B(0, \varepsilon)$. Hence $C = 0$, a contradiction. □

Lemma 2.3. *Let $h = h(A, \delta)$ be convex, $\xi \in \mathbf{R}^{2\times 2}$ and suppose that $h(A, \det A) = 0$ for $|A - \xi|$ sufficiently small. Then $h(A, \delta) = 0$ for $|(A, \delta) - R(\xi)|$ sufficiently small.*

Proof. Suppose τ is such that $h(R(A)) = 0$ if $|A - \xi|_2 < \tau$. By Lemma 2.2, for sufficiently small $|(C, \delta)|$ we can write $(\xi + C, \det\xi + \delta)$ and $(\xi - C, \det\xi - \delta)$ as convex combinations of points in $\{R(A),\ |A - \xi| < \tau\}$, so by applying convexity of h it follows that $h(\xi + C, \det\xi + \delta) \leqslant 0$ and $h(\xi - C, \det\xi - \delta) \leqslant 0$. But $0 = h(\xi, \det\xi) \leqslant \frac{1}{2}(h(\xi + C, \det\xi + \delta) + h(\xi - C, \det\xi - \delta))$, so that both terms on the right hand side are zero, concluding the proof. □

Theorem 2.4. *No convex representative of W is differentiable at $R(Y)$.*

Proof. Let φ be any convex representative of W. We will show that φ must be differentiable at (in fact, smooth in a neighbourhood of) $R(0)$ and $R(X)$ by applying Lemma 2.3 twice, from which it will follow that $\partial\varphi(R(Y))$ is not a singleton. Hence by the remarks in the introduction φ cannot be differentiable at $R(Y)$.

Since $W = \omega$ on $K(\varepsilon)$ we know $\varphi(R(A)) = 0$ for $A \in B(0, \varepsilon)$ and $\varphi(R(A)) = \phi(R(A))$ for $A \in B(X, \varepsilon)$, where the affine function ϕ was defined in Eq. (3). Apply Lemma 2.3 to $h = \varphi$, $\xi = 0$ to conclude that $\varphi(A, \delta) = 0$ in a neighbourhood of $R(0)$. By convexity this implies $\varphi \geqslant 0$ everywhere. Hence, since we have $W(Y) = 0$ it follows that $R(0) \in \partial\varphi(R(Y))$. Next apply Lemma 2.3 to $h(A, \delta) = \varphi(A, \delta) - \phi(A, \delta)$, $\xi = X$ to deduce $\varphi(A, \delta) \geqslant W(Y) - \operatorname{cof} Y : (A - Y) + \delta - \det Y$ everywhere (again by convexity). In particular this gives $(-\operatorname{cof} Y, 1) \in \partial\varphi(R(Y))$. □

Remark 2.1. Since φ is differentiable in a neighbourhood of $R(0)$ and $R(X)$, the tangent hyperplanes T_0 and T_X, say, to the graph of φ at $(R(0), W(0))$ and $(R(X), W(X))$ respectively are uniquely defined by $D\varphi(R(0))$ and $D\varphi(R(X))$ respectively. Thus an alternative way of concluding the proof would be simply to check $(R(Y), W(Y)) \in T_0 \cap T_X$.

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