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A Schwarz-type formula for minimal surfaces in Euclidean space $\mathbb{R}^n$

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Abstract This paper introduces a complex representation for minimal surfaces in $\mathbb{R}^n$, based on the Schwarz formula which solves the classical Björling problem for minimal surfaces in $\mathbb{R}^3$. As an application, it is shown that a k-dimensional plane of $\mathbb{R}^n$ is a plane of symmetry of a minimal surface in $\mathbb{R}^n$ provided it intersects the surface orthogonally. A procedure for the construction of minimal surfaces is also described. This procedure introduces minimal surfaces with prescribed geometric properties, starting from real analytic curves in $\mathbb{R}^n$. ***To cite this article: L.J. Alías, P. Mira, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 389–394.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une formule de type Schwarz pour les surfaces minimales de l'espace euclidien $\mathbb{R}^n$

Résumé Cet article présente une représentation complexe des surfaces minimales de $\mathbb{R}^n$, basée sur la formule de Schwarz qui résout le problème classique de Björling pour les surfaces minimales de $\mathbb{R}^3$. Comme application, nous montrons qu'un plan de dimension k de $\mathbb{R}^n$ est un plan de symetrie d'une surface minimale de $\mathbb{R}^n$ s'il lui est orthogonal. Nous décrivons aussi un procédé de construction de surfaces minimales ayant des propriétés géométriques prédéterminées, à partir de courbes analytiques réelles. ***Pour citer cet article : L.J. Alías, P. Mira, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 389–394.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

En 1844 E.G. Björling [3] proposa de déterminer s'il était possible de construire d'une façon explicite une surface minimale de $\mathbb{R}^3$ qui contienne une bande analytique donnée. Cette question, connue sous le nom de *problème de Björling* pour les surfaces minimales, a été résolue par Schwarz [11] en 1890 grâce à une formule en variable complexe. Cette formule se revela un outil très utile aussi bien pour l'étude des propriétés des surfaces minimales que pour l'obtention de nouveaux exemples.

Dans ce travail, nous étendons la formule de Schwarz aux surfaces minimales de $\mathbb{R}^n$, en prouvant :

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THÉORÈME 1. – *Soit* $x:\Omega\subseteq\mathbb{C}\to\mathbb{R}^n$ *une surface minimale. Notons* $z=s+\mathrm{i}t$ *et définissons* $\beta(s)=x(s,0)$, $B(s)=(\partial x/\partial t)(s,0)$ *sur un intervalle réel* $I\subset\Omega$. *Soit* $\Delta\subseteq\Omega$ *un ouvert quelconque de* Ω *contenant* I *et sur lequel on peut définir des extensions holomorphes* $\beta(z)$, $B(z)$ *de* β, B. *Alors, on a pour tout* $z\in\Delta$

$$x(z)=\operatorname{Re}\beta(z)+\operatorname{Im}\int_{s_0}^{z}B(w)\,\mathrm{d}w. \tag{1}$$

Ici, $s_0\in I$ *est fixe mais arbitraire et l'intégrale ne possède pas de périodes imaginaires en* Δ.

Utilisant ce résultat, nous étudions le problème de Björling de $\mathbb{R}^n$, et nous le formulons de la façon suivante : *soit* $\beta:I\to\mathbb{R}^n$ *une courbe regulière analytique, et attribuons analytiquement à chaque* $s\in I$ *un plan* $\Pi(s)$ *de façon que* $\beta'(s)\in\Pi(s)$. *Construire toutes les surfaces minimales* $S\subset\mathbb{R}^n$ *qui contiennent* β *et telles que* $\Pi(s)=T_{\beta(s)}S$ *pour tout* $s\in I$.

Il se trouve que le problème de Björling en $\mathbb{R}^n$ a une seule solution. Ceci est démontré dans le résultat suivant, exprimé en une représentation complexe des surfaces minimales.

THÉORÈME 2. – *Soit* $\beta(s),B(s):I\to\mathbb{R}^n$ *des courbes analytiques telles que* $|\beta'(s)|=|B(s)|>0$ *pour tout* s, *avec des extensions holomorphes* $\beta(z),B(z):\Omega\subseteq\mathbb{C}\to\mathbb{C}^n$. *Ici* Ω *est simplement connexe et contient* I. *Supposons que* $\langle\beta'(z),B(z)\rangle\equiv 0$. *Alors, l'application* $x(z):\Omega\subseteq\mathbb{C}\to\mathbb{R}^n$ *définie par* (1) *est une surface minimale de* $\mathbb{R}^n$ (*eventuellement avec des points de ramification isolés loin de* β).

Celle-ci est la seule surface minimale en $\mathbb{R}^n$ *qui contienne* β *et dont le plan tangent en* $\beta(s)$ *soit engendré par* $\{\beta'(s),B(s)\}$ *pour tout* $s\in I$.

La dernière section présente quelques applications du Théorème 2. Tout d'abord, il est nécessaire d'introduire le concept suivant.

DÉFINITION 3. – Soit $x:M^2\to\mathbb{R}^n$ une surface de $\mathbb{R}^n$ et considérons une sousvariété de dimension k $\Sigma^k\subset\mathbb{R}^n$. On dit que Σ^k coupe orthogonalement cette surface si pour tout point $x(p)$ de $x(M^2)\cap\Sigma^k\neq\emptyset$, aussi bien $T_pM^2\cap T_{x(p)}\Sigma^k$ que $T_pM^2\cap(T_{x(p)}\Sigma^k)^\perp$ sont de dimension 1.

Dans cette situation $x(M^2)\cap\Sigma^k$ est une courbe régulière de $\mathbb{R}^n$. Le résultat suivant est un principe de réflexion général pour les surfaces minimales de $\mathbb{R}^n$ qui généralise celui déjà connu pour $\mathbb{R}^3$.

THÉORÈME 4. – *Tout* k*-plan de* $\mathbb{R}^n$ *qui coupe orthogonalement une surface minimale est un plan de symétrie de cette surface.*

Le Théorème 2 procure aussi une preuve simple du résultat suivant : *Soit* β *une courbe régulière analytique d'une hypersurface analytique* $\Sigma^{n-1}\subset\mathbb{R}^n$, *alors, il existe une seule surface minimale en* $\mathbb{R}^n$ *qui est coupée orthogonalement par* Σ^{n-1} *le long de* $\beta(s)$. *Cette surface peut être construite explicitement.*

La fin de ce travail montre que si β est une courbe régulière analytique de $\mathbb{R}^{2n}$, il existe une seule courbe complexe de $\mathbb{R}^{2n}$ qui passe par β.

Une étude approfondie du problème de Björling de $\mathbb{R}^n$ et ses applications apparaîtra en [1].

1. Presentation

In 1844 E.G. Björling [3] asked whether, given an analytic strip in $\mathbb{R}^3$, it is possible to construct explicitly a minimal surface in $\mathbb{R}^3$ containing that strip in its interior. The question, known as *Björling problem* for minimal surfaces, was solved in 1890 by H.A. Schwarz [11] by means of a complex variable formula which describes minimal surfaces in terms of analytic strips. That formula happened to be quite useful to establish results about minimal surfaces, as well as to construct particular examples of minimal surfaces in $\mathbb{R}^3$ with interesting geometric properties. For instance, an easy application of this Schwarz formula shows that the catenoid is the only minimal surface in $\mathbb{R}^3$ that contains a piece of a circle as a geodesic. One further application gives a construction of Henneberg's minimal surface starting from Neil's parabola. Modern approaches to this classical Björling problem can be found in [5,8].

Besides, the interest of the geometry of minimal surfaces in the Euclidean space $\mathbb{R}^n$ is well known (*see* [4,6,9,10]). Hence, a natural question is to establish an extension to $\mathbb{R}^n$ of the Schwarz formula for minimal surfaces in $\mathbb{R}^3$, and study its applications, particularly its relation with Björling-type problems for minimal surfaces in $\mathbb{R}^n$ and its usefulness in the construction of examples. This will be treated in [1].

The present paper gives a prelude to the work [1]. Here we present a Schwarz-type formula in $\mathbb{R}^n$ (Theorem 1), and show how it can be seen as a complex representation for minimal surfaces (Theorem 2). As an application, we also provide an extension to $\mathbb{R}^n$ of the well-known reflection principles for minimal surfaces in $\mathbb{R}^3$ (Theorem 4). This classical result in $\mathbb{R}^3$ was set by Schwarz, and extended later on to minimal surfaces in $\mathbb{R}^4$ by Eisenhart (*see* also [2]). Another extensions of Schwarz reflection principle can be found in [7] and [9]. Besides, we use Theorem 4 to give a procedure for constructing minimal surfaces in $\mathbb{R}^n$. The resulting surfaces satisfy a certain type of geometric properties, such as to intersect orthogonally a given analytic hypersurface of $\mathbb{R}^n$, or to be a complex curve in an even-dimensional Euclidean space $\mathbb{R}^{2n}$.

2. A representation formula

We begin with some basic preliminaries. The details may be found in [9]. Let us consider a regular surface in $\mathbb{R}^n$, that is, a smooth immersion $x : M^2 \to \mathbb{R}^n$ of a connected 2-dimensional manifold in $\mathbb{R}^n$, being $x = (x_1, \dots, x_n)$. Then M^2 inherits via x a Riemannian metric from the one of $\mathbb{R}^n$. We say that $x : M^2 \to \mathbb{R}^n$ is a *minimal surface* in $\mathbb{R}^n$ if the coordinate functions x_k are harmonic with respect to the induced Riemannian metric of M^2.

Since M^2 is a 2-dimensional Riemannian manifold, it is well known that we can define around any point of M^2 isothermal parameters s, t. Therefore any surface in $\mathbb{R}^n$ can be seen, at least locally, as a conformal map $x : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$. Here by writing $\Omega \subseteq \mathbb{C}$ we are emphasizing that the complex parameter $z = s + \mathrm{i}t$ provides isothermal parameters s, t for Ω.

Let now $x : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ be a minimal surface. Then we can define on Ω the complex function $\phi(z) : \Omega \to \mathbb{C}^n$ given by

$$\phi(z) = 2\frac{\partial x}{\partial z} = \frac{\partial x}{\partial s} - \mathrm{i}\frac{\partial x}{\partial t}.$$

Since the immersion x is minimal, $\phi(z)$ is holomorphic and has no real periods. In that situation the minimal immersion x can be recovered via ϕ by means of the formula

$$x(z) = \operatorname{Re}\int_{\gamma_z} \phi(w)\,\mathrm{d}w, \tag{2}$$

where γ_z is any path in Ω from a fixed base point to $z \in \Omega$.

Once here, let us note that for a minimal surface $x : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ we can suppose without losing generality that $\Omega \cap \mathbb{R}$ is non empty. Hence we may choose a real interval $I \subset \Omega$. Consider the functions $\beta(s) = x(s, 0)$ and $B(s) = (\partial x/\partial t)(s, 0)$, both defined on I. Therefore we have $\phi(s, 0) = \beta'(s) - \mathrm{i}B(s)$. Since β, B are real analytic, they admit holomorphic extensions $\beta(z)$, $B(z)$ on a certain open set $\Delta \subseteq \Omega$ with $I \subset \Delta$. Thus $\phi(z) = \beta'(z) - \mathrm{i}B(z)$ for all $z \in \Delta$, what tells that Eq. (2) can be written on Δ as

$$x(z) = \operatorname{Re}\beta(z) + \operatorname{Im}\int_{s_0}^{z} B(w)\,\mathrm{d}w.$$

Here s_0 is any point of I. The integral in the above formula does not depend on s_0, and it has no imaginary periods, since $\phi(z)$ has no real periods. Summarizing,

THEOREM 1. – *Let $x : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ be a minimal surface. Let $z = s + \mathrm{i}t$ and define $\beta(s) = x(s, 0)$, $B(s) = (\partial x/\partial t)(s, 0)$ on a real interval $I \subset \Omega$. Let $\Delta \subseteq \Omega$ be any open set in Ω containing I over which* *we can define holomorphic extensions* $\beta(z)$, $B(z)$ *of* β, B. *Then it holds for all* $z \in \Delta$

$$x(z) = \operatorname{Re} \beta(z) + \operatorname{Im} \int_{s_0}^{z} B(w)\,\mathrm{d}w. \tag{3}$$

Here $s_0 \in I$ *is fixed but arbitrary and the integral has no imaginary periods in* Δ.

It is possible to give an interesting variation of formula (3). For this we first recall the *cross-product* in $\mathbb{R}^n$, which assigns to any vectors $u_1, \ldots, u_{n-1}$ of $\mathbb{R}^n$ the only vector $u_1 \wedge \cdots \wedge u_{n-1}$ that makes the identity

$$\langle u_1 \wedge \cdots \wedge u_{n-1}, v\rangle = \det(u_1, \ldots, u_{n-1}, v)$$

hold for all $v \in \mathbb{R}^n$.

Consider a minimal surface $x : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ with $z = s + \mathrm{i}t$, and define $\beta(s) = x(s, 0)$ on a real interval $I \subset \Omega$. Let $\eta_1(s), \ldots, \eta_{n-2}(s)$ be real analytic functions defined on I such that for all $s \in I$ they form an orthonormal basis of $(T_{(s,0)}\Omega)^{\perp}$. We will assume that the condition $\partial x/\partial t = \eta_1 \wedge \cdots \wedge \eta_{n-2} \wedge \partial x/\partial s$ is fulfilled, what can be achieved reordering if necessary the η_k's. Then there exists an open set $\Delta \subseteq \Omega$ containing I over which the functions η_k admit holomorphic extensions $\eta_k(z)$. In this manner we find that (3) can be written alternatively as

$$x(z) = \operatorname{Re}\left(\beta(z) - \mathrm{i} \int_{s_0}^{z} \eta_1(w) \wedge \cdots \wedge \eta_{n-2}(w) \wedge \beta'(w)\,\mathrm{d}w\right).$$

This formula generalizes the classical solution to Björling problem in $\mathbb{R}^3$.

Theorem 1 is the key tool in the study of Björling problem for minimal surfaces in $\mathbb{R}^n$. We formulate this problem as follows: *let* $\beta : I \to \mathbb{R}^n$ *be a regular analytic curve, and let us assign analytically to each* $s \in I$ *a plane* $\Pi(s)$ *satisfying* $\beta'(s) \in \Pi(s)$. *Construct all minimal surfaces* $S \subset \mathbb{R}^n$ *containing* β *so that* $\Pi(s) = T_{\beta(s)}S$ *for all* $s \in I$.

The next result shows that there exists a unique solution to Björling problem in $\mathbb{R}^n$, and provides a complex representation for minimal surfaces.

THEOREM 2. – *Let* $\beta(s), B(s) : I \to \mathbb{R}^n$ *be analytic curves such that* $|\beta'(s)| = |B(s)| > 0$ *for all* s, *with holomorphic extensions* $\beta(z), B(z) : \Omega \subseteq \mathbb{C} \to \mathbb{C}^n$. *Here* Ω *is simply connected and contains* I. *Assume that* $\langle \beta'(z), B(z)\rangle \equiv 0$. *Then the mapping* $x(z) : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ *defined by* (3) *is a minimal surface in* $\mathbb{R}^n$ (*possibly with isolated branch points away from* β).

This is the unique minimal surface in $\mathbb{R}^n$ *containing* β *whose tangent plane at* $\beta(s)$ *is spanned by* $\beta'(s)$ *and* $B(s)$ *for all* $s \in I$.

The above uniqueness is to be understood in the following way: two minimal surfaces in $\mathbb{R}^n$ in the conditions of the Theorem must overlap on a non-empty open set.

Proof. – Consider the map $\phi(z) = \beta'(z) - \mathrm{i}B(z)$ from Ω to $\mathbb{C}^n$. Then $\langle \phi(z), \phi(z)\rangle = 0$, where $\langle, \rangle$ stands for the usual inner product in $\mathbb{C}^n$. Applying Lemma 4.3 in [9] we get that $x(z) : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ defined as in (3) is a minimal surface in $\mathbb{R}^n$. This finishes the first claim. Moreover, this minimal surface clearly contains β and its tangent plane at $\beta(s)$ is spanned by $\beta'(s)$ and $B(s)$ for all $s \in I$. This follows readily if we take into account that $\phi(z) = x_s - \mathrm{i}x_t = \beta'(z) - \mathrm{i}B(z)$ for $x(z)$ as in (3).

To check the uniqueness part of the second claim we start with a minimal surface $x(z) : \Omega \subseteq \mathbb{C} \to \mathbb{R}^n$ whose image contains $\beta(I)$. It follows that there is a regular analytic curve $\gamma(s) : I \to \Omega$ satisfying $x(\gamma(s)) = \beta(s)$ for all $s \in I$. Thus it admits a holomorphic extension $\gamma(w)$ to an open set $W \subseteq \Omega$ with $I \subset W$. Choose now $s_0 \in I$. Since $\gamma(s)$ is regular, $\gamma(w)$ defines a biholomorphic mapping $\gamma(w) : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$. Here U is an open subset of W containing a real interval of the form $(s_0 - \delta, s_0 + \delta)$, and V is an open subset of Ω. All of this shows that an open piece of the minimal surface can be expressed as $\psi : U \subseteq \mathbb{C} \to \mathbb{R}^n$, being $\psi(w) = x(\gamma(w))$. Furthermore, ψ satisfies $\psi(s, 0) = \beta(s)$ on U. At this point, Theorem 1 shows that *every* minimal surface in $\mathbb{R}^n$ containing $\beta(s)$ is locally determined by $\beta(s)$ and its tangent plane distribution along $\beta(s)$. Thus the uniqueness follows, simply by recalling that two distinct minimal surfaces cannot overlap on a non-empty open set. □

3. Symmetries of minimal surfaces

We shall say that a k-plane Π^k of $\mathbb{R}^n$ is a *plane of symmetry* of a surface $x : M^2 \to \mathbb{R}^n$ if for all $p \in M^2$ there exists a certain $q \in M^2$ such that $x(p), x(q)$ are symmetric with respect to Π^k, that is, such that $x(q) - x(p) \in (\Pi^k)^\perp$ and $(x(p) + x(q))/2 \in \Pi^k$. Besides, we give the following

DEFINITION 3. – Let $x : M^2 \to \mathbb{R}^n$ be a regular surface in $\mathbb{R}^n$, and consider a k-dimensional embedded submanifold $\Sigma^k \subset \mathbb{R}^n$. We say that Σ^k intersects the surface orthogonally provided at any point $x(p)$ of $x(M^2) \cap \Sigma^k \neq \emptyset$ both $T_pM^2 \cap T_{x(p)}\Sigma^k$ and $T_pM^2 \cap (T_{x(p)}\Sigma^k)^\perp$ are 1-dimensional.

In this situation $x(M^2) \cap \Sigma^k$ is a regular curve in $\mathbb{R}^n$. All these definitions extend naturally the usual ones for surfaces in $\mathbb{R}^3$. The last result of this work is stated in terms of these notions, and generalizes the classical Schwarz reflection principles for minimal surfaces in $\mathbb{R}^3$.

THEOREM 4. – *Every k-plane of $\mathbb{R}^n$ that intersects orthogonally a minimal surface is a plane of symmetry of the surface.*

Remark 1. – We have stated this result as it usually appears in the $n = 3$ case. However, the statement is not true as it stands now, even for the 3-dimensional situation. In fact, if we are given a minimal surface in $\mathbb{R}^n$ which is symmetric with respect to a k-plane Π^k, and such that Π^k intersects the surface orthogonally, we may delete an open piece of the surface away from Π^k. In this manner Π^k would not be anymore a plane of symmetry of the resulting minimal surface, but however Π^k would still intersect it orthogonally.

What the theorem actually states is that if a k-plane Π^k of $\mathbb{R}^n$ intersects orthogonally a minimal surface, then this surface can be *minimally* extended so that it is symmetric with respecto to Π^k.

Proof. – Let $x : M^2 \to \mathbb{R}^n$ be a minimal surface, and suppose that the k-plane Π^k intersects the surface orthogonally along a regular curve $\beta : I \to \mathbb{R}^n$. We may assume that Π^k is the coordinate $e_1, \ldots, e_k$-plane. Besides, let us consider the function $\widehat{x} : M^2 \to \mathbb{R}^n$ defined as

$$\widehat{x}(p) = \big(x_1(p), \ldots, x_k(p), -x_{k+1}(p), \ldots, -x_n(p)\big).$$

In this way $x(p)$ and $\widehat{x}(p)$ are symmetric with respect to Π^k for all $p \in M^2$. It is clear that $\widehat{x} : M^2 \to \mathbb{R}^n$ is a minimal surface. Also note that $\beta(I) \subset S \cap \widehat{S}$, where $S = x(M^2)$ and $\widehat{S} = \widehat{x}(M^2)$.

With these definitions, what the Theorem asserts is that we can extend S as a minimal surface so that the extension S' satisfies $S' = \widehat{S'}$. To prove this, it suffices to show that $S \cap \widehat{S}$ has non-empty interior since, due to the fact that both S and $\widehat{S}$ are real analytic, this would mean that S can be extended as a minimal surface to $S' = S \cup \widehat{S}$.

So, let us verify that there exists an open set of the surface that is contained in $S \cap \widehat{S}$. First, taking into account that $\beta(I)$ is contained in Π^k, we have $\beta(s) = \big(a_1(s), \ldots, a_k(s), 0, \ldots, 0\big)$, where each $a_j(s)$ is real analytic. Since Π^k intersects the surface orthogonally along β, for all $s \in I$ the vector $B(s) = J\beta'(s)$ is tangent to the surface at $\beta(s)$ and lies in $(\Pi^k)^\perp$. Here J stands for the usual almost complex structure of our oriented surface. Therefore $B(s)$ may be written as $B(s) = (0, \ldots, 0, B_{k+1}(s), \ldots, B_n(s))$. At this point Theorem 2 shows that an open piece of the surface is given by

$$\chi(z) = \left(\operatorname{Re} a_1(z), \ldots, \operatorname{Re} a_k(z), \operatorname{Im} \int^z B_{k+1}(w)\,\mathrm{d}w, \ldots, \operatorname{Im} \int^z B_n(w)\,\mathrm{d}w\right),$$

where $a_j(z)$, $B_j(z)$ are holomorphic extensions of $a_j(s)$, $B_j(s)$ respectively, defined on an open set $\Omega \subseteq \mathbb{C}$ containing I. Since $a_j(z), B_j(z)$ take real values when restricted to I, we have over $\Omega \cap \Omega^*$ that $a_j(z) = \overline{a_j(\bar{z})}$ and $B_j(z) = \overline{B_j(\bar{z})}$. Here, as usual, $\Omega^* = \{\bar{z} : z \in \Omega\}$. This tells that for all $z \in \Omega \cap \Omega^*$

1. $\chi_j(z) = \operatorname{Re} a_j(z) = \operatorname{Re} a_j(\bar{z}) = \chi_j(\bar{z})$ for $j = 1, \ldots, k$.
2. $\chi_j(z) = \operatorname{Im} \int^z B_j(w)\,\mathrm{d}w = -\operatorname{Im} \int^{\bar{z}} B_j(w)\,\mathrm{d}w = -\chi_j(\bar{z})$ for $j = k+1, \ldots, n$.

Therefore $S \cap \widehat{S}$ contains an open subset, and the proof is finished. $\square$

To end up we show how the solution to Björling problem obtained in Theorem 2 can be used to construct minimal surfaces with interesting prescribed geometric properties.

First of all, consider a regular analytic curve $\beta(s)$ on a real analytic oriented hypersurface $\Sigma^{n-1} \subset \mathbb{R}^n$. Then Theorem 2 tells that there exists a unique minimal surface in $\mathbb{R}^n$ that is orthogonally intersected by Σ^{n-1} along $\beta(s)$. This minimal surface is the one determined by the data $\{\beta(s), B(s)\}$, where $B(s)$ is defined for all s as the Gauss map of Σ^{n-1} at $\beta(s)$.

In particular, if Σ^{n-1} is the unit sphere $\mathbb{S}^{n-1}$ and $\beta(s)$ is arclengthwise parametrized, the only minimal surface in $\mathbb{R}^n$ that is orthogonally intersected by $\mathbb{S}^{n-1}$ along $\beta(s)$ is given by the data $\{\beta(s), \beta(s)\}$.

This last remark presents a way of constructing minimal surfaces by means of analytic curves. To get further into this direction let us recall the concept of a *complex curve* in $\mathbb{R}^{2n}$ (*see* [6]). This can be defined as a map $x(z) : \Omega \subseteq \mathbb{C} \to \mathbb{R}^{2n}$ of the form

$$x(z) = \left(\operatorname{Re} f_1(z), \operatorname{Im} f_1(z), \ldots, \operatorname{Re} f_n(z), \operatorname{Im} f_n(z)\right),$$

where here each $f_k(z)$ is a holomorphic function on Ω. In this way every non-constant complex curve is a minimal surface.

Let now $\beta(s), B(s) : I \to \mathbb{R}^{2n}$ be as in the conditions of Theorem 2. It is easily shown that the minimal surface determined by $\{\beta(s), B(s)\}$ is a complex curve in $\mathbb{R}^{2n}$ if and only if

$$B(s) = \left(-\beta_2'(s), \beta_1'(s), \ldots, -\beta_{2n}'(s), \beta_{2n-1}'(s)\right). \tag{4}$$

In other words, given a regular analytic curve $\beta(s)$ in $\mathbb{R}^{2n}$, there exists a unique complex curve in $\mathbb{R}^{2n}$ that passes through β. This complex curve may be constructed by means of (3), where $B(s)$ is defined by (4).

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