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Analyse numérique/*Numerical Analysis*

A three field stabilized finite element method for the Stokes equations

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Abstract We consider in this work the boundary value problem for Stokes equations on a two dimensional domain in cases where non-standard boundary conditions are given. We study the cases where pressure and normal or tangential components of the velocity are given in different parts of the boundary and solve the problem with a minimal regularity. We introduce the problem and its variational formulation which is a mixed one. The principal unknowns are the pressure and the vorticity, the multiplier is the velocity. We present the numerical discretization which needs some stabilization. We prove the convergence and the behavior of the a priori error estimates. Some numerical tests are also presented. ***To cite this article: M. Amara et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 603–608.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une méthode d'éléments finis stabilisée à trois champs pour les équations de Stokes

Résumé On propose dans ce travail, une formulation vitesse-tourbillon-pression pour le problème de Stokes bidimensionnel dans lequel on impose des conditions au bord non standard. On s'intèresse plus précisément aux cas où, sur certaines parties du bord, sont données la pression et la composante tangentielle de la vitesse ou bien le tourbillon et la composante normale de la vitesse. En partant d'une formulation mixte variationnelle le problème est résolu avec des hypothèses minimales sur la régularité. Dans cette formulation, les inconnues principales sont la pression et le tourbillon, tandis que la vitesse joue le rôle du multiplicateur. Nous présentons le problème discrétisé associé, pour lequel nous rajoutons un terme de stabilisation. Un résultat de convergence, décrivant le comportement de l'erreur d'approximation a priori, est démontré. Nous terminons par quelques résultats numériques. ***Pour citer cet article : M. Amara et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 603–608.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On considère le problème de Stokes formulé en vitesse-tourbillon-pression, dans un domaine Ω polygonal simplement connexe, de frontière $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$ où Γ_1, Γ_2 et Γ_3 sont trois ouverts disjoints :

$$\begin{cases} \mathbf{curl}\,\omega + \nabla p = \mathbf{f} & \text{dans } \Omega, \\ \omega = \operatorname{curl}\mathbf{u} & \text{dans } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{dans } \Omega, \end{cases} \qquad \begin{cases} \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{u}\cdot\mathbf{t} = 0 & \text{sur } \Gamma_1, \\ \mathbf{u}\cdot\mathbf{t} = 0, \quad p = p_0 & \text{sur } \Gamma_2, \\ \mathbf{u}\cdot\mathbf{n} = 0, \quad \omega = \omega_0 & \text{sur } \Gamma_3, \end{cases}$$

avec $\mathbf{f} \in \mathbb{L}^2(\Omega) = (\mathrm{L}^2(\Omega))^2$ et $p_0 \in \mathrm{L}^2(\Gamma_2)$, $\omega_0 \in \mathrm{L}^2(\Gamma_3)$. On propose la formulation variationnelle mixte suivante :

$$\begin{cases} \text{trouver } \boldsymbol{\sigma} = (\omega, p) \in \mathbb{X} \text{ et } \mathbf{u} \in \mathbb{M} \text{ tel que :} \\ (\omega, \operatorname{curl}\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) = F(\mathbf{v}) & \forall \mathbf{v} \in \mathbb{M}, \\ (\omega, \theta) - (\theta, \operatorname{curl}\mathbf{u}) = 0 & \forall \theta \in \mathrm{L}^2(\Omega), \\ (q, \operatorname{div}\mathbf{u}) = 0 & \forall q \in \mathrm{L}^2(\Omega), \end{cases} \quad \text{i.e.,} \quad \begin{cases} \text{trouver } (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X} \times \mathbb{M} \text{ tel que :} \\ a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}) = 0 & \forall \tau \in \mathbb{X}, \\ b(\boldsymbol{\sigma}, \mathbf{v}) = -F(\mathbf{v}) & \forall \mathbf{v} \in \mathbb{M}, \end{cases} \tag{1}$$

où a, b et F sont définies par (7) et (6) et les espaces $\mathbb{X}$ et $\mathbb{M}$ par (3) et (4). En caractérisant les éléments du noyau $\mathbb{V}$ de b par :

$$\mathbb{V} = \left\{ \boldsymbol{\tau} = (\theta, q) \in \mathbb{X};\ \mathbf{curl}\,\theta + \nabla q = 0 \text{ p.p. dans } \Omega,\ q = 0 \text{ sur } \Gamma_2,\ \theta = 0 \text{ sur } \Gamma_3 \right\},$$

on montre que le problème admet une solution et une seule. On remarquera que tout élément $(\theta, q) \in \mathbb{V}$ vérifie $\Delta\theta = \Delta q = 0$, ce qui permet de définir la trace de θ et de q dans $\mathrm{H}_{00}^{-1/2}(\Gamma_3)$ et $\mathrm{H}_{00}^{-1/2}(\Gamma_2)$ respectivement.

On introduit alors les espaces de dimension finie $\mathbb{X}_h$ et $\mathbb{M}_h$ définis par (9). En notant $\mathbb{V}_h$ le noyau de b dans $\mathbb{X}_h$, on montre que la forme bilinéaire a n'est pas $\mathbb{V}_h$-elliptique. En ce qui concerne l'unicité de la solution du problème discret, on vérifie que celle-ci est assurée pour $\mathbf{u}_h$ et ω_h, mais pas pour p_h. La caractérisation de $\mathbb{V}$ qui nous permet de montrer que le problème continu a une solution unique, n'est plus valable en discret. Par contre, pour tout $(\theta_h, q_h) \in \mathbb{V}_h$, la norme de $\mathbf{curl}\,\theta_h + \nabla q_h$ dans $\mathrm{H}^{-1}(\Omega)$ peut être majorée en fonction des sauts de θ_h et q_h au travers des arêtes internes du maillage. En utilisant cette remarque et en s'inspirant de [1] et [2], on modifie alors la formulation en rajoutant un terme de stabilisation à la forme bilinéaire a. De manière plus précise, on considère le problème :

$$\begin{cases} \text{trouver } (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_h \times \mathbb{M}_h \text{ avec } \boldsymbol{\sigma}_h = (\omega_h, p_h) \text{ tel que :} \\ a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}_h) = \beta_h\, G_h(\boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \mathbb{X}_h, \\ b(\boldsymbol{\sigma}_h, \mathbf{v}_h) = -F(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbb{M}_h, \end{cases}$$

où β_h est un réel positif et

$$a_h(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h) = a(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h) + \beta_h\, A_h(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h).$$

Les expressions exactes de $G_h(\boldsymbol{\tau}_h)$ et $A_h(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h)$ sont données par (12), (14) et (11). Le problème discrétisé admet alors une solution et une seule pour tout $\beta_h > 0$. De plus, dans le cas où $\beta_h = \beta$ est indépendant de h, on établit la majoration d'erreur d'approximation a priori suivante :

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbb{X}} + |\mathbf{u} - \mathbf{u}_h|_{\mathbb{M}} \leqslant C \left\{ \inf_{\boldsymbol{\tau}_h \in \mathbb{X}_h} \left(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbb{X}} + E_c(\boldsymbol{\tau}_h) \right) + \inf_{\mathbf{v}_h \in \mathbb{M}_h} |\mathbf{u} - \mathbf{v}_h|_{\mathbb{M}} \right\},$$

où le terme $E_c(\boldsymbol{\tau}_h)$ est l'erreur de consistance définie par (16). Cette majoration permet alors de conclure que dans le cas où β ne dépend pas de h, la méthode est inconditionnellement convergente. De plus, si la solution vérifie les hypothèses de régularité suivantes :

$$\mathbf{u} \in \mathbb{H}^{k+1}(\Omega) = \mathrm{H}^{k+1}(\Omega) \times \mathrm{H}^{k+1}(\Omega), \quad \omega \in \mathrm{H}^k(\Omega) \quad \text{et} \quad p \in \mathrm{H}^k(\Omega),$$

où k est le degré des éléments finis définissant les espaces $\mathbb{M}_h$ et $\mathbb{X}_h$, on obtient alors une erreur d'approximation optimale i.e. en $\mathrm{O}(h^k)$. Quelques résultats numériques sont présentés pour $k = 1$, dans le Tableau 1.

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1. Introduction

We consider in this Note the stationary Stokes equations with non-standard boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^2$, simply connected and with a polygonal boundary $\Gamma = \partial\Omega$ such that Ω is on one side only of its boundary. We suppose that we have three open and disjoint subsets Γ_1, Γ_2, Γ_3 such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$. We introduce the outward normal vector $\mathbf{n}$ and the tangent vector $\mathbf{t}$ to the boundary Γ. For any 2D vector field $\mathbf{v} = (v_1, v_2)^t$, we use the divergence $\operatorname{div}\mathbf{v} = \partial_1 v_1 + \partial_2 v_2$, the scalar rotational $\operatorname{curl}\mathbf{v} = \partial_1 v_2 - \partial_2 v_1$ and the vector rotational of any scalar field ϕ, $\mathbf{curl}\,\phi = (\partial_2\phi, -\partial_1\phi)^t$. We want to find a 2D velocity field $\mathbf{u}$ and two scalar fields ω and p in Ω such that:

$$\begin{cases} \mathbf{curl}\,\omega + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \omega = \operatorname{curl}\mathbf{u} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \end{cases} \qquad \begin{cases} \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{u}\cdot\mathbf{t} = 0 & \text{on } \Gamma_1, \\ \mathbf{u}\cdot\mathbf{t} = 0, \quad p = p_0 & \text{on } \Gamma_2, \\ \mathbf{u}\cdot\mathbf{n} = 0, \quad \omega = \omega_0 & \text{on } \Gamma_3, \end{cases} \tag{2}$$

where $\mathbf{f} \in \mathbb{L}^2(\Omega) = (\mathrm{L}^2(\Omega))^2$ is the density of external forces. We propose a mixed formulation in which the principal unknows are the pressure and the vorticity and the Lagrange multiplier is the velocity (*see* [4]). In order to insure the coercivity of the form in the discrete problem, we add to the formulation a stabilization term. We prove then that the method is unconditionally convergent. This method can be extended to the 3D case and the extension to the Navier–Stokes case is under way.

2. Functional framework and variational formulation

We consider the Hilbert space:

$$\begin{aligned} \mathbb{X} &= \mathrm{L}^2(\Omega) \times \mathrm{L}^2_0(\Omega) \quad \text{when } |\Gamma_2| = 0, \\ \mathbb{X} &= \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega) \quad \text{when } |\Gamma_2| > 0, \end{aligned} \tag{3}$$

normed by $\|(\theta, q)\|_{\mathbb{X}} = (\|\theta\|^2_{0,\Omega} + \|q\|^2_{0,\Omega})^{1/2}$, where $\mathrm{L}^2_0(\Omega) = \{q \in \mathrm{L}^2(\Omega);\ \int_\Omega q \,\mathrm{d}\Omega = 0\}$.

Let $\mathbb{M}$ be the closed subspace of $\mathbb{H}(\mathrm{div}, \mathrm{curl}; \Omega)$ given by:

$$\mathbb{M} = \{\mathbf{v} \in \mathbb{L}^2(\Omega),\ \operatorname{div}\mathbf{v} \in \mathrm{L}^2(\Omega),\ \operatorname{curl}\mathbf{v} \in \mathrm{L}^2(\Omega);\ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_1\cup\Gamma_3} = \mathbf{v}\cdot\mathbf{t}|_{\Gamma_1\cup\Gamma_2} = 0\}. \tag{4}$$

The boundary condition $\mathbf{v}\cdot\mathbf{n}|_{\Gamma_1\cup\Gamma_3} = 0$ is taken in the weak sense, i.e., $\mathbf{v}\cdot\mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$ and $\langle \mathbf{v}\cdot\mathbf{n}, \mu\rangle = 0\ \forall\mu \in \mathrm{H}^{1/2}_{00}(\Gamma_2)$. The space $\mathbb{M}$ is normed by:

$$\|\mathbf{v}\|_{\mathbb{M}} = \left(|\mathbf{v}|^2_{\mathbb{M}} + \|\mathbf{v}\|^2_{0,\Omega}\right)^{1/2} \quad \text{with } |\mathbf{v}|_{\mathbb{M}} = \left(\|\operatorname{div}\mathbf{v}\|^2_{0,\Omega} + \|\operatorname{curl}\mathbf{v}\|^2_{0,\Omega}\right)^{1/2}.$$

LEMMA 1. – *There exists* $s \in\,]1/2, 1]$ *such that* $\mathbb{M}$ *is continuously imbedded in* $\mathbb{H}^s(\Omega)$. *Then, any* $\mathbf{v} \in \mathbb{M}$ *satisfies* $\mathbf{v}\cdot\mathbf{n} \in \mathrm{L}^2(\Gamma)$ *and* $\mathbf{v}\cdot\mathbf{t} \in \mathrm{L}^2(\Gamma)$ (*see* [5]).

HYPOTHESIS H. – We assume in this Note that if $\mathbf{v} \in \mathbb{M}$ satisfies $\operatorname{div}\mathbf{v} = \operatorname{curl}\mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$.

Remark 1. – For example, if $|\Gamma_1| > 0$ or $|\Gamma_1| = 0$ and Γ_2 has only one connected component, Hypothesis H is verified.

LEMMA 2. – *Under Hypothesis* H, *the semi norm* $|\cdot|_{\mathbb{M}}$ *is equivalent to the norm* $\|\cdot\|_{\mathbb{M}}$ *in* $\mathbb{M}$.

Throughout this section we will denote by $(\cdot,\cdot)$ the scalar product in $\mathrm{L}^2(\Omega)$, by $\langle\cdot,\cdot\rangle$ the duality in the space $\mathbb{M}$ and by $\langle\cdot,\cdot\rangle_{\Gamma_i}$ the scalar product in $\mathrm{L}^2(\Gamma_i)$ for $i = 2, 3$. Integration by parts gives the following variational formulation for problem (2):

$$\begin{cases} \text{Find } \boldsymbol{\sigma} = (\omega, p) \in \mathbb{X} \text{ and } \mathbf{u} \in \mathbb{M} \text{ such that:} \\ (\omega, \operatorname{curl} \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{M}, \\ (\omega, \theta) - (\theta, \operatorname{curl} \mathbf{u}) = 0 \quad \forall \theta \in \mathrm{L}^2(\Omega), \\ (q, \operatorname{div} \mathbf{u}) = 0 \quad \forall q \in \mathrm{L}^2(\Omega), \end{cases} \tag{5}$$

where

$$F \in \mathbb{M}' \text{ is given by } F(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle \omega_0, \mathbf{v} \cdot \mathbf{t} \rangle_{\Gamma_3} - \langle p_0, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_2} \quad \forall \mathbf{v} \in \mathbb{M}. \tag{6}$$

Consider now the bilinear forms $a : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $b : \mathbb{X} \times \mathbb{M} \to \mathbb{R}$ defined for all $\boldsymbol{\sigma} = (\omega, p)$, $\boldsymbol{\tau} = (\theta, q) \in \mathbb{X}$ and $\mathbf{v} \in \mathbb{M}$ by:

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\omega, \theta) \quad \text{and} \quad b(\boldsymbol{\tau}, \mathbf{v}) = -(\theta, \operatorname{curl} \mathbf{v}) + (q, \operatorname{div} \mathbf{v}). \tag{7}$$

We obtain then from (5) a saddle point formulation for problem (2):

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{X} \times \mathbb{M} \text{ such that:} \\ a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \mathbf{u}) = 0 \quad \forall \boldsymbol{\tau} \in \mathbb{X}, \\ b(\boldsymbol{\sigma}, \mathbf{v}) = -F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{M}. \end{cases} \tag{8}$$

Denoting by $\mathbb{V}$ the kernel of b, we have the following characterization:

$$\mathbb{V} = \left\{ \boldsymbol{\tau} = (\theta, q) \in \mathbb{X};\ \mathbf{curl}\, \theta + \nabla q = 0 \text{ a.e. in } \Omega,\ q = 0 \text{ on } \Gamma_2,\ \theta = 0 \text{ on } \Gamma_3 \right\}.$$

Using this characterization, we can prove the next result:

THEOREM 3. – *Let* $\mathbf{f} \in \mathrm{L}^2(\Omega)$, $p_0 \in \mathrm{L}^2(\Gamma)$ *and* $\omega_0 \in \mathrm{L}^2(\Gamma)$. *Under Hypothesis* H, *the saddle point problem* (8) *has a unique solution* $\boldsymbol{\sigma} = (\omega, p) \in \mathbb{X}$ *and* $\mathbf{u} \in \mathbb{M}$ *which verifies* (2).

3. The discrete problem and error estimate

Let $(\mathcal{T}_h)_h$ be a regular family of triangulations of $\overline{\Omega}$. For each triangle K, we denote by h_K its diameter, and by $|K|$ its area and for each edge e, h_e is its length. With each triangulation $\mathcal{T}_h$ we associate the sets:

- $\mathcal{E}_h$ of the internal edges,
- $\mathcal{F}_h^i$ of the edges which belong to the part Γ_i of the boundary ($i = 1, 2, 3$),
- $\mathcal{C}_h = \mathcal{E}_h \cup \mathcal{F}_h^1 \cup \mathcal{F}_h^2 \cup \mathcal{F}_h^3$ of all the edges of $\mathcal{T}_h$.

For every $l \in \mathbb{N}$ and $K \in \mathcal{T}_h$, we denote by $P_l(K)$ the space of the polynomial functions defined on K, with degree less or equal to l and $\mathbb{P}_l(K) = P_l(K) \times P_l(K)$.

We consider an integer $k \geqslant 1$ and we define the following discrete spaces:

$$\begin{aligned} L_h &= \left\{ q_h \in \mathrm{L}^2(\Omega);\ q_{h|K} \in P_{k-1}(K)\ \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{X}_h &= L_h \times L_h \cap \mathbb{X}, \\ \mathbb{M}_h &= \left\{ \mathbf{v}_h \in \left(\mathrm{C}^0\left(\overline{\Omega}\right)\right)^2; \mathbf{v}_{h|K} \in \mathbb{P}_k(K)\ \forall K \in \mathcal{T}_h \right\} \cap \mathbb{M}. \end{aligned} \tag{9}$$

The numerical approximation for the saddle point problem (8) goes as follows:

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_h \times \mathbb{M}_h \text{ such that:} \\ a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in \mathbb{X}_h, \\ b(\boldsymbol{\sigma}_h, \mathbf{v}_h) = -F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{M}_h. \end{cases} \tag{10}$$

The choice of the spaces $\mathbb{X}_h$ and $\mathbb{M}_h$ permits to the bilinear form b to inherit the inf–sup condition satisfied in the continuous case. In order to follow the standard analysis (*see*, for example, [6]), we need to obtain the coercitivity of a on the discrete kernel:

$$\mathbb{V}_h = \left\{ \boldsymbol{\tau}_h \in \mathbb{X}_h;\ b(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0\ \forall \mathbf{v}_h \in \mathbb{M}_h \right\}.$$

However, one can prove that this coercivity is not verified in $\mathbb{V}_h$.

We follow [1] and [2] and introduce a modified version of a that enhances its coercivity. In the continuous case the elements (θ, q) of $\mathbb{V}$ are characterized by $\mathbf{curl}\,\theta + \nabla q = 0$. Here, we are only able to estimate the norm of $\|\mathbf{curl}\,\theta + \nabla q\|_{-1,\Omega}$ in terms of the jumps across the edges of the elements. This is enough for our purpose.

Let $\mathbf{n}_e = \mathbf{n}_e^K$ and $\mathbf{t}_e = \mathbf{t}_e^K$ be the outward normal and tangent vector to the edge e with respect to the triangle K.

DEFINITION 1. – For any $\boldsymbol{\tau}_h = (\theta_h, q_h) \in \mathbb{X}_h$ we define the jump $[\boldsymbol{\tau}_h]_e$ across any edge e of $\mathcal{T}_h$ as follows:

$$\begin{array}{llll}
e = \partial K \cap \partial K', & [\boldsymbol{\tau}_h]_e = [\theta_h]_e\, \mathbf{t}_e^K + [q_h]_e\, \mathbf{n}_e^K, & [\theta_h]_e = \left(\theta_h^K - \theta_h^{K'}\right), & [q_h]_e = \left(q_h^{K'} - q_h^K\right), \\
e \in \mathcal{F}_h^1, & [\boldsymbol{\tau}_h]_e = 0, & [\theta_h]_e = 0, & [q_h]_e = 0, \\
e \in \mathcal{F}_h^2, & [\boldsymbol{\tau}_h]_e = -q_h^K\, \mathbf{n}_e, & [\theta_h]_e = 0, & [q_h]_e = -q_h^K, \\
e \in \mathcal{F}_h^3, & [\boldsymbol{\tau}_h]_e = \theta^K\, \mathbf{t}_e, & [\theta_h]_e = \theta^K, & [q_h]_e = 0.
\end{array} \tag{11}$$

DEFINITION 2. – For any $\boldsymbol{\tau} = (\theta, q) \in \mathbb{X}$, we define $\mathbf{R}\boldsymbol{\tau} = \mathbf{curl}\,\theta + \nabla q$.

We consider the symmetric bilinear form $A_h : \mathbb{X}_h \times \mathbb{X}_h \to \mathbb{R}$ given for all $\boldsymbol{\delta}_h, \boldsymbol{\tau}_h \in \mathbb{X}_h$ by:

$$A_h(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h) = \sum_{K \in \mathcal{T}_h} h_K^2\, (\mathbf{R}\boldsymbol{\delta}_h, \mathbf{R}\boldsymbol{\tau}_h)_K + \sum_{e \in \mathcal{C}_h} h_e \left([\boldsymbol{\delta}_h]_e, [\boldsymbol{\tau}_h]_e\right)_e \tag{12}$$

and the associated semi norm on $\mathbb{X}_h$ defined for all $\boldsymbol{\tau}_h \in \mathbb{X}_h$ by:

$$|\boldsymbol{\tau}_h|_h = \sqrt{A_h(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)} = \left(\sum_{K \in \mathcal{T}_h} h_K^2\, \|\mathbf{R}\boldsymbol{\delta}_h\|_{0,K}^2 + \sum_{e \in \mathcal{C}_h} h_e \left\|[\boldsymbol{\delta}_h]_e\right\|_{0,e}^2\right)^{1/2}. \tag{13}$$

We consider also the linear form $G_h : \mathbb{X}_h \to \mathbb{R}$ given for all $\boldsymbol{\tau}_h = (\theta_h, q_h) \in \mathbb{X}_h$ by:

$$G_h(\boldsymbol{\tau}_h) = \sum_{K \in \mathcal{T}_h} h_K^2\, (\mathbf{f}, \mathbf{R}\boldsymbol{\tau}_h)_K - \sum_{e \in \mathcal{F}_h^2} h_e \left(p_0\, \mathbf{n}_e, [\boldsymbol{\tau}_h]_e\right)_e + \sum_{e \in \mathcal{F}_h^3} h_e \left(\omega_0\, \mathbf{t}_e, [\boldsymbol{\tau}_h]_e\right)_e. \tag{14}$$

Then, for a fixed parameter $\beta_h > 0$, we define the stabilized bilinear form $a_h : \mathbb{X}_h \times \mathbb{X}_h \to \mathbb{R}$ such that for all $\boldsymbol{\delta}_h, \boldsymbol{\tau}_h \in \mathbb{X}_h$, we have,

$$a_h(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h) = a(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h) + \beta_h\, A_h(\boldsymbol{\delta}_h, \boldsymbol{\tau}_h).$$

The discrete problem (10) has to be changed and we will consider the following one:

$$\left\{\begin{array}{ll}
\text{Find } (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{X}_h \times \mathbb{M}_h \text{ with } \boldsymbol{\sigma}_h = (\omega_h, p_h) \text{ such that:} & \\
a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \mathbf{u}_h) = \beta_h\, G_h(\boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \mathbb{X}_h, \\
b(\boldsymbol{\sigma}_h, \mathbf{v}_h) = -F(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbb{M}_h.
\end{array}\right. \tag{15}$$

For any $\beta_h > 0$, this problem has a unique solution.

We want now to estimate the error $E_h = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbb{X}} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbb{M}}$ where $(\boldsymbol{\sigma} = (\omega, p), \mathbf{u})$ is the solution of the continuous problem (8) and $(\boldsymbol{\sigma}_h = (\omega_h, p_h), \mathbf{u}_h)$ is the solution of the discrete one (15). From the standard analysis, we obtain:

THEOREM 4. – *We assume that $\beta_h = \beta$ is independent of h then there exists a constant $C > 0$ dependent on β and independent of h such that for every $\mathbf{v}_h \in \mathbb{M}_h$ and every $\boldsymbol{\tau}_h \in \mathbb{X}_h$:*

$$E_h = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbb{X}} + |\mathbf{u} - \mathbf{u}_h|_{\mathbb{M}} \leqslant C\left\{E_c(\boldsymbol{\tau}_h) + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbb{X}} + |\mathbf{u} - \mathbf{v}_h|_{\mathbb{M}}\right\},$$

where $E_c(\boldsymbol{\tau}_h)$ represents the consistency error given by:

$$E_c(\boldsymbol{\tau}_h) = \sup_{\boldsymbol{\delta}_h \in \mathbb{X}_h} \frac{|A_h(\boldsymbol{\tau}_h, \boldsymbol{\delta}_h) - G_h(\boldsymbol{\delta}_h)|}{|\boldsymbol{\delta}_h|_h}. \tag{16}$$

Table 1. – Numerical results.

Tableau 1. – Résultats numériques.

Mesh	5 × 5	10 × 10	20 × 20	40 × 40	80 × 80	$\beta = 1$	$\beta = 0.1$	$\beta = 0.05$	$\beta = 0.01$
e_ω	2.4	1.08	0.5	0.25	0.12	0.57	0.2	0.2	0.32
e_p	3.95	2.0	1.1	0.56	0.28	1.78	0.55	1.07	4.97
e_{u_1}	2.15	1.1	0.55	0.27	0.13	1.06	0.24	0.23	0.27
e_{u_2}	2.08	1.04	0.52	0.26	0.13	1.06	0.24	0.23	0.29

COROLLARY 1. – *Under the hypothesis of Theorem* 4*, we have*

- $\lim_{h\to 0} E_h = 0$*, i.e., the method is unconditionally convergent*;
- *if* $\mathbf{u} \in \mathbb{H}^{k+1}(\Omega)$, $\omega \in \mathrm{H}^k(\Omega)$ *and* $p \in \mathrm{H}^k(\Omega)$ *then* $E_h = \mathrm{O}(h^k)$*, i.e., the method is optimal in terms of finite elements.*

4. Numerical results

In this section, we present the Bercovier–Engelman test (*see* [5]) which will allow us to compute the error between the exact solution and the numerical approximation calculated in the case $k = 1$. For this test, we take $\Omega =]0,1[^2$ and the boundary condition: $\mathbf{u} = 0$ on Γ. The right-hand side of equation composed of f_1 and f_2 is chosen appropriately such that the exact solution is either:

$$u_1(x,y) = -256y(y-1)(2y-1)x^2(x-1)^2, \quad u_2(x,y) = -u_1(y,x), \quad p(x,y) = (x-0.5)(y-0.5).$$

Besides, in Table 1, we present the absolute error in L^2-norm for the unknowns ω and p and in H^1-norm for (u_1,u_2). These errors are calculated first on structured meshes, for β fixed to 0.1 and then on an unstructured mesh (composed by 1922 triangles and 1022 nodes) for different values of β. The results show that β must be chosen correctly, not too big but not too small neither. We note that when β is fixed to 0.1, the error is divided by two when the mesh size is also divided by two. We obtain numerically an $\mathrm{O}(h)$ behavior for these errors. When the mesh is fixed, we note the sensitivity of the errors to the parameter β, particularly for the pressure. The error for the vorticity and the velocity remains good for β between 0.1 and 0.01 but the error on the pressure increases quickly as β goes away from 0.1. This confirms that p_h cannot be computed when $\beta = 0$.

Remark 2. – The boundary conditions considered in this example are of Dirichlet type. In this context, some classical methods are available and they can be applied and compared to this one. In terms of CPU time, a discretization technique using the mini-element for instance is certainly cheaper than this one, but can not be extended to more physical boundary conditions.

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