C. R. Acad. Sci. Paris, Ser. I 334 (2002) 31–35

Analyse fonctionnelle/*Functional Analysis*

Hecke algebras associated with induced representations

Robyn Curtis

Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, 1211 Genève 24, Switzerland

Received 10 October 2001; accepted 29 October 2001

Note presented by Alain Connes.

Abstract We define the Hecke von Neumann algebra $\mathcal{L}(G, H, \sigma)$ associated with a group G, a subgroup H and a unitary representation σ of H. We show that when σ is finite dimensional, $\mathcal{L}(G, H, \sigma)$ can be seen as a corner algebra of the tensor product of the group von Neumann algebra of a locally compact group and a matrix algebra. To cite this article: R. Curtis, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 31–35. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les algèbres de Hecke associées aux représentations induites

Résumé Nous définissons l'algèbre de Hecke von Neumann $\mathcal{L}(G, H, \sigma)$ associée à un groupe G, un sous-groupe H et une représentation unitaire σ de H. Nous montrons que, si σ est de dimension finie, alors $\mathcal{L}(G, H, \sigma)$ peut se voir comme l'algèbre de coin du produit tensoriel de l'algèbre de von Neumann associée à un groupe localement compact et d'une algèbre de matrices. Pour citer cet article : R. Curtis, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 31–35. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soient G un groupe, H un sous-groupe de G et σ une représentation unitaire de H sur un espace de Hilbert $\mathcal{K}$. *L'algèbre de Hecke von Neumann* $\mathcal{L}(G, H, \sigma)$ correspondante se définit comme suit : soit $\mathcal{H}$ l'espace de Hilbert des fonctions $\xi : H \to \mathcal{K}$ telles que $\xi(hx) = \sigma(h)\xi(x)$ pour tous $h \in H$ et $x \in G$ et

$$\sum_{Hy \in H\backslash G} \left\|\xi(y)\right\|^2 < \infty.$$

Soit $\mathcal{B}(\mathcal{K})$ l'algèbre involutive des opérateurs linéaires bornés sur $\mathcal{K}$. Soit $\mathbb{C}(G, H, \sigma)$ l'espace des fonctions $f : G \to \mathcal{B}(\mathcal{K})$ telles que $f(h_1 x h_2) = \sigma(h_1) f(x)\sigma(h_2)$ pour tous $h_1, h_2 \in H$ et $x \in G$ et telles que le support de f soit fini dans les espaces quotients G/H et $H\backslash G$. Pour la convolution, cet espace est *l'algèbre de Hecke* du triple (G, H, σ). On a des représentations $\lambda : \mathbb{C}(G, H, \sigma) \to \mathcal{B}(\mathcal{H})$ et $\rho : G \to \mathcal{U}(\mathcal{H})$, où $\mathcal{U}(\mathcal{H})$ désigne le groupe des opérateurs unitaires sur $\mathcal{H}$. Alors $\mathcal{L}(G, H, \sigma)$ est l'adhérence faible de $\lambda(\mathbb{C}(G, H, \sigma))$ dans $\mathcal{B}(\mathcal{H})$. Il résulte des définitions que $\mathcal{L}(G, H, \sigma)$ est dans le commutant de $\rho(G)$; quand σ est de dimension finie, cette inclusion devient une égalité : $\mathcal{L}(G, H, \sigma) = \rho(G)'$.

E-mail address: Robyn.Curtis@math.unige.ch (R. Curtis).

S1631-073X(02)02195-7/FLA

L'objet principal de cette Note est de généraliser à la situation décrite ci-dessus deux résultats pour le cas particulier du caractère unité $\sigma = 1$ qui apparaissent dans la thèse de K. Tzanev [9] (*voir* aussi le théorème 5 de [8]) : d'abord, si H est un sous-groupe compact ouvert d'un groupe localement compact G, alors $\mathcal{L}(G, H, 1)$ est un coin de l'algèbre de von Neumann $\mathcal{L}(G)$ du groupe G ; ensuite, étant donné une paire (G, H) quelconque (sans topologie sur G), on peut construire un groupe localement compact $\overline{G}$ et un sous-groupe compact ouvert $\overline{H}$ tels que $\mathcal{L}(\overline{G}, \overline{H}, 1)$ et $\mathcal{L}(G, H, 1)$ sont isomorphes.

Dans le cas plus général d'une représentation unitaire quelconque de H, nous montrons d'abord que, si H est un sous-groupe compact ouvert d'un groupe localement compact G et $\sigma : H \to \mathcal{U}(\mathcal{K})$ est une représentation unitaire continue, alors $\mathcal{L}(G, H, \sigma)$ est un coin du produit tensoriel $\mathcal{L}(G) \otimes \mathcal{B}(\mathcal{K})$; ensuite, étant donné un triple (G, H, σ) tel que σ est de dimension finie (sans topologie sur G ni condition de continuité sur σ), nous construisons un groupe localement compact $\overline{G}_\sigma$, un sous-groupe compact ouvert $\overline{H}_\sigma$ et une représentation unitaire continue $\bar{\sigma}$ de $\overline{H}_\sigma$ tels que $\mathcal{L}(\overline{G}_\sigma, \overline{H}_\sigma, \bar{\sigma})$ et $\mathcal{L}(G, H, \sigma)$ sont isomorphes.

Nous illustrons cette construction par des exemples, l'un avec le produit en couronne $F \wr \mathbb{Z}$ d'un groupe abelien fini F par le groupe cyclique $\mathbb{Z}$, qui s'appelle *le groupe de l'allumeur de réverbères*, d'autres avec des variantes de la paire $(\mathbb{Q} \rtimes \mathbb{Q}^*, \mathbb{Z})$ qui apparaît dans [3].

Let G be a group with subgroup H and let σ be a unitary representation of H on a Hilbert space $\mathcal{K}$. Denote by $\mathcal{H}$ the Hilbert space of functions $\xi : G \to \mathcal{K}$ such that $\xi(hx) = \sigma(h)\xi(x)$ for all $h \in H$, $x \in G$ and

$$\sum_{Hy \in H \backslash G} \left\|\xi(y)\right\|^2 < \infty,$$

together with the natural addition, scalar multiplication and scalar product. The *induced representation* $\rho : G \to \mathcal{U}(\mathcal{H})$ of σ from H to G is defined by $(\rho(g)\xi)(x) = \xi(xg)$ for all $g, x \in G$.

Denote by $\mathbb{C}(G, H, \sigma)$ the vector space of all functions $f : G \to \mathcal{B}(\mathcal{K})$ such that $f(h_1 x h_2) = \sigma(h_1) f(x) \sigma(h_2)$ for all $h_1, h_2 \in H$ and $x \in G$ and such that the support of f is finite when seen in the quotient spaces G/H and $H\backslash G$ respectively. Endowed with the convolution product:

$$(f_1 * f_2)(g) = \sum_{xH \in G/H} f_1(x) f_2\left(x^{-1}g\right) = \sum_{Hy \in H\backslash G} f_1\left(gy^{-1}\right) f_2(y) \quad \forall g \in G,$$

and involution $f^*(g) = f(g^{-1})^*$ for all $g \in G$, the vector space $\mathbb{C}(G, H, \sigma)$ is an involutive algebra, called the *Hecke algebra* of the triple (G, H, σ). If $\mathbb{C}(G, H, 1)$ is commutative then (G, H) is said to be a *Gelfand pair* [7]. A classical example is $G = \mathrm{GL}_n(\mathbb{Q})$ and $H = \mathrm{GL}_n(\mathbb{Z})$ (*see* Section 3.2.1 of [1]); more recent examples involving the Grigorchuk group appear in [2].

Denote by λ the natural (left) action of $\mathbb{C}(G, H, \sigma)$ on $\mathcal{H}$ and define the *Hecke von Neumann algebra* of the triple (G, H, σ) to be the weak closure of $\lambda(\mathbb{C}(G, H, \sigma))$ in $\mathcal{B}(\mathcal{H})$. Clearly, $\mathcal{L}(G, H, \sigma)$ is contained in the commutant $\rho(G)'$ of $\rho(G)$; when σ is finite dimensional the reverse inclusion also holds [6]. Example 1 below shows that the inclusion can be strict when σ is infinite dimensional.

For each $g \in G$, let $L(g)$ denote the index of $H \cap gHg^{-1}$ in H and let $R(g) = L(g^{-1})$. The *quasi-normaliser* $Q = Q_G(H)$ of H in G is the subgroup of all $g \in G$ such that both $L(g)$ and $R(g)$ are finite. When $Q = G$ we say that H is *quasi-normal* in G, or equivalently, that (G, H) is a *Hecke pair*. Functions in $\mathbb{C}(G, H, \sigma)$ are automatically 0 outside Q, so $\mathbb{C}(G, H, \sigma) = \mathbb{C}(Q, H, \sigma)$. The map $g \mapsto R(g)/L(g)$ is a group homomorphism from Q to $\mathbb{Q}_+^*$, which we will denote by Δ.

Example 1. – Let G be a group with a subgroup H such that $Q_G(H) = H$ and let σ be the right regular representation of H. Then $\mathcal{L}(G, H, \sigma) = \mathcal{L}(H, H, \sigma) \simeq \mathcal{L}(H)$ and $\rho(G)' \simeq \mathcal{L}(G)$. These algebras are in general not isomorphic; for example, take H to be an infinite amenable group and G the (non-amenable) free-product of any non-trivial group with H.

For each $g \in Q$, denote by $\mathcal{C}(g)$ the subspace of operators $A \in \mathcal{B}(\mathcal{K})$ such that $A\sigma(h) = \sigma(ghg^{-1})A$ for all $h \in H \cap g^{-1}Hg$. If $f \in \mathbb{C}(G, H, \sigma)$ then $f(g) \in \mathcal{C}(g)$ for all $g \in Q$; conversely, if $g \in Q$ and $A \in \mathcal{C}(g)$ then there is a function $f \in \mathbb{C}(G, H, \sigma)$ such that $f(g) = A$.

The subset B_σ of Q, consisting of all $g \in Q$ such that $\mathcal{C}(g)$ is not reduced to $\{0\}$, is a union of double cosets of Q with respect to H which contains H and is closed under inverses. Corwin showed in [6] that when σ is finite dimensional, ρ is irreducible if and only if σ is irreducible and $B_\sigma = H$. Examples of this case (with $Q = H$) can be found in [4].

Example 2. – Let $H = A_3$, the alternating group on 3 letters, let $G = S_4$, the symmetric group on 4 letters, and let σ be one of the two non-trivial characters on H. Then $Q = G$ and B_σ is the union of A_3 and the complement of S_3, which shows that B_σ need not be a subgroup of Q.

Example 3. – If σ is the trivial character 1 on a normal subgroup H of G, then $B_\sigma = G$ and $\mathcal{L}(G, H, 1)$ is just the standard von Neumann group algebra $\mathcal{L}(G/H)$ of the quotient group G/H.

PROPOSITION 1. – *Suppose that G is a group with quasi-normal subgroup H and σ is a unitary representation of H on a finite dimensional Hilbert space $\mathcal{K}$. Let ϕ be the canonical faithful normal state on $\mathcal{L}(G, H, \sigma)$ defined by $\phi(\lambda(f)) = \frac{1}{n}\mathrm{tr}(f(1))$ for all $f \in \mathbb{C}(G, H, \sigma)$, where n is the dimension of $\mathcal{K}$, and let Δ_ϕ be the modular operator of ϕ. The spectrum of Δ_ϕ is the closure of $\Delta(B_\sigma)$ in $\mathbb{R}_+^*$; in particular, if $\mathcal{L}(G, H, \sigma)$ is a factor then*

$$S\big(\mathcal{L}(G, H, \sigma)\big) \subseteq \overline{\Delta(B_\sigma)}, \tag{*}$$

where S is Connes' invariant [5]. *The modular automorphism group (σ_t) of $\mathcal{L}(G, H, \sigma)$ associated with Δ_ϕ is given by $\sigma_t^\phi(\lambda(f)) = \lambda(f_t)$ for all $f \in \mathbb{C}(G, H, \sigma)$ and $t \in \mathbb{R}$, where $f_t \in \mathbb{C}(G, H, \sigma)$ is the function $g \mapsto \Delta(g)^{it} f(g)$.*

In Example 5(i) below, the inclusion at (*) is an equality; in Example 6(i), it is strict.

When G is a topological group such that H is compact and open and $\sigma : H \to \mathcal{U}(\mathcal{K})$ is continuous, the Hecke von Neumann algebra $\mathcal{L}(G, H, \sigma)$ is easy to describe. In this case, H is quasi-normal in G and the functions in $\mathbb{C}(G, H, \sigma)$ are automatically continuous on G with compact support. Let μ be a right invariant Haar measure on G, scaled so that $\mu(H) = 1$. The modular homomorphism Δ_G of G, defined by the equation $\mathrm{d}\mu(gx) = \Delta_G(g)\,\mathrm{d}\mu(x)$ for all $x, g \in G$, is such that $\Delta_G(g) = \Delta(g)$ for all $g \in G$; in particular, $\Delta_G(G) \subseteq \mathbb{Q}_+^*$. Consider the Hilbert space $\mathcal{H}$ of the induced representation ρ as a subspace of $\mathrm{L}^2(G, \mu) \otimes \mathcal{K}$, the Hilbert space of L^2-integrable functions from G to $\mathcal{K}$. Denote by $C_c(G) \otimes \mathcal{B}(\mathcal{K})$ the involutive algebra of functions $f : G \to \mathcal{B}(\mathcal{K})$ of compact support, and consider the Hecke algebra $\mathbb{C}(G, H, \sigma)$ as a subalgebra of $C_c(G) \otimes \mathcal{B}(\mathcal{K})$. The action λ of $\mathbb{C}(G, H, \sigma)$ on $\mathcal{H}$ can then be extended to an action $\tilde\lambda$ of $C_c(G) \otimes \mathcal{B}(\mathcal{K})$ on $\mathrm{L}^2(G, \mu) \otimes \mathcal{K}$; the von Neumann algebra $\mathcal{L}(G) \otimes \mathcal{B}(\mathcal{K})$ is the weak closure of $\tilde\lambda(C_c(G) \otimes \mathcal{B}(\mathcal{K}))$ in $\mathcal{B}(\mathrm{L}^2(G, \mu) \otimes \mathcal{K})$. Let $q_\sigma : G \to \mathcal{B}(\mathcal{K})$ be the function equal to σ on H and 0 elsewhere. Then q_σ is a projection in $C_c(G) \otimes \mathcal{B}(\mathcal{K})$ and $\tilde\lambda(q_\sigma)$ is a projection in $\mathcal{L}(G) \otimes \mathcal{B}(\mathcal{K})$, with range space $\mathcal{H}$.

PROPOSITION 2. – *Suppose that G is a topological group with compact open subgroup H and σ is a continuous unitary representation of H on a Hilbert space $\mathcal{K}$; then*

$$\mathcal{L}(G, H, \sigma) = \tilde\lambda(q_\sigma)\big(\mathcal{L}(G) \otimes \mathcal{B}(\mathcal{K})\big)\tilde\lambda(q_\sigma),$$

where $\tilde\lambda(q_\sigma)$ is the projection in $\mathcal{L}(G) \otimes \mathcal{B}(\mathcal{K})$ defined in the paragraph above.

Given any triple (G, H, σ) such that $\sigma : H \to \mathcal{K}$ is finite dimensional (without any topology on G nor continuity condition on σ), it is possible to construct another triple $(\overline{G}_\sigma, \overline{H}_\sigma, \bar\sigma)$ which satisfies the conditions of Proposition 2 and whose Hecke von Neumann algebra $\mathcal{L}(\overline{G}_\sigma, \overline{H}_\sigma, \bar\sigma)$ is isomorphic to $\mathcal{L}(G, H, \sigma)$. We present this construction below.

Without loss of generality, suppose that H is quasi-normal in G. Let $\sim$ be the equivalence relation on $G \times \mathcal{U}(\mathcal{K})$ defined by $(g,u) \sim (gh^{-1}, \sigma(h)u)$ for all $h \in H$ and let S_σ be the resulting quotient space. Consider $G \times \mathcal{U}(\mathcal{K})$ with the product of the discrete topology on G and the compact topology on $\mathcal{U}(\mathcal{K})$, and S_σ with the corresponding quotient topology. Denote by φ_σ the natural action of G on S_σ given by $\varphi_\sigma(g)[x,u] = [gx,u]$. Let $\overline{G}_\sigma$ and $\overline{H}_\sigma$ be the closures of $\varphi_\sigma(G)$ and $\varphi_\sigma(H)$ respectively in the compact-open topology on the space of continuous functions from S_σ to itself. The representation of $\varphi_\sigma(H)$ on $\mathcal{K}$ given by $\varphi_\sigma(h) \mapsto \sigma(h)$ extends continuously to a unitary representation of $\overline{H}_\sigma$ on $\mathcal{K}$, which we will denote by $\bar{\sigma}$.

THEOREM 3. – *Let G be a group with subgroup H, let $\sigma : H \to \mathcal{U}(\mathcal{K})$ be a finite dimensional unitary representation and let $\overline{G}_\sigma$, $\overline{H}_\sigma$ and $\bar{\sigma}$ be as defined in the paragraph above. Then $\overline{G}_\sigma$ is a topological group with compact open subgroup $\overline{H}_\sigma$ and $\bar{\sigma}$ is a continuous unitary representation of $\overline{H}_\sigma$ on $\mathcal{K}$. Moreover,*

$$\mathcal{L}(G,H,\sigma) \simeq \mathcal{L}\big(\overline{G}_\sigma, \overline{H}_\sigma, \bar{\sigma}\big).$$

Below we present three examples of triples (G,H,σ) whose Hecke von Neumann algebras we can under-stand using the ideas presented above.

Example 4 *(*The lamplighter group*)*. – Let F be a finite Abelian group and let:

$$Y_- = \bigoplus_{\ell=-\infty}^{0} F_\ell, \quad Y_+ = \bigoplus_{\ell=1}^{\infty} F_\ell \quad \text{and} \quad Y = Y_- \oplus Y_+,$$

where $F_\ell \simeq F$ for all $\ell \in \mathbb{Z}$. Denote by α the action of $\mathbb{Z}$ on Y given by the shift to the right, i.e., $(\alpha_k(y))_\ell = y_{\ell-k}$ for all $k, \ell \in \mathbb{Z}$ and $y \in Y$. Let G be the semi-direct product $Y \rtimes_\alpha \mathbb{Z}$, let H be the subgroup Y_+ of G and let σ be a unitary representation of H on $\mathbb{C}$.

Write $\sigma = (\sigma_\ell)_{\ell \geqslant 1}$, where $\sigma_\ell : F_\ell \to \mathcal{U}(\mathbb{C}) = S^1$ for all $\ell \geqslant 1$. To understand the triple $(\overline{G}_\sigma, \overline{H}_\sigma, \bar{\sigma})$, we will need the compact group

$$X_+ = \prod_{\ell=1}^{\infty} F_\ell,$$

the locally compact group $X = Y_- \oplus X^+$ and the finite subgroup $\Gamma = \sigma(Y_+)$ of S^1.

Case (i). Suppose that $\sigma_\ell = \sigma_1$ for all $\ell \geqslant 1$, i.e., σ is periodic of period 1. We denote by α (again) the action of $\mathbb{Z}$ on X given by the shift to the right and by β the action of $\mathbb{Z}$ by topological automorphisms on $X \oplus \Gamma$ given by

$$\beta_k(y,u) = \big(\alpha_k(y), \sigma\big(\alpha_k(y)\big)\big)\sigma\big(y^{-1}\big)u\big) \quad \forall (y,u) \in Y \oplus \Gamma,\ k \in \mathbb{Z}.$$

Then $\overline{G}_\sigma$ is isomorphic to the locally compact group $(X \oplus \Gamma) \rtimes_\beta \mathbb{Z}$; under this isomorphism, $\overline{H}_\sigma$ is mapped to the compact open subgroup $X_+ \oplus \Gamma$ and $\bar{\sigma}$ becomes the representation $(x,u) \mapsto u$ of $X_+ \oplus \Gamma$ on $\mathbb{C}$. By Proposition 2, $\mathcal{L}(G,H,\sigma)$ can be seen as a corner of the group von Neumann algebra $\mathcal{L}((X \oplus \Gamma) \rtimes_\beta \mathbb{Z})$, which is isomorphic to the crossed product $\mathrm{L}^\infty(X \oplus \Gamma) \otimes_\beta \mathbb{Z}$. It turns out that the centre of $\mathcal{L}(G,H,\sigma)$ is isomorphic to $\mathrm{L}^\infty(D)$, where D is a fundamental domain of X with respect to α.

Case (ii). Suppose that σ is periodic of period N, i.e., $\sigma_{\ell+N} = \sigma_\ell$ for all $\ell \geqslant 1$ and N is the smallest positive integer for which this is true. Then $B_\sigma = Y \rtimes_\alpha N\mathbb{Z}$. Let α' be the action of $\mathbb{Z}$ on X given by $\alpha'_k = \alpha_{kN}$ for all $k \in \mathbb{Z}$ and let β' be the corresponding action of $\mathbb{Z}$ on $X \oplus \Gamma$. The results of case (i), where F is replaced by

$$F' = \bigoplus_{\ell=1}^{N} F_\ell,$$

show that $\mathcal{L}(G, H, \sigma)$ is isomorphic to the crossed product $\mathrm{L}^\infty(X \oplus \Gamma) \otimes_{\beta'} \mathbb{Z}$ and the centre of $\mathcal{L}(G, H, \sigma)$ is isomorphic to $\mathrm{L}^\infty(D)$, where D is a fundamental domain of X with respect to α'.

Case (iii). If σ is not periodic then $B_\sigma = Y$ and $\mathcal{L}(G, H, \sigma)$ is isomorphic to $\mathrm{L}^\infty(X^+)$.

Example 5 *(*The Bost–Connes example with an arbitrary character*)*. – Let $G = \mathbb{Q} \rtimes \mathbb{Q}^*$, where $\mathbb{Q}^*$ acts on $\mathbb{Q}$ by multiplication, let $H = \mathbb{Z}$ and let σ be a character on $\mathbb{Z}$.

Case (i). Suppose that $\sigma(\mathbb{Z})$ is finite. Denote by $\mathcal{A}$ the finite adele group $\prod' \mathbb{Q}_p$ (where the product is over all finite places p and the dash indicates a restricted direct product, i.e., elements of $\mathcal{A}$ are sequences in $\prod \mathbb{Q}_p$ whose entries lie in $\mathbb{Z}_p$ for all but a finite number of prime numbers $p \in \mathbb{N}$). Let $\mathcal{R}$ be the subgroup $\prod \mathbb{Z}_p$ of $\mathcal{A}$. Then $\overline{G}_\sigma$ is isomorphic to the locally compact group $\mathcal{A} \rtimes \mathbb{Q}^*$; under this isomorphism, $\overline{H}_\sigma$ is mapped to the compact open subgroup $\mathcal{R}$ of $\mathcal{A}$ and $\bar{\sigma}$ becomes the character on $\mathcal{R}$ defined by $x \mapsto \lim \sigma(b^{(m)})$, where $\{b^{(m)}\}$ is a sequence in $\mathbb{Z}$ which converges to x in $\mathcal{A}$. By Proposition 2, $\mathcal{L}(G, H, \sigma)$ can be seen as a corner of $\mathcal{L}(\mathcal{A} \rtimes \mathbb{Q}^*)$, which is a type III_1 factor [3]; it follows that $\mathcal{L}(G, H, \sigma)$ is a type III_1 factor too. Note that $S(\mathcal{L}(G, H, \sigma)) = \mathbb{R}_+^* = \overline{\Delta(B_\sigma)}$.

Case (ii). If $\sigma(\mathbb{Z})$ is dense in the unit circle, then $B_\sigma = \mathbb{Q}$ and $\mathcal{L}(G, H, \sigma) \simeq \mathcal{L}(\mathbb{Q}/\mathbb{Z}) \simeq \mathrm{L}^\infty(K)$, where K denotes the Pontryagin dual of $\mathbb{Q}/\mathbb{Z}$.

Example 6 *(*Variant of the Bost–Connes example*)*. – Again, let $G = \mathbb{Q} \rtimes \mathbb{Q}^*$, where $\mathbb{Q}^*$ acts on $\mathbb{Q}$ by multiplication. Let p be a fixed prime number and let $H = \mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$.

Case (i). Let σ be a character on H, which can be extended to $\mathbb{Z}_p$. Then $\overline{G}_\sigma \simeq \mathbb{Q}_p \rtimes \mathbb{Q}^*$ and under this isomorphism, $\overline{H}_\sigma$ is mapped to $\mathbb{Z}_p$ and $\bar{\sigma}$ becomes the extension of σ to $\mathbb{Z}_p$. By Proposition 2, $\mathcal{L}(G, H, \sigma)$ is isomorphic to a corner of the von Neumann algebra $\mathcal{L}(\mathbb{Q}_p \rtimes \mathbb{Q}^*)$, which is a type II_∞ factor. It follows that $\mathcal{L}(G, H, \sigma)$ is a type II factor. If $\sigma \neq 1$ then the canonical faithful normal state ϕ of Proposition 1 is tracial, so $\mathcal{L}(G, H, \sigma)$ is of type II_1. If $\sigma = 1$ then $\mathcal{L}(G, H, \sigma)$ is of type II_∞, since if $f : G \to \mathbb{C}$ is the function equal to $p^{-1/2}$ on the double coset $H(0, p)H$ and 0 elsewhere, then the element $A = \lambda(f)$ of $\mathcal{L}(G, H, \sigma)$ is such that $A^*A = 1 \neq AA^*$; in this case $S(\mathcal{L}(G, H, \sigma)) = \{1\}$ is strictly contained in $\overline{\Delta(B_\sigma)} = \{p^m : m \in \mathbb{Z}\} \cup \{0\}$.

Case (ii). Let σ be a character on H which cannot be extended to $\mathbb{Z}_p$. Then $B_\sigma = \mathbb{Q}$ and $\mathcal{L}(G, H, \sigma) \simeq \mathrm{L}^\infty(K)$, where K denotes the Pontryagin dual of $\mathbb{Q}/\mathbb{Z}_{(p)}$.

Acknowledgements. My sincere thanks to Pierre de la Harpe, Vaughan Jones and Kroum Tzanev for their help and patience.

References

[1] Andrianov A., Quadratic Forms and Hecke Operators, Grundlehren der mathematischen Wissenschaften, Vol. 286, Springer-Verlag, 1987.
[2] Bartholdi L., Grigorchuk R.I., Sous-groupes paraboliques et représentations de groupes branchés, C. R. Acad. Sci. Paris, Série I 332 (2001) 789–794.
[3] Bost J.-B., Connes A., Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, Selecta Math. (N.S.) 1 (1995) 411–457.
[4] Burger M., de la Harpe P., Constructing irreducible representations of discrete groups, Proc. Indian Acad. Sci. (Math. Sci.) 107 (1997) 223–235.
[5] Connes A., Une classification des facteurs de type III, Ann. Sci. École Norm. Sup. 6 (4) (1973) 133–252.
[6] Corwin L., Induced representations of discrete groups, Proc. Amer. Math. Soc. 47 (1975) 279–287.
[7] Dieudonné J., Eléments d'analyse, Tome VI, Gauthier-Villars, 1975, pp. 44–57.
[8] Schlichting G., Polynomidentitäten und Permutationsdarstellungen lokalkompakter Gruppen, Inventiones Math. 55 (1979) 97–106.
[9] Tzanev K., C*-algèbres de Hecke et K-théorie, Thèse de doctorat, Université Paris-7, 2000.