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Existence and uniqueness of C_0-semigroup in L^∞: a new topological approach

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Abstract A sub-Markov semigroup in L^∞ is in general not strongly continuous with respect to the norm topology. We introduce a new topology on L^∞ for which the usual sub-Markov semigroups in the literature become C_0-semigroups. This is realized by a natural extension of the Phillips theorem about dual semigroup. A simplified Hille–Yosida theorem is furnished. Moreover this new topological approach will allow us to introduce the notion of L^∞-uniqueness of pre-generator. We present several important pre-generators for which we can prove their L^∞-uniqueness. ***To cite this article: L. Wu, Y. Zhang, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 699–704.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Existence et unicité de C_0-semigroupe sur L^∞

Résumé Un semigroupe sous-Markovien sur L^∞ n'est pas, en général, fortement continu par rapport à la topologie de norme. Nons allons introduire une nouvelle topologie sur L^∞ par rapport à laquelle les semigroupes sous-Markoviens dans la litterature deviennent C_0-semigroupes. Ce sera réalisé par une extension naturelle du théorème de Phillips pour semigroupe dual. Un théorème de Hille–Yosida simplifié est fourni. Cette nouvelle topologie nous permet d'introduire la notion d'unicité dans L^∞ d'un prégénérateur. Nous présentons plusieurs important opérateurs dont l'unicité dans L^∞ est établie. ***Pour citer cet article : L. Wu, Y. Zhang, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 699–704.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Le lien entre processus de Markov et semigroupes d'opérateurs est bien connu. Mais comme W. Feller [7, 8], E. Dynkin [4], G. Hunt (parmi tant d'autres) l'ont observé, un semigroupe de noyaux sous-Markoviens n'est, en général, fortement continu ni sur L^∞, ni sur $C_b(E)$, par rapport à la topologie de norme. Différentes théories sortant du cadre des C_0-semigroupes ont été élaborées, mais le théorème de Hille–Yosida devient très compliqué, voir, e.g. [8–10,2] etc. Le but de cette Note est d'introduire une nouvelle topologie sur L^∞ par rapport à (p.r.à.) laquelle les semigroupes usuels deviennent C_0-semigroupes (*voir* Yosida [14] ou

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définition 1 en bas pour la définition précise). Nous sommes inspirés par l'extension suivante du théorème de Phillips,

THÉORÈME F1. – *Soient* $(X, \|\cdot\|)$ *un espace de Banach, et* $(T_t)_{t\geqslant 0}$ *un semigroupe d'opérateurs bornés sur* $(X, \|\cdot\|)$. *Les propriétés suivantes sont équivalentes* :

(a) (T_t) *est un* C_0*-semigroupe sur* $(X, \|\cdot\|)$;

(b) *le semigroupe dual* (T_t^*) *est de classe* C_0 *sur* X^* *p.r.à. la topologie* $\mathcal{C}(X^*, X)$ *de convergence uniforme sur les compacts de* X ;

(c) $\lim_{t\to 0+} T_t^* y = y$ *p.r.à. la topologie faible* $\sigma(X^*, X)$, $\forall y \in X^*$.

Dans ce cas, le dual $\mathcal{L}^*$ *du générateur* $\mathcal{L}$ *de* (T_t) *est exactement le générateur de* (T_t^*) *sur* $(X^*, \mathcal{C}(X^*, X))$, *et* $y \in \mathbb{D}(\mathcal{L}^*)$ *ssi* $\liminf_{t\to 0}(1/t)\|T_t^* y - y\|^* < +\infty$.

Ce résultat nous dit que sur $\mathrm{L}^\infty(E, \mathcal{B}, \mu)$ où E est un espace polonais et μ une mesure σ-finie sur sa tribu borélienne $\mathcal{B}$, une topologie naturelle pour étudier les C_0-semigroupes est $\mathcal{C}(\mathrm{L}^\infty, \mathrm{L}^1)$, désignée simplement par $\mathcal{C}$. Ceci est en plus justifié par le théorème de Hille–Yosida simplifié suivant pour les semigroupes sous-Markoviens de classe C_0,

THÉORÈME F2. – *Un opérateur linéaire* $\mathcal{L}$ *sur* $\mathrm{L}^\infty(E, \mu)$ *est le générateur d'un* C_0*-semigroupe sous-Markovien sur* (L^∞ *p.r.à. la topologie* $\mathcal{C}$), *si et seulement si*

(i) $\mathcal{L}$ *est fermé, densement défini dans* $(\mathrm{L}^\infty(\mu), \sigma(\mathrm{L}^\infty, \mathrm{L}^1))$;

(ii) *si* $\lambda > 0$, $f \in \mathbb{D}(\mathcal{L})$ (*domaine de définition de* $\mathcal{L}$), $f - \lambda\mathcal{L}f = g$, *alors* $\|f\|_\infty \leqslant \|g\|_\infty$ *et* $f \geqslant 0$, μ*-a.e. si* $g \geqslant 0$, μ*-a.e.* ;

(iii) *l'ensemble d'images de* $1 - \lambda\mathcal{L}$ *est* $\mathrm{L}^\infty(\mu)$ *pour tout* $\lambda > 0$.

Cette nouvelle topologie $\mathcal{C}$ sur L^∞ nous permet d'introduire la notion d'unicité d'un pré-générateur dans L^∞ (qui n'existait pas à cause de non-existence de C_0-semigroupe), dont différentes characterisations (surtout celles en problème de Cauchy) peuvent être obtenues. On peut montrer que plusieurs opérateurs importants sont uniques dans L^∞. Voir la version anglaise.

Dans une prochaine Note nous allons étudier les semigroupes de Feller sur l'espace $C_b(E)$.

1. Introduction

Let E be a Polish space equipped with a σ-finite measure μ on its Borel σ-field $\mathcal{B}$. The links between Markov processes and semigroups of operators, established by W. Feller [7,8], E. Dynkin [4], G. Hunt among many others are now well-known. But as observed by those famous works, a semigroup of sub-Markov (resp. sub-Markov and Feller) kernels on $(E, \mathcal{B}, \mu)$ is in general not strongly continuous on $\mathrm{L}^\infty(E, \mu) := \mathrm{L}^\infty(\mu)$ (resp. on the space $C_b(E)$ of real bounded and continuous functions on E) with respect to (w.r.t. in short) its norm-topology. Without that strong continuity, the theory of semigroups becomes quite complicated and the Hille–Yosida theorem becomes very difficult, see, e.g., [7,8,4] and Jefferies [9,10], Cerrai [2] etc. The probabilists and analytists work then in $\mathrm{L}^p(E, \mu)$ for $1 \leqslant p < +\infty$ in which the usual sub-Markov semigroups are C_0-semigroups once μ is excessive. This last theory is so rich both in PDE's theory for elliptic operators and in probability (e.g., Dirichlet forms), that we quote here only [5,6,11] where the reader can find a large number of references.

The main purpose of this Note is to furnish a new topology on L^∞ for which the usual semigroups in the literature become C_0-semigroups, and a simplified Hille–Yosida theorem together with applications to the uniqueness of Cauchy problem.

2. A new variant of Phillips theorem and a Hille–Yosida theorem on L^∞

Let X be a real linear vector space endowed with some locally convex (l.c. in short) Hausdorff topology β, which is assumed to be *sequentially complete.* To emphasize the role of β, we write often X_β instead of X. Following [14], Chapter IX, we introduce

DEFINITION 1. – Given a family of continuous linear operators $(T_t)_{t\geqslant 0}$ on X_β. It is called a C_0-semigroup on X, if

(i) $T_0 = I$; $T_sT_t = T_{s+t}$ for all $s, t \geqslant 0$ (semigroup property);
(ii) $\forall x \in X$, $t \to T_t x$ is continuous from $\mathbb{R}^+$ to X_β (strong continuity);
(iii) for some $\lambda_0 \in \mathbb{R}$, $(e^{-\lambda_0 t}T_t)_{t\geqslant 0}$ are equicontinuous.

Recall that the generator $\mathcal{L}$ of (T_t) is defined as follows: $x \in \mathbb{D}(\mathcal{L})$ iff $\lim_{t\to 0+}(1/t)(T_t x - x)$ exists in X_β, and the last limit is $\mathcal{L}x$.

Let $Y = X_\beta^*$, the topological dual space of X_β. We introduce on Y:

• *the topology of uniform convergence on compact subsets of* X_β, denoted by $\mathcal{C}(Y, X)$. More precisely, for an arbitrary point $y_0 \in Y$, a basis of neighborhoods of y_0 w.r.t. $\mathcal{C}(Y, X)$ is given by $N\ (y_0; \mathbf{K}, \varepsilon) := \{y \in Y;\ \sup_{x\in\mathbf{K}} |\langle x, y\rangle - \langle x, y_0\rangle| < \varepsilon\}$, where $\mathbf{K}$ runs over all compact subsets of X_β and $\varepsilon > 0$.

The following result is a satisfactory variant of the Phillips theorem.

THEOREM 1. – *Let (T_t) be a C_0-semigroup on X_β with generator $\mathcal{L}$, then its dual semigroup (T_t^*) is a C_0-semigroup on $Y = X_\beta^*$ w.r.t. $\mathcal{C}(Y, X)$; the dual operator $\mathcal{L}^*$ of $\mathcal{L}$ is the generator of (T_t^*) on $(Y, \mathcal{C}(Y, X))$, and $\mathbb{D}(\mathcal{L}^*)$ is dense in Y w.r.t. the Mackey topology $\tau(Y, X)$ (the strongest l.c. topology on Y such that its dual is X), and $y \in \mathbb{D}(\mathcal{L}^*)$ iff $\lim_{t\to 0}(1/t)(T_t^* y - y)$ exists w.r.t. the weak topology $\sigma(Y, X)$. In the case where X_β is quasi-complete (i.e., the bounded and closed subsets are complete), $\mathbb{D}(\mathcal{L}^*)$ is dense in Y w.r.t. $\mathcal{C}(Y, X)$.*

The above result takes a very pleasant form in the following special setting (which will be very useful later):

COROLLARY 1. – *Let $(X, \|\cdot\|)$ be a Banach space. Assume that $(T_t)_{t\geqslant 0}$ is a semigroup of bounded operators on $(X, \|\cdot\|)$. Then the following properties are equivalent*:

(a) *(T_t) is a C_0-semigroup on $(X, \|\cdot\|)$;*
(b) *its dual semigroup (T_t^*) is a C_0-semigroup on $(X^*, \mathcal{C}(X^*, X))$;*
(c) *$\lim_{t\to 0+} T_t^* y = y$ w.r.t. $\sigma(X^*, X)$, $\forall y \in X^*$.*

In that case, the dual $\mathcal{L}^*$ of the generator $\mathcal{L}$ of (T_t) is exactly the generator of (T_t^*) on $(X^*, \mathcal{C}(X^*, X))$, and $y \in \mathbb{D}(\mathcal{L}^*)$ iff $\liminf_{t\to 0}(1/t)\|T_t^* y - y\|^* < +\infty$.

Since $L^\infty = (L^1)^*$, the previous result tells us that a natural topology for studying C_0-semigroups on L^∞ is $\mathcal{C}(L^\infty, L^1)$, denoted simply by $\mathcal{C}$, and the usual semigroups of bounded operators on L^∞ in the literature, satisfying condition (c), are C_0-semigroups on $(L^\infty, \mathcal{C})$. We can prove that $(L^\infty, \mathcal{C})$ is complete by means of the Grothendick theorem. Hence the Hille–Yosida theorem (*see* [13], Chapter IX) is available: a densely defined closed operator $\mathcal{L}$ is the generator of some C_0-semigroup on $(L^\infty, \mathcal{C})$, iff for some $\lambda_0 \in \mathbb{R}$, $[1 - (1/n)(\mathcal{L} - \lambda_0)]^{-m}$, $n, m \geqslant 1$ are equicontinuous on $(L^\infty, \mathcal{C})$. This criterion can be largely simplified for characterizing a sub-Markov C_0-semigroup (recalling that a bounded operator P is sub-Markov iff P is nonnegative and $P1 \leqslant 1$, μ-a.e.):

THEOREM 2. – *A linear operator $\mathcal{L}$ on $L^\infty(E, \mu)$ is the generator of some sub-Markov C_0-semigroup on $(L^\infty, \mathcal{C})$, iff*

(i) *$\mathcal{L}$ is a densely defined closed operator in $(L^\infty(\mu), \sigma(L^\infty, L^1))$;*
(ii) *if $\lambda > 0$, $f \in \mathbb{D}(\mathcal{L})$, $f - \lambda\mathcal{L}f = g$, then $\|f\|_\infty \leqslant \|g\|_\infty$ and $f \geqslant 0$, μ-a.e. if $g \geqslant 0$, μ-a.e.*;
(iii) *the range of $1 - \lambda\mathcal{L}$ is $L^\infty(\mu)$ for any $\lambda > 0$.*

3. Uniqueness of pre-generators

Let $A : X \to X$ be a linear operator with domain $\mathcal{D}$ dense in X_β. A is said to be a *pre-generator*, if there exists some C_0-semigroup on X_β such that its generator $\mathcal{L}$ extends A.

A is said to be an *essential generator* in X_β (or X_β-e.gr in short), if A is closable and its closure $\bar{A}$ (w.r.t. β) is the generator of some C_0-semigroup on X_β. In that case we say also that A or $-A$ is X_β-unique. This uniqueness notion was used in Arendt [1], Eberle [5], L. Wu [13] etc. in the Banach space setting.

THEOREM 3. – *Let $Y = X^*$ (the topological dual space of X_β) and A a linear operator on X with domain $\mathcal{D}$ (it is often called the test-functions space), which is assumed to be dense in X_β. Assume that there is a C_0-semigroup (T_t) on X_β such that its generator $\mathcal{L}$ is an extension of A (the existence assumption). Let λ_0 be the constant in Definition 1(iii) for (T_t). Then the following properties are all equivalent:*

(i) *A is a X_β-e.gr (or X_β-unique);*

(ii) *the closure of A in X_β is exactly $\mathcal{L}$;*

(iii) *$A^* = \mathcal{L}^*$ which is the generator of the dual C_0-semigroup (T_t^*) on $(Y, \mathcal{C}(Y, X))$;*

(iv) *for some $\lambda > \lambda_0$, the range $(\lambda - A)(\mathcal{D})$ is dense in X_β;*

(v) *(Liouville property.) for some (or equivalently for all) $\lambda > \lambda_0$, $\mathrm{Ker}(\lambda - A^*) = \{0\}$, i.e., if $y \in \mathbb{D}(A^*)$ satisfies $(\lambda - A^*)y = 0$, then $y = 0$;*

(v′) *for all $\lambda > \lambda_0$ and for all $y \in Y$, the resolvent equation below has a unique solution $z \in \mathbb{D}(A^*)$:*

$$\big(\lambda - A^*\big)z = y \tag{1}$$

and the unique solution is given by $z = ((\lambda - A)^{-1})^* y$;

(vi) *(Uniqueness of strong solutions for the Cauchy problem.) for each $x \in \mathbb{D}(\bar{A})$, there is a unique strong solution $v(t)$ of*

$$\partial_t v(t) = \bar{A} v(t), \quad v(0) = x \tag{2}$$

(i.e., $t \mapsto v(t)$ is differentiable from $\mathbb{R}^+$ to X_β and its derivative $\partial_t v(t)$ coincides with $\bar{A}v(t)$. And the solution is given by $v(t) = T_t x$);

(vii) *(Uniqueness of weak solutions for the dual Cauchy problem.) for every $y_0 \in Y$, the equation*

$$\partial_t u(t) = A^* u(t), \quad u(0) = y_0 \tag{3}$$

has a unique $\mathcal{C}(Y, X)$-continuous weak solution ($u(t) \in Y$); more precisely there is a unique ($t \mapsto u(t) \in Y$) satisfying

(vii.1) *$t \to u(t)$ is continuous from $\mathbb{R}^+$ to $(Y, \mathcal{C}(Y, X))$;*

(vii.2) *for every $x \in \mathcal{D}$, $\langle x, u(t) - y_0\rangle = \int_0^t \langle Ax, u(s)\rangle \,\mathrm{d}s$.*

And the unique solution is given by $u(t) := T_t^* y_0$.

(vii′) *A^* is the generator of some C_0-semigroup on $(Y, \mathcal{C}(Y, X))$.*

If any one of the above properties is true, then

(viii) *there is only one C_0-semigroup such that its generator extends A.*

Moreover if $X_\beta = (X, \|\cdot\|)$ is a Banach space or if $X_\beta = (Z^*, \mathcal{C}(Z^*, Z))$ for some Banach space Z, then (viii) *is equivalent to each of the above properties.*

Remarks. – (a) If A is a second order elliptic operator with domain $\mathrm{C}_0^\infty(D)$ (D being an open domain in $\mathbb{R}^d$), the solutions of (1) and (3) are in the distribution sense of Schwartz, i.e., the weak solution. And equation (2) and (3) are respectively Kolmogorov's backward and forward equation. In that case, A is L^p-unique iff the dual Cauchy problem (3) has a unique weak solution in $\mathrm{L}^{p'}$ where $1/p + 1/p' = 1$ (for $p \in [1, +\infty]$). The uniqueness of L^1-weak solution of (3) is a quite recent object studied in the PDE theory and in differential geometry, but always with some auxilary condition (such as 'locally in H^1').

(b) Without the existence assumption, (vi) is no longer sufficient to (i): one should assume moreover that the solution $v(t)$ of (2) is continuously dependent of the initial data, see [1]. The uniqueness of the pre-generator A implies not only the existence and uniqueness of solution (1)–(3), but also that continuous dependence: which means in the PDE theory that (1), (2) and (3) are *well-posed*.

Example 1. – Let M be a complete connected Riemannian manifold with volume measure $\mathrm{d}x$. The Laplace operator Δ defined on $\mathrm{C}_0^\infty(M)$ is a pre-generator in $\mathrm{L}^p(M, \mathrm{d}x)$ for $1 \leqslant p < +\infty$, and in L^∞ w.r.t. $\mathcal{C}$. It is always L^p-unique for $1 < p < +\infty$ by S.T. Yau and Stricharz. But it is L^1-unique iff M is *stochastically complete*, by E.B. Davies. The sharp sufficient condition for L^∞-uniqueness is: $\mathrm{Ric}(x) \geqslant -C(1 + d(x, o)^2)$, $\forall x \in M$ ($o \in M$ fixed). That is a translation of a result due to P. Li.

4. Uniqueness of unidimensional Sturm–Liouville operators

Consider the unidimensional Sturm–Liouville operator

$$Af(x) = a(x) f'' + b(x) f', \quad \forall f \in \mathrm{C}_0^\infty(x_0, y_0), \tag{4}$$

where $-\infty \leqslant x_0 < y_0 \leqslant +\infty$, the coefficients a, b are assumed to satisfy

$$a(x) > 0 \quad \mathrm{d}x\text{-a.e.}, \quad a(x), b(x) \in \mathrm{L}^\infty_{\mathrm{loc}}(x_0, y_0; \mathrm{d}x), \quad \frac{1}{a(x)} \in \mathrm{L}^1_{\mathrm{loc}}(x_0, y_0; \mathrm{d}x), \quad \frac{b(x)}{a(x)} \in \mathrm{L}^1_{\mathrm{loc}}(x_0, y_0; \mathrm{d}x).$$

Here $\mathrm{L}^\infty_{\mathrm{loc}}(x_0, y_0; \mathrm{d}x)$ (resp. $\mathrm{L}^1_{\mathrm{loc}}(x_0, y_0; \mathrm{d}x)$) denotes the space of functions which are essentially bounded (resp. integrable) w.r.t. Lebesgue measure on any compact sub-interval of (x_0, y_0). Fix a point $c \in (x_0, y_0)$ and let

$$\rho(x) = \frac{1}{a(x)} \exp \int_c^x \frac{b(t)}{a(t)}\, \mathrm{d}t, \qquad \alpha(x) = a(x)\rho(x) = \exp \int_c^x \frac{b(t)}{a(t)}\, \mathrm{d}t.$$

They are respectively the density of the speed measure and the derivative of the scale function of Feller. Through the *Feller classification* and the speed measure and scale function of Feller, Eberle [5] has completely characterized the L^p-uniqueness of A for $1 \leqslant p < \infty$ (*see* also Djellout [3]). They did not discuss the L^∞-uniqueness because of the lack of a natural definition of such uniqueness. To fill this gap, let us recall

DEFINITION 2. – y_0 (resp. x_0) is called a *no entrance boundary* if

$$\int_c^{y_0} \rho(y)\, \mathrm{d}y \left\{ \int_c^y \frac{1}{\alpha(t)}\, \mathrm{d}t \right\} = +\infty; \quad \left(\text{resp. } \int_{x_0}^c \rho(y)\, \mathrm{d}y \left\{ \int_y^c \frac{1}{\alpha(t)}\, \mathrm{d}t \right\} = +\infty\right).$$

THEOREM 4. – *Under the previous assumption on $a(x)$, $b(x)$, the Sturm–Liouville operator A given by* (4) *is $\mathrm{L}^\infty(x_0, y_0; \rho\, \mathrm{d}x)$-unique iff both x_0 and y_0 are no entrance boundary.*

Remarks. – By [13], A is $\mathrm{L}^1(x_0, y_0; \rho\, \mathrm{d}x)$-unique iff a particle starting from any point inside (x_0, y_0) can not reach the boundary $\{x_0, y_0\}$. But the $\mathrm{L}^\infty(\mathrm{d}x)$-uniqueness (equivalent to the uniqueness in $\mathrm{L}^\infty(x_0, y_0; \rho\, \mathrm{d}x)$) means that a particle starting from the boundary $\{x_0, y_0\}$ cannot enter inside (x_0, y_0), by Theorem 4 and the probabilistic meaning of the no entrance boundary [7].

Example 2. – Let $\rho(x)$ be locally lipchitzian continuous and strictly positive over $\mathbb{R}$, and $Af = (\rho f')'/\rho$ for all $f \in \mathbb{D}(A) = \mathrm{C}_0^\infty(\mathbb{R})$. A is always unique in $\mathrm{L}^p(\rho\, \mathrm{d}x)$ for every $p \in (1, +\infty)$, by [5] or [3]. But it is not always unique in $\mathrm{L}^\infty(\rho\, \mathrm{d}x)$. For example, letting $\rho(x) = |x|^{1-s} \mathrm{e}^{-|x|^s}$ for $|x| > 1$ (and $\in \mathrm{C}^\infty(\mathbb{R} \to (0, +\infty))$) where $s \geqslant 0$, then A is $\mathrm{L}^\infty(\rho\, \mathrm{d}x)$-unique iff $s \leqslant 2$.

5. Uniqueness of Schrödinger operators

THEOREM 5. – *Let* $p \in [1, +\infty]$. *Let* $V \in L^p_{loc}(\mathbb{R}^d, dx)$ *such that* V^- *belongs to the Kato class on* $\mathbb{R}^d$. *Then* $(\Delta - V, C_0^\infty(\mathbb{R}^d))$ *is a* $L^p(\mathbb{R}^d, dx)$*-essential generator where* L^∞ *is equipped with the* $\mathcal{C}$*-topology. Moreover the closure of* $(\Delta - V, C_0^\infty(\mathbb{R}^d))$ *is the generator of the Feynman–Kac semigroup*

$$P_t^V f(x) := \mathbb{E}^x f(B_t) \exp\left(-\int_0^t V(B_s)\, ds\right),$$

where (B_t) *is the Brownian motion in* $\mathbb{R}^d$ *with generator* Δ, *starting from* x.

Proof in the case $V \geqslant 0$. – By Theorem 3 (v), it is enough to show that if $h \in L^{p'}(dx)$ satisfies $(1 - \Delta + V)h = 0$ in the distribution sense, then $h = 0$. But by Kato's inequality, we have in the distribution sense that $\Delta|h| \geqslant \mathrm{sgn}(h)\Delta h = \mathrm{sgn}(h)[1 + V]h = (1 + V)|h| \geqslant 0$. Thus h is a subharmonic function. By the Liouville theorem, $|h| = 0$.

In the general case, the proof of this result is based partially on [12].

6. Uniqueness of Nelson's pre-generator

THEOREM 6. – *Let* $\beta : \mathbb{R}^d \to \mathbb{R}^d$ *be in* L^∞_{loc}. *Let* $r = |x|$ (*the Euclidean norm*), $e_r = x/|x|$ *and* $\beta_r := \beta \cdot e_r$. *If there is some increasing function* $h : \mathbb{R}^+ \to (1, +\infty)$ *such that* $\int_0^\infty (1/h(r))\, dr = +\infty$ *and*

$$\beta_r^-(x) \leqslant h(|x|), \quad dx\text{-a.e.}$$

then $(\Delta + \beta \cdot \nabla,\ C_0^\infty(\mathbb{R}^d))$ *is* $(L^\infty(\mathbb{R}^d, dx), \mathcal{C})$*-unique.*

In Example 2 with $\rho(x) = |x|^{1-s} e^{-|x|^s}$ for $|x| > 1$, then $\beta = \rho'(x)/\rho(x)$ satisfies the increasingness condition of this theorem iff $s \leqslant 2$. Then the increasingness condition in this theorem is sharp.

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