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On Laplace–Varadhan's integral lemma

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Abstract In this Note we propose a complement to an integral lemma of Laplace–Varadhan arising in the literature of large deviations. We examine a situation in which the state space may depend on the rate of deviation. We have used this extension to analyze in a new way large deviation principles for the empirical measures on path space associated to interacting particle systems. We prove new large deviation principles on path space for an abstract class of discrete generation, as well as pure jump or McKean–Vlasov interacting particle systems. ***To cite this article: P. Del Moral, T. Zajic, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 693–698.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une note sur le lemme intégral de Laplace–Varadhan

Résumé Dans cette Note nous présentons un complément d'un lemme intégral de Laplace–Varadhan sur les grandes déviations. Nous examinons la situation où l'espace d'état dépend du taux de déviation. Cette extension nous a permis d'analyser par une nouvelle approche des principes de grandes déviations pour les mesures empiriques trajectorielles de systèmes de particules en interaction. Nous démontrons de nouveaux principes de grandes déviations trajectoriels pour des modèles particulaires à temps discret puis à sauts purs et diffusifs de type McKean–Vlasov. ***Pour citer cet article : P. Del Moral, T. Zajic, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 693–698.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Le lemme intégral de Laplace–Varadhan fait partie des outils classiques permettant de tranferer des principes de grandes déviations (PGD) satisfaits par une suite de mesures de probabilités $\{\overline{\mathbb{Q}}^N;\ N \geqslant 1\}$ à une nouvelle séquence de mesures $\{\overline{\mathbb{P}}^N;\ N \geqslant 1\}$. On rappelle qu'une suite de mesures de probabilités $\{\overline{\mathbb{Q}}^N;\ N \geqslant 1\}$ sur un espace métrique (M, d) satisfait un (PGD) lorsque les bornes supérieures et inférieures décrites dans (2) sont satisfaites. On précise que la fonction de taux est bonne lorsque ses lignes de niveau sont compactes.

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La relation intégrale de Laplace–Varadhan s'exprime plus précisément par la donnée de mesures absolument continues $\overline{\mathbb{P}}^N \sim \overline{\mathbb{Q}}^N$ sur un espace métrique (M, d) et liées par la formule (3). Lorsque la fonction $C : M \to \mathbb{R}$ est continue et bornée ce lemme peut s'énoncer de la façon suivante :

Si la suite de mesures $\{\overline{\mathbb{Q}}^N;\ N \geqslant 1\}$ satisfait un PGD pour une bonne fonction de taux H alors la suite de mesures $\{\overline{\mathbb{P}}^N;\ N \geqslant 1\}$ satisfait un PGD avec une bonne fonction de taux $H - C$. Pour plus d'information concernant les grandes déviations et d'autres méthodes de tranfert de PGD tel le lemme de contraction nous renvoyons le lecteur à l'excellent ouvrage de Dembo et Zeitouni [2].

Les mesures sous-jacentes sont très fréquemment définies sous forme de mesures images

$$\overline{\mathbb{P}}^N = \mathbb{P}^N \circ \pi_N^{-1} \quad \text{et} \quad \overline{\mathbb{Q}}^N = \mathbb{Q}^N \circ \pi_N^{-1}$$

par des transformation mesurable $\pi_N : \Omega^N \to M$ et de mesures de probabilités $(\mathbb{P}^N, \mathbb{Q}^N)$ définies sur des espaces Ω^N pouvant dépendre du paramètre N.

Dans [1] nous présentons des exemples liés aux systèmes de particules en interaction. Dans un tel cadre d'applications l'espace Ω^N est un espace produit dans lequel vivent les N trajectoires $\omega = (\omega^1, \ldots, \omega^N)$ des particules et $\pi_N(\omega) = \frac{1}{N}\sum_{i=1}^N \delta_{\omega^i}$ representent les mesures empiriques dans l'espace M des mesures de probabilités sur les trajectoires. Lorsque les mesures $\mathbb{P}^N$ et $\mathbb{Q}^N$ sont absolument continues et les dérivées de Radon–Nykodim de la forme

$$\frac{\mathrm{d}\mathbb{P}^N}{\mathrm{d}\mathbb{Q}^N}(x) = \exp\bigl(NC\bigl(\pi_N(x)\bigr)\bigr) \tag{1}$$

pour $\mathbb{Q}^N$-presque tous les $x \in \Omega^N$ alors $\overline{\mathbb{P}}^N$ et $\overline{\mathbb{Q}}^N$ sont aussi absolument continues et leurs dérivées satisfont pour $\overline{\mathbb{Q}}^N$-presque tout $u \in M$ la formule

$$\frac{\mathrm{d}\overline{\mathbb{P}}^N}{\mathrm{d}\overline{\mathbb{Q}}^N}(u) = \exp\bigl(NC(u)\bigr).$$

Dans cette situation on peut appliquer le lemme de Laplace–Varadhan dès que la fonction C est continue bornée. Dans cette Note nous proposons une nouvelle stratégie permettant de relaxer l'hypothèse de représentation analytique (1) des dérivées de Radon–Nykodim de $\mathbb{P}^N$ par rapport à $\mathbb{Q}^N$. Notre démarche consiste à remplacer $\mathbb{P}^N$ et $\mathbb{Q}^N$ par des collections de mesures $\mathbb{P}^N_{\alpha,m}$ et $\mathbb{Q}^N_m$ indexées respectivement par un paramètre réel $\alpha \in \mathbb{R}$ et par un paramètre $m \in M$. Plutôt que (1) nous supposerons que $\mathbb{P}^N_{1,m} = \mathbb{P}^N_1$ est indépendante de m et $\mathbb{P}^N_{\alpha,m} \sim \mathbb{Q}^N_m$ pour chaque couple de paramètres $(\alpha, m) \in (\mathbb{R} \times M)$ et pour $\mathbb{Q}^N_m$-presque tous les $x \in \Omega^N$ les dérivées sont liées selon la formule

$$\frac{\mathrm{d}\mathbb{P}^N_{\alpha,m}}{\mathrm{d}\mathbb{Q}^N_m}(x) = \exp\bigl(N\bigl[\alpha S_N(x, m) + C_\alpha\bigl(\pi_N(x), m\bigr)\bigr]\bigr).$$

Comme précédemment on définit les mesures images

$$\overline{\mathbb{P}}^N_{\alpha,m} = \mathbb{P}^N_{\alpha,m} \circ \pi_N^{-1} \quad \text{et} \quad \overline{\mathbb{Q}}^N_m = \mathbb{Q}^N_m \circ \pi_N^{-1}.$$

Notre principal résultat peut être énoncé de la façon suivante : supposons que la suite de mesures de probabilités $\{\overline{\mathbb{Q}}^N_m;\ N \geqslant 1\}$ satisfait pour chaque $m \in M$ un PGD de bonne fonction de taux $H_m : M \to [0, \infty]$. Supposons de plus que les applications $\{C_\alpha(\cdot, m);\ \alpha \in \mathbb{R}\}$ sont continues en chaque point m avec $C_\alpha(m, m) = 0$ et satisfont la condition de moment exponentiel

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_{\Omega^N} \exp \gamma N\bigl[S_N(x, m) + C_1\bigl(\pi_N(x), m\bigr)\bigr]\, \mathrm{d}\mathbb{Q}^N_m(x) < \infty$$

pour un couple de paramètres $(m, \gamma) \in M \times (1, \infty)$. Dans ces conditions la suite $\{\overline{\mathbb{P}}^N_1;\ N \geqslant 1\}$ satisfait un PGD de bonne fonction de taux

$$I : m \in M \to I(m) = H_m(m) \in [0, \infty].$$

Laplace–Varadhan's integral lemma is a powerful change of reference probability technique which allows to transfert a large deviations principle (LDP) from a sequence of probability measures $\{\overline{\mathbb{Q}}^N;\ N \geqslant 1\}$ to another $\{\overline{\mathbb{P}}^N;\ N \geqslant 1\}$. A collection of probability measures $\{\overline{\mathbb{Q}}^N;\ N \geqslant 1\}$ on some metric space (M,d) is said to satisfy a LDP with rate function H if there exists a lower semi-continuous function $H : M \to [0,\infty]$ such that for each open set $\mathcal{A}$ and for each closed subset $\mathcal{B}$

$$-\inf_{m\in\mathcal{A}} H(m) \leqslant \liminf_{N\to\infty} \frac{1}{N}\log\overline{\mathbb{Q}}^N(\mathcal{A}) \quad \text{and} \quad \limsup_{N\to\infty}\frac{1}{N}\log\overline{\mathbb{Q}}^N(\mathcal{B}) \leqslant -\inf_{m\in\mathcal{B}} H(m). \tag{2}$$

The rate function is good if it has compact level sets, that is for each $h \in [0,\infty)$ the level set $\{m \in M;\ H(m) \leqslant h\}$ is compact.

Consider a pair $(\overline{\mathbb{P}}^N, \overline{\mathbb{Q}}^N)$ of absolutely continuous measures on some metric space (M,d) such that

$$\frac{d\overline{\mathbb{P}}^N}{d\overline{\mathbb{Q}}^N}(u) = \exp\bigl(NC(u)\bigr) \quad \overline{\mathbb{Q}}^N\text{-a.e.} \tag{3}$$

for some measurable mapping $C : M \to \mathbb{R}$. When C is bounded continuous the Laplace–Varadhan lemma can be stated as follows: if the sequence $\{\overline{\mathbb{Q}}^N;\ N \geqslant 1\}$ satisfies a LDP with good rate function H then $\{\overline{\mathbb{P}}^N;\ N \geqslant 1\}$ satisfies a LDP with good rate function $H - C$. For more information related to large deviations and ways to transfer LDPs, such as the contraction lemma, we refer the reader to the seminal book of Dembo and Zeitouni [2].

The above sequences are frequently defined in terms of the image measures

$$\overline{\mathbb{P}}^N = \mathbb{P}^N \circ \pi_N^{-1} \quad \text{and} \quad \overline{\mathbb{Q}}^N = \mathbb{Q}^N \circ \pi_N^{-1}$$

for some probabilities $\mathbb{P}^N$ and $\mathbb{Q}^N$ on some measurable space Ω^N which may depend on N and for some measurable mapping $\pi_N : \Omega^N \to M$. It is clear that if $\mathbb{P}^N$ and $\mathbb{Q}^N$ are absolutely continuous and for $\mathbb{Q}^N$-almost every $x \in \Omega^N$

$$\frac{d\mathbb{P}^N}{d\mathbb{Q}^N}(x) = \exp\bigl(NC\bigl(\pi_N(x)\bigr)\bigr) \tag{4}$$

then the probability images $\overline{\mathbb{P}}^N$ and $\overline{\mathbb{Q}}^N$ are absolutely continuous and their Radon–Nykodim derivative satisfies (3) and Laplace–Varadhan's lemma applies as soon as C is a bounded continuous mapping. In this Note we propose a strategy to relax the analytic representation (4) of the Radon–Nykodim derivative of $\mathbb{P}^N$ with respect to $\mathbb{Q}^N$.

Our approach consists in replacing $\mathbb{P}^N$ and $\mathbb{Q}^N$ by a pair of sequences $\mathbb{P}^N_{\alpha,m}$ and $\mathbb{Q}^N_m$ indexed respectively by a real number $\alpha \in \mathbb{R}$ and a parameter $m \in M$. Instead of (4) we suppose that for any index pair $(\alpha,m) \in (\mathbb{R}\times M)$, $\mathbb{P}^N_{1,m} = \mathbb{P}^N_1$ doesn't depend on m and $\mathbb{P}^N_{\alpha,m} \sim \mathbb{Q}^N_m$ and for $\mathbb{Q}^N_m$-almost every $x \in \Omega^N$

$$\frac{d\mathbb{P}^N_{\alpha,m}}{d\mathbb{Q}^N_m}(x) = \exp\bigl(N\bigl[\alpha S_N(x,m) + C_\alpha\bigl(\pi_N(x),m\bigr)\bigr]\bigr) \tag{5}$$

for some measurable functions $S_N : \Omega^N \times M \to \mathbb{R}$ and $C_\alpha : M \times M \to \mathbb{R}$. For any $(\alpha,m) \in (\mathbb{R}\times M)$ we define the image measures

$$\overline{\mathbb{P}}^N_{\alpha,m} = \mathbb{P}^N_{\alpha,m} \circ \pi_N^{-1} \quad \text{and} \quad \overline{\mathbb{Q}}^N_m = \mathbb{Q}^N_m \circ \pi_N^{-1}.$$

We are now in a position to state our main result.

INTEGRAL LEMMA. – *Suppose the sequence of probability measures $\{\overline{\mathbb{Q}}^N_m;\ N \geqslant 1\}$ satisfies an LDP with good rate function $H_m : M \to [0,\infty]$, for each $m \in M$. Also assume that the mappings $\{C_\alpha(\cdot,m);\ \alpha \in \mathbb{R}\}$, are continuous at each m, $C_\alpha(m,m) = 0$ and the following exponential moment condition holds*

$$\limsup_{N\to\infty}\frac{1}{N}\log\int_{\Omega^N}\exp\gamma N\bigl[S_N(x,m) + C_1\bigl(\pi_N(x),m\bigr)\bigr]\,d\mathbb{Q}^N_m(x) < \infty \tag{6}$$

for some $(m,\gamma)\in M\times(1,\infty)$. *Then* $\{\overline{\mathbb{P}}_1^N;\ N\geqslant 1\}$ *satisfies an LDP with good rate function*

$$I : m\in M\to I(m)=H_m(m)\in[0,\infty].$$

We remark that (6) holds when the mappings $C_1(\cdot,m)$ and $C_\gamma(\cdot,m)$ are bounded for some $(m,\gamma)\in M\times(1,\infty)$. Notice also that when $S_N=0$ are the null mappings then we have for any $u\in M$ and $N\geqslant 1$

$$\frac{\mathrm{d}\overline{\mathbb{P}}_1^N}{\mathrm{d}\overline{\mathbb{Q}}_m^N}(u)=\exp N\big[C_1(u,m)\big].$$

If $C_1(\cdot,m)$ is continuous and (6) holds, Laplace–Varadhan's integral lemma says that the family of probability measures $\overline{\mathbb{P}}_1^N=\mathbb{P}_1^N\circ\pi_N^{-1}$ satisfies the large deviation principle with rate function $I : M\to[0,\infty]$ given for any $u\in M$ by $I(u)=H_m(u)-C_1(u,m)$. In case $C_1(\cdot,m)$ is continuous and (6) holds for all m, since $C_1(u,u)=0$ we have that $I(u)=H_u(u)$. Note that under our assumptions we only require the continuity of $C_1(\cdot,m)$ at the point m and that (6) holds for one m. Therefore, even in the case that $S_N=0$, our result does not follow from Laplace–Varadhan's integral lemma. The proof of the above integral lemma is described in detail in [1]. The exponential tightness property and the two key estimates needed to transfer the large deviations principles are stated in the next proposition.

PROPOSITION. – *Under the assumptions of the above integral lemma the sequence of probability measures* $\overline{\mathbb{P}}_1^N$ *on* M *is exponentially tight. For any Borel subset* $A\subset M$ *and for any* $1/n+1/n'=1$, $1<n,n'<\infty$, *and* $m\in M$ *we have*

$$\overline{\mathbb{P}}_1^N(A)\leqslant\overline{\mathbb{Q}}_m^N(A)^{1/n'}\,\overline{\mathbb{P}}_{n,m}^N(A)^{1/n}\exp\big[N\delta_n(m,A)\big],\tag{7}$$

$$\overline{\mathbb{Q}}_m^N(A)\leqslant\overline{\mathbb{P}}_1^N(A)^{1/n}\overline{\mathbb{P}}_{\alpha(n),m}^N(A)^{1/n'}\exp\big[N\ \delta_{\alpha(n)}(m,A)/n\big]\tag{8}$$

with $\alpha(n)=-n'/n$ *and for any* $\alpha\neq 0$

$$\delta_\alpha(m,A)=\sup_{u\in A}\big|C_1(u,m)-C_\alpha(u,m)/\alpha\big|.$$

To illuminate the structure of the Radon–Nykodim derivative formula (5) we discuss the different roles played by the two parameters $(\alpha,m)\in(\mathbb{R}\times M)$. One natural and very useful strategy in many applications of large deviations is to find judicious reference probability measures under which the random sequence at hand satisfies an LDP with a good rate function. The next stage consists in transferring this result to the desired sequence of distributions. The choice of this reference sequence $\mathbb{Q}_m^N$, $m\in M$, is often dictated by the problem at hand. In the context of interacting particle systems (IPS) $\mathbb{Q}_m^N$ is often chosen as an N-fold tensor product measure so that the particle are $\mathbb{Q}_m^N$-independent. In this situation Sanov's theorem tells us that an LDP holds with a good rate function. Using the integral lemma the LDP transfer is guaranteed as soon as we can find a collection of distributions $\mathbb{P}_{\alpha,m}^N$, $\alpha\in\mathbb{R}$, satisfying (5). Intuitively speaking the parameter α can be regarded as a deformation parameter of the sequence of measures $\mathbb{P}_1^N$. In the context of IPS each $\mathbb{P}_{\alpha,m}^N$ is the distribution law of an N-IPS model with an interaction function depending of the parameter $\alpha\in\mathbb{R}$. In some sense α measures the strength of interaction. For instance, in the forthcoming examples when $\alpha\to 0$ the particles become independent.

As mentioned, we illustrate the impact of the integral lemma in the context of IPS models. Our general and abstract context is ideally suited to treat in the same frame discrete generation IPS as well as McKean–Vlasov IPS diffusions and pure jumps (*cf.* [1]). We let $I=\{0,1,\ldots,T\}\subset\mathbb{N}$ or $I=\mathbb{R}_+=[0,T]\subset\mathbb{R}_+$ be the discrete or continuous time index with a finite time horizon T. For E a complete separable metric space we denote $\mathcal{P}(E)$ the set of all probability measures on E furnished with the weak topology. By $\Omega=D(I,E)$ we denote the set of all càdlàg paths from I into E with the Skorohod metric. We also denote by $\phi(\eta)=(\eta_t)_{t\in I}$ the distribution flow of the marginals with respect to time of a given measure $\eta\in\mathcal{P}(\Omega)$.

In the discrete time situation we start with a distribution $\eta_0\in\mathcal{P}(E)$ and a collection of Markov transitions $K=\{K_{n,\eta};\ n\in I,\ \eta\in\mathcal{P}(E)\}$. For any distribution flow $\gamma=(\gamma_n)_{n\in I}$ we denote $\mathbb{Q}_\gamma$ the measure on Ω

($= E^{T+1}$) defined by

$$\mathbb{Q}_\gamma\big(\mathrm{d}(x_0,\ldots,x_T)\big) = \eta_0(\mathrm{d}x_0)K_{1,\gamma_0}(x_0,\mathrm{d}x_1)\cdots K_{T,\gamma_{T-1}}(x_{T-1},\mathrm{d}x_T).$$

It is important to notice that the McKean distribution defined by $\mathbb{P} = \mathbb{Q}_{(\eta_n)_{n\in I}}$ and associated to the distribution flow $\eta_n = \eta_{n-1}K_{n,\eta_{n-1}}$ is a fixed point of the mapping $\eta \to \mathbb{Q}_{\phi(\eta)}$.

The N-IPS associated to the collection K and the initial distribution $\eta_0 \in \mathcal{P}(E)$ is a Markov chain $\xi_n \in E^N$, $n \in I$, with initial distribution $\eta_0^{\otimes N}$ and elementary transitions

$$\mathrm{Prob}\big(\xi_n \in \mathrm{d}\big(y^1,\ldots,y^N\big) \mid \xi_{n-1} = x\big) = \prod_{i=1}^N K_{n,m(x)}\big(x^i,\mathrm{d}y^i\big) \quad \text{with } m(x) = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}.$$

In the continuous time situation we start with a distribution $\eta_0 \in \mathcal{P}(E)$ and a collection of generators $L = \{L_{t,\mu};\ t \in I,\ \mu \in \mathcal{P}(E)\}$ defined on some dense domain $D(L)$ in the space of bounded continuous functions. For any distribution flow $\gamma = (\gamma_t)_{t\in I}$ with $\gamma_t \in \mathcal{P}(E)$, $t \in I$, we suppose there exist a solution $\mathbb{Q}_\gamma \in \mathcal{P}(\Omega)$ of the non-homogeneous martingale problem associated to $\{L_{t,\gamma_t};\ t \in I\}$ and starting at η_0. In this framework the McKean measure $\mathbb{P}$ is again characterized as the fixed point of the mapping $\eta \to \mathbb{Q}_{\phi(\eta)}$. The N-interacting particle system associated to L and $\eta_0 \in \mathcal{P}(E)$ is an E^N-valued Markov process $\xi_t = (\xi_t^1,\ldots,\xi_t^N)$ having the initial distribution $\eta_0^{\otimes N}$ and generator $\mathcal{L}_t(f)(x^1,\ldots,x^N) = \sum_{i=1}^N L^{(i)}_{t,m(x)}(f)(x^1,\ldots,x^N)$ where $L^{(i)}_{t,\mu}$ is used instead of $L_{t,\mu}$ when it acts on the i-th variable.

Let $M = \mathcal{P}(\Omega)$ and let $\mathbb{P}^N$ be the probability measure induced by the N-IPS process $(\xi_t)_{t\in I}$ on the product path space Ω^N. By $\overline{\mathbb{P}}^N = \mathbb{P}^N \circ \pi_N^{-1}$ we denote the image probability measure of the empirical measure on path space with $\pi_N(\omega) = \frac{1}{N}\sum_{i=1}^N \delta_{\omega^i}$. For each $\eta \in M$ we also denote by $\mathbb{Q}_\eta^N = (\mathbb{Q}_{\phi(\eta)})^{\otimes N}$ the N-fold tensor product of the measure $\mathbb{Q}_{\phi(\eta)}$ and by $\overline{\mathbb{Q}}_\eta^N = \mathbb{Q}_\eta^N \circ \pi_N^{-1}$ the corresponding image measure. Sanov's theorem tells us that the sequence $\overline{\mathbb{Q}}_{\phi(\eta)}^N$, $N \geqslant 1$, satisfies a LDP with good rate function $\eta' \to H_\eta(\eta') = \mathrm{Ent}(\eta' \mid \mathbb{Q}_{\phi(\eta)})$. To transfer this result and conclude that $\overline{\mathbb{P}}^N$ satisfies a LDP with good rate function $\eta \to H_\eta(\eta) = \mathrm{Ent}(\eta \mid \mathbb{Q}_{\phi(\eta)})$ we proceed as follows.

In the discrete time situation we suppose that $K_{n,\mu}(x,\cdot) \sim K_{n,\eta}(x,\cdot)$. Under appropriate regularity conditions [1] if we take in (5)

$$S_N(\omega,\eta) = \sum_{n=1}^T \int m(\omega_{n-1},\omega_n)\big(\mathrm{d}(u,v)\big)\log\left[\frac{\mathrm{d}K_{n,m(\omega_{n-1})}(u,\cdot)}{\mathrm{d}K_{n,\eta_{n-1}}(u,\cdot)}(v)\right],$$

$$C_\alpha(\mu,\eta) = -\sum_{n=1}^T \int \mu_{n-1}(\mathrm{d}u)\log\left[\int \frac{\mathrm{d}K_{n,\mu_{n-1}}(u,\cdot)}{\mathrm{d}K_{n,\eta_{n-1}}(u,\cdot)}(v)\right]^\alpha K_{n,\eta_{n-1}}(u,\mathrm{d}v)$$

we define the desired collection of distributions $\mathbb{P}^N_{\alpha,\eta}$, $\alpha \in \mathbb{R}$. Under $\mathbb{P}^N_\alpha$ the N-IPS model is the N-IPS model associated to the collection of Markov transitions $K^{(\alpha)}_{n,\mu}$ defined up to a normalizing constant $Z_n^{(\alpha)}(\mu,\eta)(x)$ by

$$K^{(\alpha)}_{n,\mu}(x,\mathrm{d}y) = \frac{1}{Z_n^{(\alpha)}(\mu,\eta_{n-1})(x)}\left[\frac{\mathrm{d}K_{n,\mu}(x,\cdot)}{\mathrm{d}K_{n,\eta_{n-1}}(x,\cdot)}(y)\right]^\alpha K_{n,\eta_{n-1}}(x,\mathrm{d}y).$$

The second example is the N-IPS associated to a McKean diffusion in $E = \mathbb{R}^d$ with drift function $b : E^2 \to E$ and generator defined for any $\mu \in \mathcal{P}(E)$ and $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$ by

$$L_\mu(\varphi)(x) = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2\varphi}{\partial x_i^2}(x) + \sum_{i=1}^d b_i[x,\mu]\frac{\partial\varphi}{\partial x_i}(x) \quad \text{with } b_i[x,\mu] = \int_{\mathbb{R}^d}\mu(\mathrm{d}y)b_i(x,y).$$

We notice that for any $\eta \in M$, the stochastic process $B_t^\eta(\omega) = \omega_t - \omega_0 - \int_0^t b[\omega_s,\eta_s]\,\mathrm{d}s$ is a $\mathbb{Q}_{\phi(\eta)}$-Brownian motion. Under appropriate regularity conditions [1] if we take in (5)

$$S_N(\omega,\eta) = \int_0^T \int m(\omega_t)(\mathrm{d}u)\big(b\big[u, m(\omega_t)\big] - b[u,\eta_t]\big)\,\mathrm{d}B_t^\eta(u),$$

$$C_\alpha(\mu,\eta) = -\frac{\alpha^2}{2} \int_0^T \int \mu_t(\mathrm{d}u)\big\|\mu_t\big[b(u,\cdot) - b(u,\eta_t)\big]\big\|^2\,\mathrm{d}t$$

we define a distribution law $\mathbb{P}_{\alpha,\eta}^N$ under which the N-IPS model $(\xi_t)_{t\in I}$ becomes the N-IPS model associated to the time inhomogeneous generators $L_{t,\mu}^{(\alpha)}$ defined for any $t \in I$, $\mu \in \mathcal{P}(E)$ and $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$ by

$$L_{t,\mu}^{(\alpha)}(\varphi)(x) = L_{\eta_t}(\varphi)(x) + \alpha \sum_{i=1}^d \big(b_i[x,\mu] - b_i[x,\eta_t]\big)\frac{\partial\varphi}{\partial x_i}(x).$$

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