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Probabilités/*Probability Theory*

Instantaneous liquidity rate, its econometric measurement by volatility feedback

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Abstract In the univariate case we show mathematical existence, in real time and model free, of the instantaneous liquidity rate, which is a measure of the market stability. We give a mathematical formula expressing the instantaneous liquidity rate in terms of self cross volatilities, which, for frequently traded assets, are econometrically measurable. ***To cite this article: P. Malliavin, M.E. Mancino, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 505–508.***

Prime instantanée de liquidité et sa mesure économétrique par les volatilités itérées

Résumé Dans le cas univarié l'existence de la prime instantanée de liquidité est démontrée, ceci indépendamment de toute spécification du modéle ; ce taux donne une mesure quantitative de la stabilité du marché. Nous établissons une formule mathématique donnant la prime instantanée de liquidité en terms de termes de volatilités itérées, qui, pour les valeurs objets d'un nombre élevé de cotations, sont économétriquement mesurables. ***Pour citer cet article : P. Malliavin, M.E. Mancino, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 505–508.***

1. Introduction

In the standard Black–Scholes setting, a lognormal diffusion process is assumed, moreover a constant volatility which does not depend on time and on asset price is assumed. These assumptions are not confirmed in real financial markets. To cope with this empirical evidence, asset price evolution is described by more sophisticated stochastic differential equations. Two main effects are detected in the literature. Asset price volatility depends on the asset price (feedback effects): volatility depends on the asset liquidity and it is decreasing in asset price. This feature has been modeled assuming that the volatility of the asset price $S(t)$ is given by $\sigma S(t)^{\delta}$, $0<\delta<1$ (CEV models, *see* [1]), instead of $\sigma S(t)$ as in the classical Black–Scholes paradigm. The second effect is given by the observation that volatility varies over time with a strong autoregressive component. This effect has been modeled assuming that the volatility follows a stochastic differential equation (continuous time GARCH models, [3]).

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In this Note we assume a feedback effect of asset price on its volatility, i.e., volatility depends on the asset price itself. We will not specify the nature of this dependence, only assuming its existence. We will deduce some econometrical approach to decipher it from the market behaviour. We *rescale*, in real time, the Malliavin calculus, using as local scale, the scale associated to the historical volatility. We prove that the rescaled variation satisfies a first order linear ordinary differential equation which determines the instantaneous feedback effect. Note that the instantaneous liquidity rate is a fundamental quantity in the computation of the so called Greeks. In fact in finance, classical Malliavin calculus considers the propagation of an initial infinitesimal perturbation along the market future evolution; taking expectation of this infinitesimal propagation on a contingent claim payoff, we solve the parabolic equation describing the Greeks of the contingent claim (called delta, gamma, vega). These quantities are useful in an applied perspective to hedge the contingent claim propagation. Actually in order to write these quantities it is sufficient to have the knowledge of the rescaled variation process. Therefore we are able to compute the Greeks in terms of the self cross-volatilities, which are econometrically measurable.

2. Main result

We consider the variation of the price of a single asset during a short period (say, few days). For this reason the actualization by the basic interest rate will be considered as negligeable. Let $x(t)$ be the logarithm of the price process, suppose it follows the SDE associated to the risk free measure

$$\mathrm{d}x_W(t) = a\big(x_W(t)\big)\,\mathrm{d}W(t) - \frac{1}{2}a^2\big(x_W(t)\big)\,\mathrm{d}t,$$

where W is a Brownian motion and where $a(x_W(\cdot))$ is an unknown function describing the feedback of the prices on the volatility. We assume that this function does not depend on t, being considered in a short time interval. Moreover, we assume that a belongs to $\mathrm{C}_b^2(\mathbb{R})$.

The meaning of this function a is the following: in the period considered every trader has fixed his strategy according to the price fluctuations. The resultant of all these individual strategies has the effect to build up, for the period considered, this function a. A statistical observation of the historical process will allow us to determine at each time the value of a together with its first and second derivative.

We emphasize that the risk-free process has an infinitesimal generator fully determined by its volatility. This is not the case of the historical process. However, the historical process and the risk free process have the same volatility. As the volatility of the historical process can be econometrically measured, the infinitesimal generator of the risk-free process can be econometrically measured.

The data of an infinitesimal deformation $x_W(t) + \varepsilon\zeta(t)$ will transform the risk free measure according to the following Girsanov factor:

$$\int \frac{\zeta(t)}{a(t)}\,\mathrm{d}W(t).$$

The rescaled variation is defined as:

$$z(t) = \frac{\zeta(t)}{a(t)}.$$

THEOREM 2.1. – *The rescaled variation is differentiable with respect to t; its logarithmic derivative $\lambda(t)$ will be called the* instantaneous liquidity rate function. *Therefore we have for every $s < t$:*

$$z(t) = \exp\left(\int_s^t \lambda(\tau)\,\mathrm{d}\tau\right) z(s). \tag{1}$$

THEOREM 2.2. – *Denoting by $*$ the Itô contraction, define the following cross volatilities*:

$$\mathrm{d}x * \mathrm{d}x := A\,\mathrm{d}t, \quad \mathrm{d}A * \mathrm{d}x := B\,\mathrm{d}t, \quad \mathrm{d}B * \mathrm{d}x := C\,\mathrm{d}t, \tag{2}$$

then the infinitesimal liquidity rate function λ has the following expression:

$$\lambda = \frac{3}{8}\frac{B^2}{A^3} - \frac{1}{4}\frac{B}{A} - \frac{1}{4}\frac{C}{A^2}. \tag{3}$$

Proof of Theorem 2.1. – The variation equation has the following expression:

$$\mathrm{d}\zeta = a'\zeta\,\mathrm{d}W - a'a\zeta\,\mathrm{d}t.$$

Using Ito calculus we have

$$\mathrm{d}(a) = a'a\,\mathrm{d}W - \frac{1}{2}a'a^2\,\mathrm{d}t + \frac{1}{2}a''a^2\,\mathrm{d}t,$$
$$\mathrm{d}\left(\frac{1}{a}\right) = -\frac{a'}{a}\,\mathrm{d}W + \frac{1}{2}a'\,\mathrm{d}t - \frac{1}{2}a''\,\mathrm{d}t + \frac{1}{a}\left(a'\right)^2\mathrm{d}t.$$

Therefore the rescaled variation has the following Ito differential:

$$\mathrm{d}z = \zeta\left(\left(\frac{a'}{a} - \frac{a'}{a}\right)\mathrm{d}W - \frac{1}{2}\left(a' + a''\right)\mathrm{d}t\right).$$

Then $z(t)$ is a differentiable function of t and

$$\dot{z}(t) = -\frac{1}{2}z(t)\left(a'\left(x_W(t)\right)a\left(x_W(t)\right) + a''\left(x_W(t)\right)a\left(x_W(t)\right)\right) = \lambda(t)z(t),$$

where we have defined

$$\lambda = -\frac{1}{2}\left(a'a + a''a\right). \qquad \square$$

Proof of Theorem 2.2. – Consider the following Ito differentials:

$$\mathrm{d}x = a\,\mathrm{d}W - \frac{1}{2}a^2\,\mathrm{d}t;$$

then $A = a^2$; B is the cross volatility of A and x and has the following expression:

$$B\,\mathrm{d}t = 2aa'\,\mathrm{d}x * \mathrm{d}x = 2a^3a'\,\mathrm{d}t.$$

Therefore we get

$$aa' = \frac{B}{2a^2} = \frac{1}{2}\frac{B}{A}.$$

The cross volatility of B and x is C, and we have

$$2\,\mathrm{d}\left(aa'\right) * \mathrm{d}x = 2\left(aa'' + \left(a'\right)^2\right)a^2\,\mathrm{d}t,$$

on the other side we have

$$2\,\mathrm{d}\big(aa'\big) * \mathrm{d}x = \frac{1}{A^2}\big(A(\mathrm{d}B * \mathrm{d}x) - B(\mathrm{d}A * \mathrm{d}x)\big) = \frac{1}{A^2}\big(AC - B^2\big)\,\mathrm{d}t,$$

it follows

$$2aa'' = \frac{C}{A^2} - \frac{3}{2}\frac{B^2}{A^3}.$$

Finally we obtain

$$\lambda = -\frac{C}{4A^2} - \frac{B}{4A} + \frac{3}{8}\frac{B^2}{A^3}. \qquad \square$$

Remark 2.3. – As x is a logarithm of a price it is dimensionless. The function A appears in the parabolic equation

$$\frac{\partial}{\partial t} + \frac{1}{2}A^2\frac{\partial^2}{\partial x^2}.$$

Therefore A has the dimension of the inverse of time, i.e., $A \simeq T^{-1}$; by Itô calculus $\mathrm{d}_t^I x\,\mathrm{d}_t^I x = A\,\mathrm{d}t$, which means that the Itô differentiation is

$$\mathrm{d}_t^I \simeq T^{-1/2}.$$

We get

$$\begin{aligned}
&B\,\mathrm{d}t = \mathrm{d}_t^I A\,\mathrm{d}_t^I x \simeq T^{-3/2}T^{-1/2} = T^{-2},\\
&\frac{B}{A} \simeq T^{-1},\\
&\frac{B^2}{A^3} \simeq T^{-4}T^3 = T^{-1},\\
&C\,\mathrm{d}t = \mathrm{d}_t^I B\,\mathrm{d}_t^I x \simeq T^{-5/2}T^{-1/2} = T^{-3},\\
&\frac{C}{A^2} \simeq T^{-3}T^2 = T^{-1}.
\end{aligned}$$

Finally λ has the dimension of the inverse of a time, as it should be in the interest rate category.

Remark 2.4. – We want to stress that the sign of $\lambda(t)$ is a pathwise indicator of the market stability. If $\lambda(t) < -c < 0$ for $t \in [0, +\infty]$ the *Delta* will vanish at an exponential rate where t_0 goes to $+\infty$. The market is memoryless (*see* [2]).

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