C. R. Acad. Sci. Paris, Ser. I 334 (2002) 585–590

Systèmes dynamiques/*Dynamical Systems*

Partially hyperbolic geodesic flows are Anosov *

Gonzalo Contreras

Cimat, PO box 402, 36.000 Guanajuato GTO, México, Mexique

Received 30 October 2001; accepted 5 November 2001

Note presented by Etienne Ghys.

Celebrating the 60th birthday of Jacob Palis.

Abstract We prove that if a $\mathbb{Z}$ or $\mathbb{R}$-action by symplectic linear maps on a symplectic vector bundle E has a weakly dominated invariant splitting $E = S \oplus U$ with $\dim U = \dim S$, then the action is hyperbolic. In particular, contact and geodesic flows with a dominated splitting with $\dim S = \dim U$ are Anosov. ***To cite this article: G. Contreras, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 585–590.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les flots géodésiques partiellement hyperboliques sont Anosov

Résumé Considérons une action de $\mathbb{R}$ ou $\mathbb{Z}$ sur un fibré vectoriel muni d'une struture symplectique par des applications linéaires préservant cette structure symplectique, et supposons que cette action possède une décomposition invariante faiblement dominée $E = S \oplus U$ avec $\dim U = \dim S$. On montre alors que cette action est nécessairement hyperbolique. ***Pour citer cet article : G. Contreras, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 585–590.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Un fibré vectoriel symplectique $\pi : \mathbf{E} \to B$ est un fibré vectoriel dont les changements de cartes $(U \cap V) \times \mathbb{R}^{2N} \hookleftarrow$ préservent la structure symplectique standard de $\mathbb{R}^{2N}$. Un tel fibré est donc équipé d'une forme symplectique continue dont la valeur sur chaque fibre est induite par la forme symplectique standard de $\mathbb{R}^{2N}$.

Soit $\mathrm{Sp}(\mathbf{E})$ l'espace des isomorphismes du fibré vectoriel symplectique $\mathbf{E}$. Considérons une $\mathbb{R}$-action $\Psi : \mathbb{R} \to \mathrm{Sp}(\mathbf{E})$ et pour tout $t \in \mathbb{R}$, notons Ψ_t l'isomorphisme de fibré associé. Cette action induit un flot continu $\psi_t : B \hookleftarrow$ qui vérifie $\psi_t \circ \pi = \pi \circ \Psi_t$.

On dit que l'action Ψ est *faiblement partiellement hyperbolique*[1] s'il existe une décomposition invariante du fibré $\mathbf{E} = S \oplus U$, non nécessairement continue, telle que pour tout $b \in B$,

(1) $\{0\} \neq S(b) \neq \mathbf{E}(b)$;

(2) $\inf_{t \geqslant 0} \|\Psi_t|_{S(b)}\| \cdot \|\Psi_{-t}|_{U(\psi_t b)}\| = 0$;

(3) $\inf_{t \geqslant 0} \|\Psi_t|_{S(\psi_{-t} b)}\| \cdot \|\Psi_{-t}|_{U(b)}\| = 0$.

Une action Ψ est dite *hyperbolique* s'il existe une décomposition invariante du fibré $\mathbf{E} = E^s \oplus E^u$ et des constantes $C > 0$, $\lambda > 0$, telles que :

(i) $|\Psi_t(\xi)| \leqslant C\, \mathrm{e}^{-\lambda t} |\xi|$ pour tout $t > 0$, $\xi \in E^s$;

(ii) $|\Psi_{-t}(\xi)| \leqslant C\, \mathrm{e}^{-\lambda t} |\xi|$ pour tout $t > 0$, $\xi \in E^u$.

E-mail address: gonzalo@cimat.mx (G. Contreras).

S1631-073X(02)02196-9/FLA

Dans ce cas il est facile de montrer que la décomposition est continue et que les sous-fibrés E^s et E^u sont lagrangiens.

Enfin rappelons que l'*ensemble non errant* $\Omega(\psi)$ de ψ est l'ensemble des points $b \in B$ tels que pour tout voisinage U de b il existe $T > 0$ vérifiant $\psi_T(U) \cap U \neq 0$.

Dans cette Note, nous montrons le résultat suivant :

THÉORÈME A. – *Soit* $\pi : \mathbf{E} \to B$ *un fibré vectoriel symplectique de base* B *compacte et* $\Psi : \mathbb{R} \to \mathrm{Sp}(\mathbf{E})$ *une* $\mathbb{R}$*-action continue. Si* Ψ *est faiblement partiellement hyperbolique,* $\dim S = \dim U$ *et* $\Psi|_{\pi^{-1}(\Omega)}$ *est la restriction à l'ensemble non-errant* $\Omega = \Omega(\psi|_B)$ *du flot induit* ψ*, alors* $\Psi|_{\pi^{-1}(\Omega)}$ *est hyperbolique.*

En fait, la restriction à l'ensemble non-errant n'est pas nécessaire si on sait a priori que la décomposition est continue :

THÉORÈME B. – *Soit* Ψ *une* $\mathbb{R}$*-action symplectique et continue sur une base compacte* B*. S'il existe deux sous-fibrés continus* S *et* U *tels que* $\dim S = \dim U$ *et tels que* $S \oplus U$ *est une décomposition invariante faiblement partiellement hyperbolique, alors* Ψ *est hyperbolique sur toute la base* B*.*

Il se trouve que lorsque la décomposition $S \oplus U$ est continue, elle coincide avec la décompostion hyperbolique : $S = E^s$ et $U = E^u$. En particulier les fibrés S et U sont lagrangiens.

Des preuves analogues permettent d'énoncer les deux théorèmes ci-dessus dans le cadre des Z-actions : $\mathbb{Z} \to \mathrm{Sp}(\mathbf{E})$, et en particulier pour la différentielle de symplectomorphismes.

On peut appliquer le théorème A lorsque $\Psi_t = \mathrm{d}\phi_t$, où ϕ_t est un flot de contact sur une variété compacte $\mathcal{N}$ préservant une 1-forme non dégénérée Θ et $\mathbf{E}(x) = \ker \Theta_x$. Puisque ϕ_t préserve la forme volume $\Theta \wedge (\mathrm{d}\Theta)^n$, $\dim \mathcal{N} = 2n + 1$, nous savons par le théorème de récurrence de Poincaré que $\Omega(\phi) = \mathcal{N}$. Nous avons alors :

COROLLAIRE. – *Un flot de contact faiblement partiellement hyperbolique vérifiant* $\dim S = \dim U$ *est Anosov.*

Un exemple de flot de contact est donné par le flot géodésique sur une variété Riemannienne compacte (M, g) avec la 1-forme $\Theta_v(\xi) = \langle v, \mathrm{d}\pi(\xi)\rangle_g$, $\xi \in T_v SM$, $SM = \{v \in TM \mid \|v\|_g = 1\}$; et la projection $\pi : SM \to M$. Ceci répond à une question de M. Herman.

Les deux théorèmes ci-dessus peuvent s'appliquer au cas d'un niveau d'energie régulier d'un champ Hamiltonien qui ne possède pas de fibré transverse continu. (*Voir* version en anglais).

---

1. Statements

A symplectic vector bundle $\pi : \mathbf{E} \to B$ is a vector bundle whose transition maps $(U \cap V) \times \mathbb{R}^{2N} \hookleftarrow$ preserve the canonical symplectic structure of $\mathbb{R}^{2N}$ on the fibers. Such bundle carries a continuous symplectic form on each fiber induced by the symplectic form on $\mathbb{R}^{2N}$.

Consider a continuous $\mathbb{R}$-action $\Psi : \mathbb{R} \to \mathrm{Sp}(\mathbf{E})$, where $\Psi_t : \mathbf{E} \to \mathbf{E}$ is a bundle map which is a symplectic linear isomorphism on each fiber and $\Psi_{s+t} = \Psi_s \circ \Psi_t$. The action Ψ induces a continuous flow $\psi_t : B \hookleftarrow$ such that $\psi_t \circ \pi = \pi \circ \Psi_t$.

We say that the action Ψ is *weakly partially hyperbolic*[2] if there exists an invariant splitting $\mathbf{E} = S \oplus U$ (not necessarily continuous) such that for each $b \in B$,

(1) $\{0\} \neq S(b) \neq \mathbf{E}(b)$;
(2) $\inf_{t \geqslant 0} \|\Psi_t|_{S(b)}\| \cdot \|\Psi_{-t}|_{U(\psi_t b)}\| = 0$;
(3) $\inf_{t \geqslant 0} \|\Psi_t|_{S(\psi_{-t} b)}\| \cdot \|\Psi_{-t}|_{U(b)}\| = 0$.

We say that Ψ is *hyperbolic* if there exists a invariant splitting $\mathbf{E} = E^s \oplus E^u$ and constants $C > 0$, $\lambda > 0$, such that

To cite this article: G. Contreras, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 585–590

(i) $|\Psi_t(\xi)| \leqslant C\,\mathrm{e}^{-\lambda t}\,|\xi|$ for all $t > 0$, $\xi \in E^s$;
(ii) $|\Psi_{-t}(\xi)| \leqslant C\,\mathrm{e}^{-\lambda t}\,|\xi|$ for all $t > 0$, $\xi \in E^u$.

It follows that the hyperbolic splitting $E^s \oplus E^u$ is necessarily continuous and that the subspaces E^s and E^u are Lagrangian.

Define the *non-wandering* set $\Omega(\psi)$ of ψ as the set of points $b \in B$ such that for every neighbourhood U of b there exists $T > 0$ such that $\psi_T(U) \cap U \neq 0$.

Here we prove:

THEOREM A. – *Let $\pi : \mathbf{E} \to B$ be a continuous symplectic vector bundle with B compact and $\Psi : \mathbb{R} \to \mathrm{Sp}(\mathbf{E})$ a continuous $\mathbb{R}$-action with induced flow ψ_t: $\pi \circ \Psi_t = \psi_t \circ \pi$.*

If Ψ is weakly partially hyperbolic with $\dim S = \dim U$ and $\Omega = \Omega(\psi|_B)$ is the non-wandering set then the restricted action $\Psi|_{\pi^{-1}(\Omega)}$ is hyperbolic.

The restriction to the non-wandering set is not needed if we know a priori that the splitting is continuous:

THEOREM B. – *Let Ψ is a continuous symplectic $\mathbb{R}$-action on a compact base B. Suppose that $S \oplus U$ is a weakly partially hyperbolic splitting with $\dim S = \dim U$. If furthermore, S and U are continuous subbundles, then Ψ is hyperbolic over all B.*

It turns out that when the splitting $S \oplus U$ is continuous, necessarily $S = E^s$ and $U = E^u$ in the hyperbolic splitting. In particular they are Lagrangian subbundles.

The theorems above, with the same proofs, hold for $\mathbb{Z}$-actions: $\mathbb{Z} \to \mathrm{Sp}(\mathbf{E})$. In particular, for the derivatives of symplectic diffeomorphisms.

We say that $\Psi : \mathbb{R} \to \mathrm{Sp}(\mathbf{E})$ is *partially hyperbolic* if there is an invariant splitting $\mathbf{E} = S \oplus U$ and $\tau > 0$, $0 < \lambda < 1$ such that

$$\left\|\Psi_t|_{S(b)}\right\| \cdot \left\|\Psi_{-t}|_{U(\psi_t b)}\right\| < \lambda, \quad \text{for all } b \in B.$$

A partially hyperbolic action is also weakly partially hyperbolic and its splitting $S \oplus U$ is necessarily continuous. Thus we get:

COROLLARY 1. – *A partially hyperbolic symplectic action with $\dim S = \dim U$ on a compact base B is hyperbolic.*

We can apply theorem A to the case when $\Psi_t = \mathrm{d}\phi_t$, where ϕ_t is a contact flow on a compact manifold $\mathcal{N}$ preserving a non-degenerate 1-form Θ and $\mathbf{E}(x) = \ker \Theta_x$, $x \in \mathcal{N}$. Since ϕ_t preserves the volume form $\Theta \wedge (\mathrm{d}\Theta)^n$, $\dim \mathcal{N} = 2n + 1$, by Poincaré recurrence, $\Omega(\phi) = \mathcal{N}$. Hence we obtain

COROLLARY 2. – *A weakly partially hyperbolic contact flow with $\dim S = \dim U$ is Anosov.*

An example of contact flow is the geodesic flow of a compact Riemannian manifold (M, g), with the 1-form $\Theta_v(\xi) = \langle v, \mathrm{d}\pi(\xi)\rangle_g$, $\xi \in T_v SM$; where $SM = \{v \in TM \mid \|v\|_g = 1\}$ and $\pi : SM \to M$ is the projection. This answers a question posed by M. Herman: is a partially hyperbolic geodesic flow Anosov?[3]

The theorems above can be applied to a regular energy level of a non-contact Hamiltonian flow which has no obvious continuous invariant transversal bundle. The problem here is that the tangent space to the energy level is odd-dimensional and the bundle $\mathbf{E} = S \oplus U$ can not contain the direction of the Hamiltonian vectorfield. One avoids this problem by projecting the derivative of the Hamiltonian flow along a continuous transversal bundle as follows. Let (M, ω) be a symplectic manifold, $H : M \to \mathbb{R}$ and e a regular value of H. Suppose that $\mathcal{N} = H^{-1}(\{e\})$ is compact. Let X be the Hamiltonian vector field for H on $\mathcal{N}$, $\omega(X, \cdot) = \mathrm{d}H$, and let ϕ be its flow. Let $\mathbf{E}$ be a continuous (non-invariant) subbundle of $T\mathcal{N}$ which is transversal to X (e.g., endow M with a Riemannian metric and let $\mathbf{E} = \{v \in T\mathcal{N} \mid \omega(\nabla H, v) = 0\}$). Then $(\mathbf{E}, \omega|_{\mathbf{E}})$ is a symplectic bundle. Let $\Lambda : T\mathcal{N} = \mathbf{E} \oplus \langle X\rangle \to \mathbf{E}$ be the projection in the splitting. Then we ask that the symplectic action $\Psi = \Lambda \circ \mathrm{d}\phi$ on $\mathbf{E}$ is weakly partially hyperbolic. Apply theorem A to show that the action Ψ is hyperbolic. Since the growth of $\mathrm{d}\phi_t$ in the direction of the Hamiltonian vectorfield is subexponential, then standard methods using a graph transformation (*cf.* Hirsch–Pugh–Shub [4], or [1], p. 930) show that if Ψ is hyperbolic then ϕ is Anosov.

2. Proofs

We say that Ψ is *quasi-hyperbolic* if $\sup_{t\in\mathbb{R}}|\Psi_t(\xi)| = +\infty$ for all $\xi \in \mathbf{E}$, $\xi \neq 0$. We shall use:

PROPOSITION 2.1. – *Let $\pi : \mathbf{E} \to B$ is a continuous vector bundle, and $\Psi : \mathbb{R} \to GL(\mathbf{E})$ a continuous $\mathbb{R}$-action of linear isomorphisms with induced flow ψ_t: $\pi \circ \Psi_t = \psi_t \circ \pi$.*

If B is compact and Ψ is quasi-hyperbolic then the restriction $\Psi|_{\pi^{-1}(\Omega)}$ to the lift of the nonwandering set $\Omega = \Omega(\psi|_B)$ is hyperbolic.

The proof is similar to that of [1], §3 Theorem 0.2, pp. 926–929, and its origins can be traced back to Eberlein [2], Sacker and Sell [8,9] and Selgrade [10].

Define

$$\mathfrak{S}(b) := \{\mathfrak{s} \in S(b) \mid \forall \mathfrak{u} \in U(b),\ \Omega_b(\mathfrak{s},\mathfrak{u}) = 0\},$$
$$\mathfrak{U}(b) := \{\mathfrak{u} \in U(b) \mid \forall \mathfrak{s} \in S(b),\ \Omega_b(\mathfrak{u},\mathfrak{s}) = 0\}.$$

Proof of Theorem A. – Fix $b \in B$ and fix sequences $\tau_n \to +\infty$ and $\sigma_n \to +\infty$ such that

$$\lim_n \|\Psi_{\tau_n}|_{S(b)}\| \cdot \|\Psi_{-\tau_n}|_{U(\psi_{\tau_n} b)}\| = 0, \tag{4}$$

$$\lim_n \|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n} b)}\| \cdot \|\Psi_{-\sigma_n}|_{U(b)}\| = 0. \tag{5}$$

Write $\mathfrak{B}(b) := \{v \in \mathbf{E}(b) \mid \sup_{t\in\mathbb{R}} |\Psi_t v| < +\infty\}$. By Proposition 2.1, we have to prove that $\mathfrak{B}(b) = \{0\}$.

LEMMA 2.2. –

(a) *If* $\liminf_n \|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n} b)}\| = 0$ *then*

$$\forall \mathfrak{s} \in S(b)\setminus\{0\}, \quad \limsup_n |\Psi_{-\sigma_n}(\mathfrak{s})| = +\infty. \tag{6}$$

(b) *If* $\liminf_n \|\Psi_{-\tau_n}|_{U(\psi_{\tau_n} b)}\| = 0$ *then*

$$\forall \mathfrak{u} \in U(b)\setminus\{0\}, \quad \limsup_n |\Psi_{\tau_n}(\mathfrak{u})| = +\infty. \tag{7}$$

(c) *If both conditions* (6) *and* (7) *hold, then* $\mathfrak{B}(b) = \{0\}$.

Proof. – (a) If $\mathfrak{s} \in S(b)\setminus\{0\}$, then

$$\liminf_n \frac{|\mathfrak{s}|}{|\Psi_{-\sigma_n}(\mathfrak{s})|} = \liminf_n \frac{|\Psi_{\sigma_n}(\Psi_{-\sigma_n}\mathfrak{s})|}{|\Psi_{-\sigma_n}\mathfrak{s}|} \leqslant \liminf_n \|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n} b)}\| = 0.$$

(b) If $\mathfrak{u} \in U(b)\setminus\{0\}$, then

$$\liminf_n \frac{|\mathfrak{u}|}{|\Psi_{\tau_n}(\mathfrak{u})|} = \liminf_n \frac{|\Psi_{-\tau_n}(\Psi_{\tau_n}\mathfrak{u})|}{|\Psi_{\tau_n}\mathfrak{u}|} \leqslant \liminf_n \|\Psi_{-\tau_n}|_{U(\psi_{\tau_n} b)}\| = 0.$$

(c) Conditions (6) and (7) imply that $\mathfrak{B}(b) \cap S(b) = \{0\}$ and $\mathfrak{B}(b) \cap U(b) = \{0\}$.

Let $v = \mathfrak{s} \oplus \mathfrak{u} \in S \oplus U = \mathbf{E}$ be such that $\mathfrak{s} \neq 0$ and $\mathfrak{u} \neq 0$. From (4) we have that

$$\frac{|\Psi_{\tau_n}\mathfrak{s}|}{|\mathfrak{s}|} \cdot \frac{|\mathfrak{u}|}{|\Psi_{\tau_n}\mathfrak{u}|} = \frac{|\Psi_{\tau_n}\mathfrak{s}|}{|\mathfrak{s}|} \cdot \frac{|\Psi_{-\tau_n}(\Psi_{\tau_n}\mathfrak{u})|}{|\Psi_{\tau_n}\mathfrak{u}|} \xrightarrow{n} 0.$$

Therefore

$$\lim_n |\Psi_{\tau_n}(\mathfrak{s})| / |\Psi_{\tau_n}(\mathfrak{u})| = 0.$$

Using (7), we have that

$$\limsup_n \left|\Psi_{\tau_n}(v)\right| \geqslant \limsup_n \left(\left|\Psi_{\tau_n}(\mathfrak{u})\right| - \left|\Psi_{\tau_n}(\mathfrak{s})\right|\right) = +\infty. \tag{8}$$

Then $v \notin \mathfrak{B}(b)$. □

Suppose first that $\mathfrak{S}(b) = \{0\}$ and $\mathfrak{U}(b) = \{0\}$. Let $\mathfrak{u} \in U(b)$, $\mathfrak{u} \neq 0$. Since $\mathfrak{U}(b) = \{0\}$, there exists $\mathfrak{s} \in S(b)$ such that $\Omega(\mathfrak{s}, \mathfrak{u}) = 1$. Since $|\Psi_{\tau_n}\mathfrak{s}| \cdot |\Psi_{\tau_n}\mathfrak{u}| \geqslant \Omega(\Psi_{\tau_n}\mathfrak{s}, \Psi_{\tau_n}\mathfrak{u}) = \Omega(\mathfrak{s}, \mathfrak{u}) = 1$, then

$$\liminf_n |\Psi_{\tau_n}\mathfrak{s}| = 0 \quad \Longrightarrow \quad \limsup_n |\Psi_{\tau_n}\mathfrak{u}| = +\infty. \tag{9}$$

From (4),

$$\frac{|\Psi_{\tau_n}\mathfrak{s}|}{|\mathfrak{s}|} \cdot \frac{|\mathfrak{u}|}{|\Psi_{\tau_n}\mathfrak{u}|} = \frac{|\Psi_{\tau_n}\mathfrak{s}|}{|\mathfrak{s}|} \cdot \frac{|\Psi_{-\tau_n}(\Psi_{\tau_n}\mathfrak{u})|}{|\Psi_{\tau_n}\mathfrak{u}|} \xrightarrow{n} 0. \tag{10}$$

So that

$$\liminf_n |\Psi_{\tau_n}\mathfrak{s}| > 0 \quad \Longrightarrow \quad \limsup_n |\Psi_{\tau_n}\mathfrak{u}| = +\infty. \tag{11}$$

From (9) and (11), we obtain that

$$\limsup_n |\Psi_{\tau_n}\mathfrak{u}| = +\infty, \quad \forall \mathfrak{u} \in U(b) \setminus \{0\}. \tag{12}$$

Similar arguments to (9) and (11) using (5), show that

$$\limsup_n |\Psi_{-\sigma_n}\mathfrak{s}| = +\infty, \quad \forall \mathfrak{s} \in S(b) \setminus \{0\}. \tag{13}$$

Then (12), (13) and Lemma 2.2(c) imply that $\mathfrak{B}(b) = \{0\}$.

The hypothesis on S and U are symmetric, so, using the inverse action Ψ_{-t} if necessary, it is enough to assume from now on that $\mathfrak{S}(b) \neq \{0\}$.

Let $v \in \mathfrak{S}(b) \setminus \{0\}$. There exists $w = \mathfrak{s} + \mathfrak{u} \in S \oplus U = \mathbf{E}$ such that $\Omega(v, w) = 1$. Since by definition of $\mathfrak{S}(b)$, we have that $\Omega(v, \mathfrak{u}) = 0$, then

$$1 = \Omega(v, \mathfrak{s}) = \Omega(\Psi_{\tau_n}v, \Psi_{\tau_n}\mathfrak{s}) \leqslant \left\|\Psi_{\tau_n}|_{S(b)}\right\|^2 |v|\,|\mathfrak{s}|.$$

Therefore $\liminf_n \|\Psi_{\tau_n}|_{S(b)}\| > 0$. From (4) we have that

$$\liminf_n \left\|\Psi_{-\tau_n}|_{U(\psi_{\tau_n}b)}\right\| = 0. \tag{14}$$

Suppose that

$$\liminf_n \left\|\Psi_{-\sigma_n}|_{U(b)}\right\| = 0.$$

If $\mathfrak{u}_1, \mathfrak{u}_2 \in U(b)$, then

$$\left|\Omega(\mathfrak{u}_1, \mathfrak{u}_2)\right| = \lim_n \left|\Omega(\Psi_{-\sigma_n}\mathfrak{u}_1, \Psi_{-\sigma_n}\mathfrak{u}_2)\right| \leqslant \liminf_n \left\|\Psi_{-\sigma_n}|_{U(b)}\right\|^2 = 0.$$

Therefore the subspace $U(b)$ is isotropic. If $0 \neq \mathfrak{s} \in \mathfrak{S}(b)$ then, by the definition of $\mathfrak{S}(b)$, the subspace $U(b) \oplus \langle\mathfrak{s}\rangle$ would be isotropic. This contradicts the hypothesis $\dim U = \dim S = \frac{1}{2}\dim \mathbf{E}$. Hence

$$\liminf_n \left\|\Psi_{-\sigma_n}|_{U(b)}\right\| > 0. \tag{15}$$

Now (15) and (5) imply that

$$\liminf_n \left\|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n}b)}\right\| = 0. \tag{16}$$

But (14), (16) and Lemma 2.2 imply that $\mathfrak{B}(b) = \{0\}$. □

Proof of Theorem B. – Define

$$E^s := \Big\{ \mathfrak{v} \in \mathbf{E} \,\Big|\, \sup_{t \geqslant 0} |\Psi_t \mathfrak{v}| < +\infty \Big\}, \qquad E^u := \Big\{ \mathfrak{v} \in \mathbf{E} \,\Big|\, \sup_{t \geqslant 0} |\Psi_{-t} \mathfrak{v}| < +\infty \Big\}.$$

LEMMA 2.3. – *If S and U are continuous, then $E^s = S$ and $E^u = U$.*

Proof. – The limits (12), [(14), 2.2(b)], (8) and (13), [(16), 2.2(a)], (8) imply that $E^s \subseteq S$ and $E^u \subseteq U$. We only prove that $E^s = S$. Let $v \in S$ and suppose that $v \notin E^s$. Then there exists $b_n \to +\infty$ such that $\lim_n |\Psi_{b_n} v| = +\infty$. Define $a_n \in [0, b_n]$ by

$$|\Psi_{a_n} v| = \max_{0 \leqslant t \leqslant b_n} |\Psi_t v|.$$

Then $|\Psi_{a_n} v| \geqslant |\Psi_{b_n} v| \xrightarrow{n} +\infty$, so that $\lim_n a_n = +\infty$.

Let $u_n := \Psi_{a_n} v / |\Psi_{a_n} v|$, $x_n := \pi\, u_n$. For $-a_n \leqslant t \leqslant 0$, we have that

$$|\Psi_t u_n| \leqslant \frac{|\Psi_{t+a_n} v|}{|\Psi_{a_n} v|} \leqslant 1.$$

Since B is compact and $|\mathfrak{u}_n| \equiv 1$, taking a subsequence we can assume that $x_n \to x \in B$ and $u_n \to u \in \mathbf{E}(x)$. Then $|\Psi_t(u)| \leqslant 1$ for all $t \leqslant 0$. Using the continuity of $x \mapsto S(x)$, we have that

$$u \in E^u(x) \cap \lim_n S(x_n) = E^u(x) \cap S(x) \subseteq U(x) \cap S(x) = \{0\}.$$

Since $|u| = 1$, this is a contradiction. □

The following lemma completes the proof of Theorem B. A proof of the lemma can be found in [1], Lemma 3.2, p. 927.

LEMMA 2.4. – *Let Ψ be a quasi-hyperbolic action with B compact. Then there exists $\tau > 0$ such that*

$$\big\| \Psi_\tau |_{E^s(b)} \big\| < \frac{1}{2} \quad \text{and} \quad \big\| \Psi_{-\tau} |_{E^u(b)} \big\| < \frac{1}{2} \quad \text{for all } b \in B.$$

Acknowledgement. The author want to thank J.M. Gambaudo for his help with the french translation.

---

* Partially supported by CONACYT-México grant # 36496-E.

[1] La définition classique de l'hyperbolicité partielle demande la condition plus forte qu'il existe $t > 0$ et $0 < \lambda < 1$ tels que : $\|\Psi_t|_{S(b)}\| \cdot \|\Psi_{-t}|_{U(\psi_t b)}\| < \lambda$ pour tout $b \in B$. Cette condition supplémentaire implique la continuité de la décomposition.

[2] The usual partial hyperbolicity requires the stronger condition that there exist $t > 0$ and $0 < \lambda < 1$ such that $\|\Psi_t|_{S(b)}\| \cdot \|\Psi_{-t}|_{U(\psi_t b)}\| < \lambda$, for all $b \in B$; which implies the continuity of the splitting $S \oplus U$.

[3] If $\dim S \neq \dim U$, a product of two manifolds of negative curvature would be a counterexample.

References

[1] Contreras G., Iturriaga R., Convex Hamiltonians without conjugate points, Ergodic Theory Dynamic Systems 19 (1999) 901–952.

[2] Eberlein P., When is a geodesic flow of Anosov type? I., J. Differential Geom. 8 (1973) 437–463.

[3] Franks J., Robinson C., A quasi-Anosov diffeomorphism that is not Anosov, Trans. Amer. Math. Soc. 223 (1976) 267–278.

[4] Hirsch M., Pugh C., Shub M., Invariant Manifolds, Lecture Notes in Math., Vol. 583, Springer, 1977.

[5] Mañé R., Quasi-Anosov diffeomorphisms and hyperbolic manifolds, Trans. Amer. Math. Soc. 229 (1977) 351–370.

[6] Robinson C., A quasi-Anosov flow that is not Anosov, Indiana Univ. Math. J. 25 (1976) 763–767.

[7] Ruggiero R.O., Persistently expansive geodesic flows, Comm. Math. Phys. 140 (1991) 203–215.

[8] Sacker R.J., Sell G.R., A note on Anosov diffeomorphisms, Bull. Amer. Math. Soc. 80 (1974) 278–280.

[9] Sacker R.J., Sell G.R., Existence of exponential dichotomies and invariant splitting I, II, III, J. Differential Equations 15 (1974) 429–458; 22 (1976) 478–496; 22 (1976) 497–522.

[10] Selgrade J., Isolated invariant sets for flows on vector bundles, Trans. Amer. Math. Soc. 203 (1975) 359–390.