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Équations aux dérivées partielles/*Partial Differential Equations*

On the maximum principle for second-order elliptic operators in unbounded domains

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Abstract We are concerned with the maximum principle for second-order elliptic operators of the kind $Lu = a_{ij}(x)u_{x_i x_j} + c(x)u$ in unbounded domains of $\mathbf{R}^n$. Using a geometric condition, already considered by Berestycki, Nirenberg and Varadhan in [2] and a weak boundary Harnack inequality due to Trudinger, Cabré [3] was able to prove the ABP (Alexandroff–Bakelman–Pucci) estimate for a large class of unbounded domains, obtaining as a consequence the maximum principle for general elliptic operators. In this Note we introduce a weak form of the above geometric condition and we show that in the case $c \leqslant 0$ this is enough to obtain the maximum principle for a larger class of domains. ***To cite this article: V. Cafagna, A. Vitolo, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 359–363.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un principe du maximum pour opérateurs elliptiques du second ordre dans des domaines non bornés

Résumé On considère le principe du maximum pour des opérateurs elliptiques du sécond ordre du type $Lu = a_{ij}(x)u_{x_i x_j} + c(x)u$ dans des domaines non bornés de $\mathbf{R}^n$. En utilisant une condition géométrique, déjà considérée par Berestycki, Nirenberg et Varadhan dans [2] et une inégalité de Harnack faible due à Trudinger, Cabré [3] est arrivé à démontrer l'estimation ABP (Alexandroff–Bakelman–Pucci) pour une large classe de domaines non bornés, en obténant le principe du maximum pour des opérateurs elliptiques généraux. Dans cette Note nous introduisons une forme faible de cette condition géométrique et nous démontrons que cela suffit à obtenir le principe du maximum pour une classe plus large de domaines. ***Pour citer cet article : V. Cafagna, A. Vitolo, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 359–363.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

On se donne un espace de fonctions $\mathbf{X}(\Omega)$ défini sur un domaine $\Omega \subset \mathbf{R}^n$.

DÉFINITION 1. – On dit qu'un opérateur L vérifie le principe du maximum dans Ω par rapport à $\mathbf{X}(\Omega)$ si

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(i) $Lw \geqslant 0$ in Ω ;
(ii) $w \in \mathbf{X}(\Omega)$;
(iii) $\limsup_{x \to \partial\Omega} w(x) \leqslant 0$
implique que $w \leqslant 0$ in Ω.

Dans cette Note on considère des opérateurs elliptiques du sécond ordre du type

$$Lu = a_{ij}(x)u_{x_i x_j} + c(x)u,$$

où les coefficients $a_{ij} = a_{ji} \in \mathrm{L}^\infty(\Omega)$ et $c \in \mathrm{L}^\infty(\Omega)$ vérifient :

$$\exists \gamma_0 > 0,\ \Gamma_0 > 0 \text{ t.q. } \gamma_0|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Gamma_0|\xi|^2,\ \forall \xi \in \mathbf{R}^n,$$
$$c \leqslant 0.$$

DÉFINITION 2. – On dit que Ω vérifie la condition **wG** par rapport à un sous-ensemble $H \subset \Omega$ si, pour des constantes positives $\sigma < 1$, $\tau < 1$, on a

$$\forall x \in H,\ \exists B_R \ni x,\ \text{boule de rayon } R,\ \text{t.q. } |B_R \setminus \Omega_x| \geqslant \sigma|B_R|.$$

Par le symbole Ω_x nous désignons la composante de $\Omega \cap B_{R/\tau}$ à laquelle x appartient. Dans le cas $H = \Omega$ on dit simplement que Ω vérifie la condition **wG**.

THÉORÈME A. – *On suppose que Ω vérifie la condition* **wG**. *Alors L vérifie le principe du maximum dans Ω par rapport à $\mathbf{X}(\Omega) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\Omega), w$ bornée supérieurement$\}$.*

COMMENTAIRE 1. – La condition **wG** est une forme faible de la condition géométrique **G** suivante, considérée par Berestycki, Nirenberg et Varadhan dans [2] et par Cabré dans [3] :

Ω vérifie la condition **G** s'il existe de constantes positives $\sigma < 1$, $\tau < 1$ and R_0 t.q.

$$\forall x \in H,\ \exists B_R \ni x,\ R \leqslant R_0,\ \text{t.q. } |B_R \setminus \Omega_x| \geqslant \sigma|B_R|.$$

En utilisant cette condition Cabré a fourni une preuve de l'estimation ABP pour des opérateurs elliptiques du type $Lu = a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u$, $c \leqslant 0$. L'idée à la base de nôtre Théorème A est que la condition faible, même si peut-être insuffisante pour donner l'estimation ABP, suffit néanmoins à prouver le principe du maximum pour une classe plus large d'opérateurs.

THÉORÈME B. – *On se donne un sous-ensemble H de Ω. On suppose que* :
(i) *l'opérateur L vérifie le principe du maximum dans toute composante $\widetilde{\Omega}$ de $\Omega \setminus H$ par rapport à $\mathbf{X}(\widetilde{\Omega}) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\widetilde{\Omega}), w$ bornée supérieurement$\}$* ;
(ii) *Ω vérifie la condition* **wG** *par rapport à H.*
Alors L vérifie le principe du maximum dans tout Ω par rapport à $\mathbf{X}(\Omega) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\Omega),\ w$ bornée supérieurement$\}$.

THÉORÈME C. – *On se donne un sous-ensemble H de Ω. On suppose que* :
(i) *toute composante $\widetilde{\Omega}$ de $\Omega \setminus H$ vérifie la condition* **wG** ;
(ii) *Ω vérifie la condition* **wG** *par rapport à H.*

Alors L vérifie le principe du maximum dans tout sous-domaine $\Omega_0 \subset \Omega$ par rapport à $\mathbf{X}(\Omega_0) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\Omega_0), w$ bornée supérieurement$\}$.

Comme exemples d'application des trois théorèmes on trouve différentes classes de domaines non bornés dans lesquels le principe du maximum est valable pour l'opérateur L :
A. Les cônes dans $\mathbf{R}^n$; le complémentaire d'une spirale logarithmique dans $\mathbf{R}^2$.
B. Le cut-plane $\mathbf{R}^2 \setminus \{(x, y) \in \mathbf{R}^2,\ x \geqslant 0\}$ et domaines analogues en dimensions supérieures.
C. L'extérieur d'une parabole dans le plan ; le complementaire d'une suite de boules de rayons $r \geqslant r_0 > 0$ centrées dans les points à coordonnées entières de la démi-droite positive et domaines analogues en dimensions supérieures.

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1. Introduction

Let $\mathbf{X}(\Omega)$ be a space of functions defined on a domain $\Omega \subset \mathbf{R}^n$.

DEFINITION 1. – We say that an operator L verifies the maximum principle in Ω with respect to $\mathbf{X}(\Omega)$ if
(i) $Lw \geqslant 0$ in Ω;
(ii) $w \in \mathbf{X}(\Omega)$;
(iii) $\limsup_{x\to\partial\Omega} w(x) \leqslant 0$
imply $w \leqslant 0$ in Ω.

In this Note we will be concerned with second-order elliptic operators of the kind

$$Lu = a_{ij}(x)u_{x_ix_j} + c(x)u,$$

where the coefficients $a_{ij} = a_{ji} \in \mathrm{L}^\infty(\Omega)$ and $c \in \mathrm{L}^\infty(\Omega)$ satisfy:

$$\exists \gamma_0 > 0,\ \Gamma_0 > 0 \text{ s.t. } \gamma_0|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \Gamma_0|\xi|^2,\ \forall \xi \in \mathbf{R}^n,$$
$$c \leqslant 0.$$

DEFINITION 2. – We say that Ω satisfies condition **wG** with respect to a subset $H \subset \Omega$ if there exist positive constants $\sigma < 1$, $\tau < 1$ s.t.

$$\forall x \in H,\ \exists B_R \ni x,\ \text{a ball of radius } R,\ \text{s.t. } |B_R \setminus \Omega_x| \geqslant \sigma|B_R|,$$

where by the symbol Ω_x we intend the component of the set $\Omega \cap B_{R/\tau}$ containing x. If $H = \Omega$ we simply say that Ω satisfies condition **wG**.

THEOREM A. – *If Ω satisfies condition* **wG**, *then the operator L satisfies the maximum principle in Ω with respect to* $\mathbf{X}(\Omega) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\Omega),\ w \text{ bounded above}\}$.

Comment 1. – Condition **wG** is a weak form of the following geometric condition **G** considered by Berestycki, Nirenberg and Varadhan in [2] and by Cabré in [3]:
Ω satisfies condition **G** if there exist positive constants $\sigma < 1$, $\tau < 1$ and R_0 s.t.

$$\forall x \in H,\ \exists B_R \ni x,\ R \leqslant R_0,\ \text{s.t. } |B_R \setminus \Omega_x| \geqslant \sigma|B_R|.$$

Using this condition Cabré was able to prove the ABP estimate for elliptic operators of the kind $Lu = a_{ij}(x)u_{x_ix_j} + b_i(x)u_{x_i} + c(x)u$, $c \leqslant 0$. The idea underlying our Theorem A is that the weaker condition, though perhaps not strong enough to imply the ABP estimate, is nonetheless sufficient to obtain the maximum principle, for a narrower class of operators.

THEOREM B (a globalising trick). – *Let H be a subset of Ω. Suppose that*:
(i) *the operator L satisfies the maximum principle in every component $\widetilde{\Omega}$ of $\Omega \setminus H$ with respect to* $\mathbf{X}(\widetilde{\Omega}) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\widetilde{\Omega}),\ w \text{ bounded above}\}$;
(ii) *Ω satisfies condition* **wG** *with respect to H*.
Then L satisfies the maximum principle in the whole of Ω with respect to $\mathbf{X}(\Omega) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\Omega),\ w \text{ bounded above}\}$.

THEOREM C (an iterative trick). – *Let H be a subset of Ω. Suppose that*:
(i) *every component of $\widetilde{\Omega}$ of $\Omega \setminus H$ satisfies condition* **wG**;
(ii) *Ω satisfies condition* **wG** *with respect to H*.
Then L satisfies the maximum principle in every subdomain $\Omega_0 \subset \Omega$ with respect to $\mathbf{X}(\Omega_0) = \{w \in \mathrm{W}^{2,n}_{\mathrm{loc}}(\Omega_0),\ w \text{ bounded above}\}$.

2. Proof of the results

Proof of Theorem A. – By considering components of the set where w might possibly be positive, it is enough to prove the theorem in the case $c=0$. Let $w \in \mathbf{X}(\Omega)$. Set

$$M = \sup_{\Omega} w$$

and consider a sequence $y_k \in \Omega$ s.t.

$$w(y_k) \geqslant M - \frac{1}{k}.$$

By condition **wG** one can choose a ball B_{R_k} s.t.

$$y_k \in B_{R_k} : |B_{R_k} \setminus \Omega_{y_k}| \geqslant \sigma |B_{R_k}|$$

Our scope is to prove that $M \leqslant 0$. This will be obtained by proving that $M \leqslant C/k$, C some constant not depending on k. Define

$$u = M - w.$$

Set

$$s = \liminf_{x \to B_{R_k/\tau} \cap \partial\Omega_{y_k}} u(x)$$

and

$$u_s^-(x) = \begin{cases} \inf\{u(x), s\} & \text{if } x \in \Omega_{y_k}, \\ s & \text{if } x \notin \Omega_{y_k}. \end{cases}$$

Remarking that in Ω, by definition $u \geqslant 0$ and by hypothesis $Lu = -Lw \leqslant 0$, one can apply the Harnack–Trudinger boundary inequality (see [4], Theorem 9.27 and [3], Theorem 2.2):

$$\left(\frac{1}{|B_{R_k}|} \int_{B_{R_k}} (u_s^-)^p \right)^{1/p} \leqslant \frac{C}{k}$$

with p and C depending only on n, λ_0, Λ_0 and τ. Remarking that

$$s = \liminf_{x \to B_{R_k/\tau} \cap \partial\Omega_{y_k}} u \geqslant \liminf_{x \to B_{R_k/\tau} \cap \partial\Omega} u = M + \liminf_{x \to B_{R_k/\tau} \cap \partial\Omega} (-w) = M - \limsup_{x \to B_{R_k/\tau} \cap \partial\Omega} w$$

one has that $s \geqslant M$ and $u_s^- \geqslant M$ in $B_{R_k} \setminus \Omega_{y_k}$. Therefore

$$\frac{C}{k} \geqslant \left(\frac{1}{|B_{R_k}|} \int_{B_{R_k}} (u_s^-)^p \right)^{1/p} \geqslant \left(\frac{1}{|B_{R_k}|} \int_{B_{R_k} \setminus \Omega_{y_k}} (u_s^-)^p \right)^{1/p} \geqslant M \left(\frac{|B_{R_k} \setminus \Omega_{y_k}|}{|B_{R_k}|} \right)^{1/p} \geqslant M\sigma^{1/p}$$

which is what we intended to prove. □

Remark. – We notice that our proof does not work for an operator $Lu = a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u$, $c \leqslant 0$, $b_i(x) \neq 0$, due to the fact that the constant in the Harnack–Trudinger inequality depends on the products $R\|b_i\|_{L^\infty}$.

Proof of Theorem B. – Let $w \in \mathbf{X}(\Omega)$ be such that $Lw \geqslant 0$ in Ω and $w \leqslant 0$ on $\partial\Omega$. To show that $\sup_\Omega w = N \leqslant 0$, we first prove that $N \leqslant K = \sup_H w$. We do this by using the maximum principle in each component of $\Omega \setminus H$. Then we show that $K \leqslant 0$ using the Harnack–Trudinger inequality, as in the proof of Theorem A. □

Proof of Theorem C. – First, we observe that condition **wG** is *hereditary*, in the sense that, if Ω satisfies condition **wG** with respect to a subset, then also every subdomain satisfies the condition with respect to its intersection with the subset. Therefore, by Theorem A, L satisfies the maximum principle in every component of $\Omega_0 \setminus H$, and Ω_0 satisfies condition **wG** with respect to $\Omega_0 \cap H$. An application of Theorem B yields the proof. □

3. Applications

It is worth remarking that there exist unbounded domains in which the Laplace operator does not verify the maximum principle. In fact the radial function $w(r) = -1/r$ is harmonic in the exterior of the unit ball $B_1 \in \mathbf{R}^3$.

We give now some examples of unbounded domains in which the operator L satisfies the maximum principle.

A. As a direct application of Theorem A we notice that L satisfies the maximum principle in every cone $\Omega \in \mathbf{R}^n$. The same result is reported by Berestycki, Caffarelli and Nirenberg in [1] for the operator $\Delta + c$, and extended by Cabré in [3] to general elliptic operators.

Another example is the complement in the plane of a logarithmic spiral defined as $\phi(t) = (t, \log(t))$, $t \geqslant 1$.

B. As an application of Theorem B we notice that L satisfies the maximum principle in the *cut plane* Ω (the complement of the nonnegative x-axis in $\mathbf{R}^2$).

To prove this we consider the subset H of the *cut plane* Ω defined by $H = \{(x, y) \in \mathbf{R}^2 : x > 0,\ y = ax\}$, for $0 < |a| < 1$. H disconnects Ω in two cones. The claim follows from remarking that Ω satisfies condition **wG** with respect to H.

An analogous result can be stated in higher dimension, i.e., L satisfies the maximum principle in the domain defined as the complement of the set $\{x \in \mathbf{R}^n,\ x_i \geqslant 0,\ i = 1, \ldots, n-1, x_n = 0\}$.

The result in the *cut plane* for the Laplace operator is reported by Hayman and Kennedy ([5], Theorem 5.16). They also discuss higher-dimensional analogues with the additional hypothesis that the subharmonic function be nonpositive at infinity.

C. As an application of Theorem C we notice that L satisfies the maximum principle in the domains defined by $\{(x, y) \in \mathbf{R}^2,\ y < |x|^n,\ n > 1\}$.

As a different application, we consider a domain defined as the complement of balls $B(x, r)$ of radii $r \geqslant r_0 > 0$ centered at the points $x = (n, 0) \in \mathbf{R}^2,\ n \in \mathbf{N}$. A higher-dimensional analogue is the complement of balls $B(x, r)$ of radii $r \geqslant r_0 > 0$ centered at the points $x = (x', 0) \in \mathbf{R}^n,\ x' \in \mathbf{N}^{n-1}$.

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