Sharp Hodge decompositions in two and three dimensional Lipschitz domains

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Abstract
We identify the optimal range of coefficients $s, p$ for which differential forms with coefficients in the Sobolev space $L^p_s(\Omega)$ admit natural Hodge decompositions in arbitrary two and three dimensional Lipschitz domains $\Omega$. To cite this article: D. Mitrea, M. Mitrea, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 109–112. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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un domaine lipschitzien arbitraire de $M$. Alors, il existe un $p_\Omega \in [1, 2]$ tel que pour tout $\ell \in \{0, 1, \ldots, n\}$, et tout couple $s, p$ satisfaisant

$$0 < \frac{1}{p} < 1, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad et$$

$$\frac{1}{2} \left( 1 - \frac{1}{p_\Omega} \right) < \frac{1}{p} < \frac{s}{2} < \frac{1}{2} \left( 1 + \frac{1}{p_\Omega} \right) \quad si \ n = 2, \quad (1.4)$$

on a

$$L^p_s(\Omega, \Lambda^\ell) = d_{\ell-1} H^{s,p}(\Omega; d_{\ell-1}) \oplus \delta_{\ell+1} H^{s,p}(\Omega; \delta_{\ell+1}) \oplus H^{s,p}_\Omega(\Omega, \Lambda^\ell), \quad (1.5)$$

$$L^p_s(\Omega, \Lambda^\ell) = \sigma(D)(\xi),$$

où les sommes directes sont topologiques.

L’indice $p_\Omega$ ci-dessus est lié à l’exposant critique du problème de Dirichlet et de Neumann pour $\Omega$ au sens de [1,2,10,8]. Grâce des contre-exemples simples dans des domaines avec des singularités coniques d’isolement, on voit que le théorème ci-dessus est optimal. La preuve utilise les intégrales singulières et a sens de [1,2,10,8]. Grâce des contre-exemples simples dans des domaines avec des singularités coniques d’isolement, on voit que le théorème ci-dessus est optimal. La preuve utilise les intégrales singulières et a

un caractère constructif.

Let $M$ be a (smooth) Riemannian manifold, and assume that $\mathcal{E, F} \rightarrow M$ are two Hermitian vector bundles. Recall that a relatively compact domain $\Omega \subset M$ is called Lipschitz provided $\partial \Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions. For such a domain, the unit outward conormal $v \in T^*M$ to $\partial \Omega$ is well defined a.e. with respect to the surface measure $dS$.

Sobolev spaces of fractional index $L^p_s(\Omega)$, $1 < p < \infty$, $s \in \mathbb{R}$, can then be introduced starting from the definitions in [7] (where the Euclidean case is discussed), and using a natural lifting procedure (based on smooth partitions of unity and pull-back; cf. also [8]). In a similar fashion, boundary Besov spaces $B^{p,q}_s(\partial \Omega)$ can be defined for $1 \leq p, q < \infty$, $0 < |s| < 1$. In particular, the trace operator $Tr: L^p_s(\Omega) \rightarrow B^{p-1/p}_s(\partial \Omega)$ is well-defined and bounded whenever $1 < p < \infty$ and $1/p < s$. We also set $L^p_s(\Omega, \mathcal{E}) := L^p_s(\Omega) \otimes \mathcal{E}$, $B^{p,q}_s(\partial \Omega, \mathcal{F}) := B^{p,q}_s(\partial \Omega) \otimes \mathcal{F}$, etc.

For a first-order differential operator $D: \mathcal{E} \rightarrow \mathcal{F}$ with sufficiently smooth coefficients we set

$$H^{s,p}(\Omega; D) := \{ u \in L^p_s(\Omega, \mathcal{E}); \quad Du \in L^p_s(\Omega, \mathcal{F}) \}, \quad (1.7)$$

where all derivatives are considered in a distributional sense, and equip it with the natural graph norm.

If in local coordinates $D = \sum_j a_j(x) \partial_{x_j} + \text{lower order terms}$, where $a_j$ are matrix valued, then we define $\sigma(D): T^*_x M \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F})$, the principal symbol of $D$ as $\sigma(D)(\xi) = \sum_j a_j(x) \xi_j$, if $x \in M$, $\xi \in T^*_x M$. Occasionally, we also write $\sigma(D, \xi)$ in place of $\sigma(D)(\xi)$. In particular, if $D'$ denotes the formal adjoint of $D$, Green’s formula

$$\int_{\Omega} \langle \sigma(D, v) u, v \rangle \ dS = \int_{\Omega} \langle Du, v \rangle - \int_{\Omega} \langle u, D'v \rangle \quad (1.8)$$

holds for sufficiently smooth sections $u, v$. In turn, for each $1 < p < \infty$, $-1 + 1/p < s < 1/p$, Green’s formula allows us to define the bounded operator

$$\sigma(D, v): H^{s,p}(\Omega; D) \rightarrow B^{p-p,0}_{s-1/p}(\partial \Omega, \mathcal{F}) \quad (1.9)$$

by insisting that

$$\langle \sigma(D, v) u, Tr \psi \rangle = \langle Du, \psi \rangle - \langle u, D' \psi \rangle \quad (1.10)$$

110
holds whenever \( u \in H^{r,p}(\Omega; D) \), \( \varphi \in L^{r',p'}(\Omega, \mathcal{F}) \), and \( 1/p + 1/p' = 1 \). Here \( \langle \cdot, \cdot \rangle \) stands for various natural duality pairings. Finally, we set

\[
H^{r,p}_{\sigma}(\Omega; D) := \{ u \in H^{r,p}(\Omega; D); \ \sigma(D, \nu)u = 0 \}.
\] (1.11)

We shall now specialize the discussion to the case when \( E, \mathcal{F} \) are exterior powers of the tangent bundle of \( M \), denoted in the sequel by \( \Lambda^\ell \), and when \( D \) is either the exterior derivative operator \( d_\ell : \Lambda^\ell \to \Lambda^{\ell+1} \), or its formal adjoint \( d_{\ell+1} : \Lambda^{\ell+1} \to \Lambda^\ell \), for \( 0 \leq \ell \leq \text{dim} \, M \). As is well known, \( \sigma(d_\ell) = \land, -\sigma(d_\ell) = \lor \), or the exterior and the interior product of forms, respectively.

The spaces of \( L^s \)-regular harmonic forms with vanishing normal or tangential traces in \( \Omega \) are, respectively,

\[
\mathcal{H}^s_l(\Omega, \Lambda^\ell) := \{ u \in L^s_p(\Omega, \Lambda^\ell); \ \delta u = 0, \ \delta u = 0 \text{ in } \Omega, \ \nabla u = 0 \}.
\] (1.12)

\[
\mathcal{H}^s_{\ell,\ell}(\Omega, \Lambda^\ell) := \{ u \in L^s_p(\Omega, \Lambda^\ell); \ \delta u = 0, \ \delta u = 0 \text{ in } \Omega, \ \nabla u = 0 \}.
\] (1.13)

Next, recall that the Laplace–Beltrami operator \( \Delta \) on \( M \) is given in local coordinates, where the metric tensor reads \( g = \sum g_{jk} \, dx_j \otimes dx_k \), by

\[
\Delta u := \left( \text{det}(g_{jk}) \right)^{-1/2} \sum_j \partial_j \left( \sum_k g^{jk} \left( \text{det}(g_{jk}) \right)^{1/2} \partial_k u \right),
\] (1.14)

where we take \( (g^{jk}) \) to be the matrix inverse of \( (g_{jk}) \). Let \( \nabla_{\text{tan}}, \partial_\nu \) denote, respectively, the tangential gradient and the normal derivative on \( \partial \Omega \). For each \( 1 < q < \infty \), consider the estimate

\[
\| \nabla u \|_{L^q(\partial \Omega)} \leq C \min \{ \| \nabla_{\text{tan}} u \|_{L^q(\partial \Omega)}, \| \partial_\nu u \|_{L^q(\partial \Omega)} \},
\] (1.15)

uniformly for \( u \) satisfying \( \Delta u = 0 \text{ in } \Omega \).

Following the work in the Euclidean context from \([1,10,2]\), it has been shown in \([8]\) that in any Lipschitz domain \( \Omega \), (1.15) always holds for some \( q > 2 \). For each Lipschitz domain \( \Omega \subset M \) it is then natural to set

\[
p_{\Omega} := \text{the Hölder conjugate exponent of the supremum of all } q\text{'s for which (1.15) holds both in } \Omega \text{ and in } M \setminus \overline{\Omega}.
\] (1.16)

Thus, in general, \( 1 \leq p_{\Omega} < 2 \); cf. \([1,2,10]\), for the flat, Euclidean setting and \([8]\) for Lipschitz subdomains of Riemannian manifolds. Also, \( p_{\Omega} = 1 \) if \( \partial \Omega \subset C^1 \) \([4]\), whereas for a Lipschitz polygon or polyhedron \( p_{\Omega} \) can be estimated in terms of the angles involved; cf. \([6]\).

We next define \( \mathcal{H}_{\Omega} \) to be the region of all points in \( \mathbb{R}^2 \) whose coordinates \( (s, 1/p) \) satisfy the following conditions:

\[
0 < \frac{1}{p} < 1, \quad \frac{1}{p} - 1 < s < \frac{1}{2}, \quad \text{and either}
\]

\[
\frac{1}{2} \left( 1 - \frac{1}{p_{\Omega}} \right) < \frac{1}{p} - \frac{s}{2} < \frac{1}{2} \left( 1 + \frac{1}{p_{\Omega}} \right) \quad \text{if } n = 2, \quad \text{or}
\]

\[
\frac{2}{3} \left( 1 - \frac{1}{p_{\Omega}} \right) < \frac{1}{p} - \frac{s}{3} < \frac{2}{3} \left( 1 + \frac{1}{p_{\Omega}} \right) \quad \text{if } n = 3.
\] (1.17)

The above set of inequalities describes the points in the interior of the hexagon in figure 1 below.

To state our main result, we denote by \( b_{1/2}(\Omega) \) the \( \ell \)-th Betti number of \( \Omega \).

**Theorem 1.2** (Hodge decompositions).— Let \( M \) be a compact, boundaryless, smooth, manifold of real dimension \( n \). Assume that \( M \) is equipped with a Riemannian metric tensor whose coefficients are of class \( C^{1,1} \). Also, fix \( \Omega \), an arbitrary Lipschitz subdomain of \( M \). Then, if \( n = 2 \) or \( n = 3 \), for any \( \ell \in \{0, 1, \ldots, n\} \), and any \( s, \ p \) as in (1.17),


111
Figure 1. – The region $\mathcal{H}_{\Omega}$ described in (1.17) for $n = 2, 3$.

\[ L^p_s(\Omega, \Lambda^\ell) = d_{\ell-1}H^{s,p}_0(\Omega; \delta_{\ell+1}) \oplus H^{s,\ell}_0(\Omega; \delta_{\ell+1}) \oplus H^{s,p}_0(\Omega, \Lambda^\ell), \quad (1.18) \]

\[ L^p_s(\Omega, \Lambda^\ell) = d_{\ell-1}H^{s,p}_0(\Omega; \delta_{\ell+1}) \oplus H^{s,\ell}_0(\Omega; \delta_{\ell+1}) \oplus H^{s,p}_0(\Omega, \Lambda^\ell), \quad (1.19) \]

where the direct sums (of closed linear subspaces of $L^p_s(\Omega, \Lambda^\ell)$) are topological. Furthermore, the last summands in (1.18), (1.19) are finite dimensional. Their dimensions are $b_\ell(\Omega)$ and $b_{n-\ell}(\Omega)$, respectively.

As a corollary, we have that Hodge decompositions for vector-fields with components in $L^p_s(\Omega)$, $1 < p < \infty$, always hold for $4/3 \leq p \leq 4$ in arbitrary two dimensional Lipschitz domains, and for $3/2 \leq p \leq 3$ in arbitrary three dimensional Lipschitz domains. The case when $s = 0$, $p = 2$, for which standard variational techniques work, is well known; see, e.g., [3,5].

By means of counterexamples it can be shown that the range of validity for (1.18)–(1.19) described in Theorem 1.2 is asymptotically sharp. Note that when $\partial \Omega \in C^1$ (or even when $v$ has vanishing mean oscillations), $p_\Omega = 1$ so that the region $\mathcal{H}_{\Omega}$ becomes the parallelogram described by $1 < p < \infty$, $-1 + 1/p < s < 1/p$. In the case when $\partial \Omega \in C^\infty$, proving (1.18)–(1.19) can be done by reducing matters to solving certain regular elliptic boundary problems for the Hodge–Laplacian; this point of view has been adopted in [9].

We obtain the decompositions (1.18)–(1.19) in a constructive fashion, relying on singular integral operators; details as well as applications to PDE’s in nonsmooth domains will appear elsewhere.

What the corresponding situation for (1.18)–(1.19) is when $n > 3$ remains an open problem at the moment.

\[ \text{References} \]