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L_p-bounds on curvature, elliptic estimates and rectifiability of singular sets

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Abstract We announce results on rectifiability of singular sets of pointed metric spaces which are pointed Gromov–Hausdorff limits on sequences of Riemannian manifolds, satisfying uniform lower bounds on Ricci curvature and volume, and uniform L_p-bounds on curvature. The rectifiability theorems depend on estimates for $|\mathrm{Hess}_h|_{L_{2p}}$, $(|\nabla \mathrm{Hess}_h \cdot |\mathrm{Hess}_h|^{p-2})_{L_2}$, where $\Delta h = c$, for some constant c. We also observe that (absent any integral bound on curvature) in the Kähler case, given a uniform 2-sided bound on Ricci curvature, the singular set has complex codimension 2. ***To cite this article: J. Cheeger, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 195–198.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Bornes L_p sur la courbure, estimées elliptiques et rectifiabilité d'ensembles singuliers

Résumé Nous annonçons des résultats de rectifiabilité des ensembles singuliers dans les espaces métriques pointés qui sont des limites au sens de Gromov–Hausdorff d'une suite de variétés riemanniennes pour lesquelles on a une borne uniforme sur la courbure de Ricci, le volume, et des bornes uniformes L_p sur la courbure. Les théorèmes de rectifiabilité dépendent d'estimations sur $|\mathrm{Hess}_h|_{L_{2p}}$, $(|\nabla \mathrm{Hess}_h \cdot |\mathrm{Hess}_h|^{p-2})_{L_2}$, où $\Delta h = c$, pour une constante c. Nous remarquons également que dans le cas Kählérien (en l'absence de toute borne intégrale sur la courbure), l'ensemble singulier est de codimension complexe 2. ***Pour citer cet article : J. Cheeger, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 195–198.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

0. Introduction

The proofs of the results contained in this Note are given in [2].

Let $(Y, \underline{y})$, denote a pointed metric space which is pointed Gromov–Hausdorff limit of a sequence of connected Riemannian manifolds, $\{(M_i^n, \underline{m}_i)\}$, such that

$$\mathrm{Ric}_{M_i^n} \geqslant -(n-1), \tag{0.1}$$

$$\mathrm{Vol}\big(B_1(\underline{m}_i)\big) \geqslant v > 0. \tag{0.2}$$

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For the following, see [4]; for additional background, see [3,6,8–10].

Let d denote the distance function on Y. A tangent cone, Y_y, at $y \in Y$, is the pointed Gromov–Hausdorff limit of a sequence, $\{(Y, y, r_i^{-1} d)\}$, where $r_i \to 0$. Since we assume (0.1), (0.2), every tangent cone is a metric cone, $C(W)$, on some length space, W, with diameter $\leqslant \pi$.

The *regular set*, $\mathcal{R}$, is the set of points, $y \in Y$, such that every tangent cone, Y_y, is isometric to $\mathbf{R}^n$.

The *singular set*, $\mathcal{S} = Y \setminus \mathcal{R}$, is the complement of the regular set. Let $\mathcal{S}_i$ denote the set of points, $y \in \mathcal{S}$, such that *no* tangent cone, Y_y splits off a factor, $\mathbf{R}^{i+1}$, isometrically. Then $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-1} = \mathcal{S}$, and in the sense of Hausdorff dimension, $\dim \mathcal{S}_i \leqslant i$. In actuality, $\mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2} = \emptyset$, so $\mathcal{S} = \mathcal{S}_{n-2}$ and $\dim \mathcal{S} \leqslant n-2$.

Let d_{GH} denote Gromov–Hausdorff distance and let 0 denote the origin in $\mathbf{R}^n$. For $\varepsilon > 0$, the *ε-regular set*, $\mathcal{R}_\varepsilon$, is the set of points, $y \in Y$, such that for every tangent cone, Y_y, with vertex, y_∞, we have

$$d_{GH}\big(B_1(y_\infty), B_1(0)\big) < \varepsilon.$$

Let $\overset{\circ}{\mathcal{R}}_\varepsilon$ denote the interior of $\mathcal{R}_\varepsilon$. Given $\varepsilon > 0$, there exists $\delta > 0$, such that $\mathcal{R}_\delta \subset \overset{\circ}{\mathcal{R}}_\varepsilon$. In particular, $\mathcal{R} \subset \overset{\circ}{\mathcal{R}}_\varepsilon$, for all $\varepsilon > 0$. Moreover, there exists $\varepsilon(n) > 0$, such that if $\varepsilon \leqslant \varepsilon(n)$, then $\overset{\circ}{\mathcal{R}}_\varepsilon$ is $\alpha(\varepsilon)$-bi-Hölder equivalent to a smooth connected Riemannian manifold. The exponent, $\alpha(\varepsilon)$, satisfies $\alpha(\varepsilon) \to 1$, as $\varepsilon \to 0$.

If in place of (0.1), we assume,

$$|\mathrm{Ric}_{M_i^n}| \leqslant n-1, \tag{0.3}$$

then there exists $\varepsilon(n) > 0$, such that $\mathcal{R}_\varepsilon = \mathcal{R}$, for $\varepsilon \leqslant \varepsilon(n)$. In that case, $\mathcal{R}$ is a $\mathrm{C}^{1,\alpha}$-Riemannian manifold. If in addition, M_i^n is Einstein, then $\mathcal{R}$ is an Einstein manifold and hence, the metric on $\mathcal{R}$ is C^∞.

In the Kähler case, further information was obtained in [6]. If (0.1), (0.2) hold, and if j is the largest integer such that Y_y splits off a factor, $\mathbf{R}^j$, isometrically, then $j = 2j'$ and $Y_y = \mathbf{C}^{j'} \times C(Z)$, isometrically. In particular, $\mathcal{S}_{2j'+1} \setminus \mathcal{S}_{2j'} = \emptyset$, for all j'. If in addition, (0.3) holds, then for any Y_y, the regular part, $\mathcal{R}(Y_y)$, has a natural Kähler structure.

1. Results on singular sets

THEOREM 1.1. – *Let* (0.2), (0.3) *hold and assume M_i^n is Kähler for all i. Then $\mathcal{S} = \mathcal{S}_{n-4}$. Thus, the singular set has complex codimension* 2.

If the M_i^n are Kähler–Einstein, then $c_1(M_i^n) = \lambda_i \cdot [\omega_i]$, where $c_1(M_i^n)$, $[\omega_i]$, denote the first Chern class and the Kähler class of M_i^n respectively. In this case, (0.3) is equivalent to a bound on the numbers $|\lambda_i|$.

M. Anderson has conjectured that if (0.2), (0.3) hold, then $\dim \mathcal{S} \leqslant n-4$, even if the Kähler condition is dropped; see [1].

Recall that a metric space, W, is called ℓ-rectfiable, if $0 < \mathcal{H}^\ell(W) < \infty$, and there exists a countable collection of subsets, C_j, with $\mathcal{H}^\ell(W \setminus \bigcup_j C_j) = 0$, such that each C_j is bi-Lipschitz equivalent to a subset of $\mathbf{R}^\ell$.

Let P denote a polynomial of degree k, with integer coefficients, in the integral Pontrjagin classes. Let $\widehat{P}$ denote the associated differential character taking values in $\mathbf{R}/\mathbf{Z}$; see [7]. A cone, $C(Z)$, is called *$(n-4k)$-exceptional* if it is of the form $\mathbf{R}^{n-4k} \times C(\mathbf{S}^{4k-1}/\Gamma)$, for some space form, $\mathbf{S}^{4k-1}/\Gamma$, such that $\widehat{P}(\mathbf{S}^{4k-1}/\Gamma) = 0$, for all P.

Let E_{n-4k} denote the set of points, $y \in \mathcal{S}_{n-4k}$, such every tangent cone of the form, $Y_y = \mathbf{R}^{n-4k} \times C(X)$, is $(n-4k)$-exceptional. Put $\mathcal{N}_{n-4k} = \mathcal{S} \setminus E_{n-4k}$.

Denote by $⨍_{B_r(m)} f$, the average of the function, f, over the ball $B_r(m)$.

THEOREM 1.2. – *For all i, let the manifolds, M_i^n, satisfy* (0.1), (0.2). *Assume that for some $1 \leqslant p \leqslant \frac{n}{2}$, curvature tensors, R_i, satisfy for all m*:

$$\limsup_i \fint_{B_1(m_i)} |R_i|^p \leqslant C < \infty. \tag{1.1}$$

(i) *If p is not an integer, then $\mathcal{H}^{n-2p}(\mathcal{S}) = 0$. In particular,* $\dim \mathcal{S} \leqslant n - 2p$, *for all p.*
(ii) *If $p = 1$, then compact subsets of $\mathcal{S}$ are $(n-2)$-rectifiable.*
(iii) *If $p = n/2$, then bounded subsets of $\mathcal{S}$ are finite.*
(iv) *If $p = 2k$ is an even integer, then bounded subsets of $\mathcal{N}_{n-4k}$ are $(n-4k)$-rectifiable.*
(v) *If p is an integer and M_i^n is Kähler, for all i, then bounded subsets of $\mathcal{S}$ are $(n-2p)$-rectifiable.*

For $1 \leqslant p \leqslant 2$, the assertions concerning finiteness of $(n-2p)$-dimensional Hausdorff measure which are implicit in Theorem 1.2, were obtained in [6]. There for $p = 2$, the 2-sided bound, (0.3), was assumed.

In the Kähler case, if (0.3) holds, then the L_2-norm of the curvature can be bounded in terms of a characteristic number involving the first two Chern classes and the Kähler class; see [12].

A result on $\mathcal{H}^{n-2p}$-a.e. uniqueness of tangent cones provides one important step in the proof of Theorem 1.2.

2. L_p-bounds on curvature and elliptic estimates

Below, we sometimes write $f_{m,r}$, for $\fint_{B_r(m)} f$.

Let $h : B_1(m) \to \mathbf{R}$ denote a solution to the equation $\Delta h = c$, for some constant c. Set $V = \sup_{B_{7/8}(m)} |\nabla h|$.

If $\mathrm{Ric}_{M^n} \geqslant -(n-1)$, then Bochner's formula, together with the cutoff function constructed in Theorem 6.33 of [3], leads to the bound,

$$c(n) \fint_{B_{15/16}(m)} \left| |\nabla h|^2 - \left(|\nabla h|^2\right)_{m,1} \right| - \mathrm{Ric}(\nabla h, \nabla h) \geqslant \fint_{B_{7/8}(m)} |\mathrm{Hess}_h|^2. \tag{2.1}$$

Clearly, (2.1) yields an estimate for the normalized L_q-norm of Hess_h, for any $1 \leqslant q \leqslant 2$.

Let $[a]$ denote the greatest integer $\leqslant a$ and let $k \in \mathbf{Z}_+$. Put $p^\dagger = [p - \frac{1}{2}]$, $p \neq \frac{2k+1}{2}$, and $p^\dagger = [p - \frac{1}{2}] - 1$, $p = \frac{2k+1}{2}$.

THEOREM 2.1. – *Assume* $\mathrm{Ric}_{M^n} \geqslant -(n-1)$, *then with* $3/2 < p$. *Let $h : B_1(m) \to \mathbf{R}$ satisfy $\Delta h = c$. Then there exists $\varepsilon(n,p) > 0$, such that*

$$V^{2p+2p^\dagger} \fint_{B_{3/4}(m)} |\mathrm{Hess}_h|^{2p-2p^\dagger} + V^{2p} \sum_{j=p-p^\dagger+1}^{p} \fint_{B_1(m)} |R|^j \geqslant \varepsilon(n,p) \fint_{B_{1/2}(m)} |\mathrm{Hess}_h|^{2p}. \tag{2.2}$$

$$V^{2p+2p^\dagger-2} \fint_{B_{3/4}(m)} |\mathrm{Hess}_h|^{2p-2p^\dagger} + V^{2p-2} \sum_{j=p-p^\dagger+1}^{p} \fint_{B_1(m)} |R|^j \geqslant \varepsilon(n,p) \fint_{B_{1/2}(m)} |\nabla \mathrm{Hess}_h|^2 \cdot |\mathrm{Hess}_h|^{2p-4}. \tag{2.3}$$

Relations (2.2), (2.3), give rise to estimates for limit functions, $h : B_1(y) \to \mathbf{R}$, i.e., $B_1(y) \subset Y$, $h_i : B_1(m_i) \to \mathbf{R}$, $\Delta h_i = c$, $h_i \to h$. Given the assumptions of Theorem 1.2, the estimates $h \,|\, \mathring{\mathcal{R}}_\varepsilon$ turn out not to involve curvature which has concentrated in the limit on $B_1(y) \setminus \mathring{\mathcal{R}}_\varepsilon$. This fact is crucial for the application to rectifiability.

Estimates generalizing those of Theorem 2.1 hold for sections of Riemannian vector bundles with connection, h, satisfying $\Delta h = f$, for arbitrary f.

References

[1] M.T. Anderson, Einstein metrics and metrics with bounds on Ricci curvature, Proceedings of ICM 1 (2) (1994) 443–452.
[2] J. Cheeger, Integral bounds on curvature, estimates on harmonic functions and rectifiability of singular sets, Preprint.
[3] J. Cheeger, T.H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products, Ann. of Math. 144 (1) (1996) 189–237.
[4] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below. I, J. Differential Geom. 46 (1997) 406–480.
[5] J. Cheeger, T.H. Colding, G. Tian, Constraints on singularities under Ricci curvature bounds, C. R. Acad. Sci. Paris, Série I 324 (1997) 645–649.
[6] J. Cheeger, T.H. Colding, G. Tian, On the singularities of spaces with bounded Ricci curvature, GAFA Geom. Funct. Anal. (submitted).
[7] J. Cheeger, J. Simons, Differential characters and geometric invariants, in: Geometry and Topology (College Park, MD, 1983/84), Lecture Notes in Math., Vol. 1167, Springer-Verlag, Berlin, 1985, pp. 50–80.
[8] T.H. Colding, Shape of manifolds with positive Ricci curvature, Invent. Math. 124 (1–3) (1996) 175–191.
[9] T.H. Colding, Large manifolds with positive Ricci curvature, Invent. Math. 124 (1–3) (1996) 193–214.
[10] T.H. Colding, Ricci curvature and volume convergence, Ann. of Math. 145 (3) (1997) 477–501.
[11] L. Simon, Rectifiability of the singular sets of multiplicity 1 minimal surfaces and energy minimizing maps, in: Surveys in Differential Geometry, Vol. II, International Press, 1993, pp. 246–305.
[12] G. Tian, Canonical Metrics in Kähler Geometry, Birkhäuser, 1990.