C. R. Acad. Sci. Paris, Ser. I 334 (2002) 43–46

Géométrie différentielle/*Differential Geometry*
(Géométrie algébrique/*Algebraic Geometry*)

Classification of positive quaternion-Kähler 12-manifolds

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Received 22 October 2001; accepted 20 November 2001

Note presented by Mikhaël Gromov.

Abstract We prove that the 12-dimensional complete quaternion-Kähler manifolds with positive scalar curvature belong to the list of symmetric spaces given by Wolf [12]. To cite this article: H. Herrera, R. Herrera, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 43–46. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Classification de variétés Kähleriennes quaternioniques positives de dimension 12

Résumé Dans cette Note, nous démontrons que les variétés complètes Kähleriennes quaternioniques de courbure scalaire positive et de dimension 12 appartiennent à la liste d'espaces symétriques donnée par Wolf [12]. Pour citer cet article : H. Herrera, R. Herrera, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 43–46. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Consider $\mathbb{R}^{4n} \cong \mathbb{H}^n$ as a right module over the quaternions $\mathbb{H}$ whose elements are column vectors with entries in $\mathbb{H}$. Let $\mathrm{Sp}(n)$ be the group of matrices whose entries are quaternions such that $AA^* = I$, the identity matrix. Let $A \in \mathrm{Sp}(n)$, $q \in \mathrm{Sp}(1)$ act on $X \in \mathbb{H}^n$ by AXq^{-1}, so that $\mathrm{Sp}(n)\,\mathrm{Sp}(1) = (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\{\pm 1\} \subset \mathrm{SO}(4n) \subset \mathrm{GL}(4n, \mathbb{R})$.

DEFINITION 1.1. – An oriented connected irreducible Riemannian $4n$-manifold M is called a *quaternion-Kähler manifold*, $n \geqslant 2$, if its linear holonomy is contained in $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$. We shall call M *positive* if its metric is complete and has positive scalar curvature. When $n = 1$ we add the condition that the manifold M must be Einstein and self-dual, since $\mathrm{Sp}(1)\,\mathrm{Sp}(1) = \mathrm{SO}(4)$.

Wolf showed in [12] that each compact centerless Lie group G is the isometry group of a positive quaternion-Kähler symmetric space given as the conjugacy class of a copy of $\mathrm{Sp}(1)$ in G, determined by a highest root of G. They are called "Wolf spaces" and are the only known examples with complete metrics of positive scalar curvature. Moreover, it is known that there are only finitely many positive quaternion-Kähler manifolds for each n [7]. These facts have given hope to the following conjecture.

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S1631-073X(02)02209-4/FLA

CONJECTURE 1.1. – *Every positive quaternionic Kähler manifold is isometric to a (symmetric) Wolf space.*

Hitchin proved it in 4 dimensions [6], and Poon and Salamon proved it in 8 dimensions [10]. We are able to produce a proof of the conjecture in 12 dimensions (*see* [4] for the full details). Our main result states that a positive quaternion-Kähler 12-manifold must be a Wolf space.

THEOREM 1.1. – *A positive* 12*-dimensional quaternion-Kähler manifold is isometric to one of the following symmetric spaces*:
(1) *The quaternionic projective space* $\mathbb{HP}^3$.
(2) *The complex Grassmannian* $\mathbb{G}r_2(\mathbb{C}^5)$.
(3) *The real Grassmannian* $\mathbb{G}r_4(\mathbb{R}^7)$.

For instance, Poon and Salamon carried out a careful study of the standard twistor space Z of M as a polarized algebraic variety and were able to pin down the few candidates of polarized varieties that can occur as twistor spaces. Our approach is different since the main ingredients turn out to be topological rather than algebro-geometrical.

In Section 2 we review preliminaries of quaternion-Kähler geometry. In Section 3 we outline the proof of Theorem 1.1.

2. Preliminaries

The existence of the $\mathrm{Sp}(3)\,\mathrm{Sp}(1)$-structure induces an isomorphism

$$T^*M \otimes \mathbb{C} \cong E \otimes H,$$

where E and H denote the locally defined vector bundles over M associated to the standard (faithful) complex representations of $\mathrm{Sp}(3)$ and $\mathrm{Sp}(1)$ on $E = \mathbb{C}^6$ and $H = \mathbb{C}^2$ respectively. The quaternionic structure of M is characterized by a 4-form u coming from the second Chern class of the quaternionic line bundle H, i.e., $u = -c_2(H)$. The multiple $4u = -c_2(S^2H)$ is integral and non-degenerate [11], and we shall call it the *quaternionic class*, for which we have the following lemma.

LEMMA 2.1. – *Let M be a compact connected quaternion-Kähler* 12*-manifold of positive scalar curvature. The symmetric bilinear form Q on $H^4(M)$ defined by*

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge (4u),$$

$\alpha, \beta \in H^4(M)$, is positive definite.

Let d be the dimension of the isometry group G of M, and define the *quaternionic volume* by

$$\mathbf{v}(M) = \big\langle (4u)^3, [M] \big\rangle.$$

Let Δ be the 2^6-dimensional faithful spin representation of $\mathrm{Spin}(12)$. The representation Δ splits

$$\Delta = \Delta_+ \oplus \Delta_-,$$

where $\Delta_\pm$ are two copies of the 2^5-dimensional irreducible representation of $\mathrm{Spin}(11) \subset \mathrm{Spin}(12)$. Clifford multiplication of an element of Δ_+ by an element of $T = T^*M$ gives one in Δ_-. Although the manifolds under consideration are not spin in general [11] , there are well defined Dirac operators with coefficients in S^qH (*cf.* [8,11])

$$D\big(S^qH\big) : \Gamma\big(\Delta_+ \otimes S^qH\big) \longrightarrow \Gamma\big(\Delta_- \otimes S^qH\big),$$

with index

$$f(q) = \operatorname{ind} D\big(S^q H\big) = \big\langle \operatorname{ch}\big(S^q H\big)\widehat{A}(M), [M]\big\rangle, \tag{1}$$

for $q \geqslant 0$ and $3+q$ even, where ch and $\widehat{A}$ denote the Chern character of a (complex) vector and the $\widehat{A}$-genus of the manifold, respectively. The parity condition ensures that the corresponding coupled Dirac operator is *globally defined*. Note that $f(q)$ is a polynomial in q.

THEOREM 2.1 ([8,11]). – *Let M be a* 12*-dimensional positive quaternion-Kähler manifold. Then*

$$\pi_1(M) = 0,$$
$$\pi_2(M) = \begin{cases} 0 & \textit{iff } M \textit{ is homothetic to } \mathbb{HP}^3, \\ \mathbb{Z} & \textit{iff } M \textit{ is homothetic to } \mathbb{G}r_2(\mathbb{C}^5), \\ \textit{finite with 2-torsion} & \textit{otherwise} \end{cases}$$

and M is spin if and only if M is homothetic to $\mathbb{HP}^3$. Furthermore,

$$f(q) = \begin{cases} 0, & \textit{if } q = -3, -1, 1, \\ 1, & \textit{if } q = 3, \\ d \geqslant 5, & \textit{if } q = 5. \end{cases}$$

3. Sketch of the proof of the main theorem

The strategy of the proof is to pin down the possible values of the pair $(d, \mathbf{v})$.

By Theorem 2.1, we can assume that $\pi_1(M) = 0$, $\pi_2(M)$ is finite, and we can write down all the characteristic numbers involved as coefficients in $f(q)$ in terms of d and $\mathbf{v}$. In fact,

$$f(q) = \frac{1}{322560}(q-1)(q+1)(q+3)(525\mathbf{v} - 1260d + 11760 \\ - 672q - 336q^2 + \mathbf{v}q^4 + 4\mathbf{v}q^3 - 46\mathbf{v}q^2 - 100\mathbf{v}q + 84dq^2 + 168dq).$$

Clearly, at this point, we have far too many possible pairs $(d, \mathbf{v})$ to deal with. The key point in our proof is to find another zero of $f(q)$ which will give us a relation between d and $\mathbf{v}$ and, therefore, reduce enormously our task. This is a consequence of the following lemma whose proof depends on a deep result concerning the $\widehat{A}$-genus of certain non-spin manifolds with finite second homotopy group and smooth S^1 actions (*see* [4,5] for the details).

LEMMA 3.1. – *Let M be a positive quaternion-Kähler* 12*-dimensional manifold different from* $\mathbb{G}r_2(\mathbb{C}^5)$. *Then*

$$\widehat{A}(M) = 0.$$

This, in turn, means that $f(0) = 0$, so that

$$-\frac{5}{1024}\mathbf{v} - \frac{7}{64} + \frac{3}{256}d = 0. \tag{2}$$

In particular, this implies that $d \geqslant 11$ which already suggests that M must be homogeneous. (2) gives us the following list of possible integral pairs $(d, \mathbf{v})$: (i) $(11, 4)$; (ii) $(16, 16)$; (iii) $(21, 28)$; (iv) $(26, 40)$; (v) $(31, 52)$; (vi) $(36, 64)$.

The pairs (i) and (ii) are ruled out since the quadratic form Q in Lemma 2.1 is positive definite, i.e., $\mathbf{v}^2 - 64\mathbf{v} - 16\mathbf{v}d + 576 - 288d + 36d^2 < 0$. The pair (iii) corresponds to the real Grassmannian as can be seen by standard geometrical considerations and the classification of cohomogeneity one manifolds in [2,9]. The pairs (iv) and (v) are also ruled out since there are no semisimple Lie groups of the given dimension, and rank less than or equal 3 (the upper bound is determined in [1]). The pair (vi) corresponds to the quaternionic projective space, since the isotropy group K at any point of M must have dimension at least 24 and is contained in $\mathrm{Sp}(3)\,\mathrm{Sp}(1)$. $\square$

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