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On a canonical placement of knots in irreducible 3-manifolds

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Abstract If M is a compact connected orientable irreducible 3-manifold and T is a minimal Jaco–Shalen–Johannson system of tori inside M, we define the *pieces* of M to be regular neighborhoods of incompressible tori in $T \cup \partial M$, the components of their complement or regular neighborhoods of Seifert fibres in those components that admit Seifert fibrations. For a given isotopy class $\mathcal{K}$ of knots inside M we describe, with some restrictions on M, the set of pieces which contain representatives of $\mathcal{K}$. If the knots of $\mathcal{K}$ are not contained in balls, we show that the isotopy class of a representative of $\mathcal{K}$ inside a piece P is independent of the chosen representative. ***To cite this article: P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 677–682.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur un placement canonique des nœuds dans les 3-variétés irréductibles

Résumé Si M est une 3-variété compacte connexe orientable irréductible et que T est un système minimal de tores de Jaco–Shalen–Johannson dans M, nous définissons les *pièces* de M comme étant des voisinages réguliers de tores incompressibles dans $T \cup \partial M$, les composantes de leur complémentaire ou encore des voisinages réguliers des fibres de Seifert des composantes qui admettent une fibration de Seifert. Pour une classe d'isotopie donnée $\mathcal{K}$ de nœuds dans M nous décrivons, avec certaines restrictions sur M, l'ensemble des pièces qui contiennent des représentants de $\mathcal{K}$. Si les nœuds de $\mathcal{K}$ ne sont pas contenus dans des boules, nous montrons que la classe d'isotopie d'un représentant de $\mathcal{K}$ à l'intérieur d'une pièce P est indépendante du représentant choisi. ***Pour citer cet article : P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 677–682.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Les variétés considérées (avec ou sans bord), ainsi que leurs fermetures dans une variété ambiante, sont supposées différentiables ou linéaires par morceaux.

Si M est une variété, un *nœud* dans M est une 1-sous-variété connexe close de l'intérieur $\overset{\circ}{M}$. Un *type de nœud* dans M est une classe d'isotopie de nœuds dans M. Un nœud d'une telle classe est appelé un *représentant* du type de nœud.

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DÉFINITION 1.1. – Un type de nœud est dit *global* si aucun de ses représentants n'est contenu dans une boule plongée dans M. Si un type de nœud n'est pas global (auquel cas tous ses représentants sont contenus dans des boules plongées), il est appelé *local*.

Soit M une 3-variété compacte connexe orientable irréductible et T un msJSJ dans M (Définition 1.2). Pour chaque tore incompressible $\tau \subset T \cup \partial M$, soit P_τ un voisinage régulier fermé de τ. Nous choisissons ces voisinages deux à deux disjoints. Nous les appelons les *pièces torales* de M. Nous appelons les composantes connexes de $M - \bigcup_\tau \overset{\circ}{P}_\tau$ les *pièces principales* de M. Pour chaque pièce principale P, nous introduisons des représentants des classes d'isotopie de fibrations de Seifert de P (ensemble de représentants qui peut être vide). Pour chacune de ces fibrations, nous considérons un ensemble formé d'une fibre régulière et de toutes les fibres exceptionnelles. Si C est l'une des fibres de cet ensemble, nous considérons un voisinage régulier fermé P_C de C dans $\overset{\circ}{P}$. Nous appelons ces tores pleins les *pièces circulaires* de M. Nous disons que P_C est *contenue* dans la pièce principale P.

Nous nous posons la question suivante : *un type de nœud $\mathcal{K}$ étant donné, quelles sont les pièces de M qui contiennent des représentants de $\mathcal{K}$* ? C'est la question de placement à laquelle il est fait référence dans le titre de la note.

Si $\mathcal{K}$ est local, on voit immédiatement que toutes les pièces contiennent des représentants de $\mathcal{K}$. Pour cette raison, nous nous restreignons aux types de nœuds globaux. De plus, nous supposons qu'il existe des pièces contenant des représentants de $\mathcal{K}$, c'est-à-dire que $\mathcal{K}$ est de plus *isolable* (Définition 1.3).

Dans la Définition 1.4, nous introduisons une relation d'ordre partiel, notée $\prec$, sur l'ensemble des pièces de M. Les résultats principaux de la note sont les théorèmes suivants :

THÉORÈME 1.5. – *Soit M une 3-variété compacte connexe orientable irréductible. On suppose que M n'est pas un tore solide, un tore épaissi ou un espace lenticulaire. Soit $\mathcal{K}$ un type de nœud global et isolable dans M. Il existe alors une pièce $P(\mathcal{K})$ avec la propriété qu'une pièce P contient un représentant de $\mathcal{K}$ si et seulement si $P \succeq P(\mathcal{K})$.*

THÉORÈME 1.6. – *Avec les mêmes hypothèses qu'auparavant, si P est une pièce qui contient deux représentants K_1 et K_2 de $\mathcal{K}$, alors K_1 est isotope à K_2 à l'intérieur de P.*

A partir de ces théorèmes nous déduisons une réinterprétation de la relation d'ordre $\preceq$ (Corollaire 1.7).

Dans la Section 1 nous définissons les notions utilisées et nous énonçons nos principaux résultats. Dans la Section 2 nous expliquons comment ces questions ont été motivées par l'étude des singularités. Dans les sections suivantes nous expliquons la structure des preuves des Théorèmes 1.5 et 1.6. Les techniques utilisées sont celles de la topologie géométrique. Remarquons que l'on ne peut pas raisonner directement dans le groupe fondamental, car les nœuds considérés peuvent être contractibles.

1. The statements of the results

In what follows, all manifolds (possibly with boundary) as well as their closures in an ambient manifold are supposed differentiable or piecewise linear. If P is a manifold, ∂P denotes its boundary, $\overset{\circ}{P} := P - \partial P$ denotes its interior and $\mathrm{cl}(P)$ denotes its closure in an ambient manifold. If P is a compact submanifold of a manifold M, we denote by $N(P)$ a closed regular neighborhood of P. Here P may be of any dimension not greater than the dimension of M. If we study the position of P with respect to another submanifold Q of M, we suppose $N(P)$ is taken in general position with respect to Q. A manifold isomorphic to $\mathbf{S}^1 \times \mathbf{D}^2$ is called a *solid torus*, one isomorphic to $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{D}^1$ is called a *thickened torus*. Two disjoint tori in a 3-manifold M are called *parallel* if they cobound a thickened torus.

If M is a manifold, a *knot* in M is a connected closed 1-submanifold of $\overset{\circ}{M}$. A *knot type* in M is an isotopy class of knots in M. A knot of such a class is called a *representative* of the knot type.

DEFINITION 1.1. – A knot type is called *global* if no representative of it is contained in an embedded ball of M. If a knot type is not global (in which case any representative of it is contained in an embedded ball), it is called *local*.

Let M be a compact connected orientable irreducible 3-manifold.

DEFINITION 1.2. – A 2-submanifold T of M all of whose components are incompressible tori is called a *system of Jaco–Shalen–Johannson* (abbreviated *sJSJ*) if each connected component of $\mathrm{cl}(M - N(T))$ either admits a Seifert fibration, is atoroidal (i.e., every incompressible embedded torus is parallel to the boundary) or is a torus bundle over the circle. If no subsystem of T has the same property, then T is called a *minimal system of Jaco–Shalen–Johannson* (abbreviated *msJSJ*).

Inspired by the unicity theorem proved by Waldhausen [10] for the class of graph-manifolds, Jaco–Shalen [4] and Johannson [5] proved that in a compact orientable irreducible 3-manifold, a msJSJ exists and is unique up to an isotopy (*see* other proofs in [1] and [8]; notice that with our definition the msJSJ of a torus bundle over the circle is empty). This shows that the following definition is independent of the choice of msJSJ:

DEFINITION 1.3. – Let M be a compact connected orientable irreducible 3-manifold. A knot type in M is called *isolable* if it admits a representative disjoint from a msJSJ of M.

Let T be a msJSJ for M. For every incompressible torus $\tau \subset T \cup \partial M$, let P_τ denote a closed regular neighborhood of τ (all tori of $T \cup \partial M$ are incompressible, excepted when M is a solid torus and $\tau = \partial M$). We choose those neighborhoods pairwise disjoint. We call them the *toral pieces* of M. We call the connected components of $M - \bigcup_\tau \mathring{P}_\tau$ the *main pieces* of M. If a toral piece P_τ and a main piece P meet along a common boundary component, we say that P is *adjacent* to P_τ or to τ. If P is a piece and $\tau \subset \partial P$, we say that P and τ are *adjacent*. If two main pieces P_1 and P_2 are adjacent to the same toral piece P_τ, then we say that P_1 and P_2 are *adjacent through* P_τ or *through* τ.

For each main piece P, we consider representatives of the isotopy classes of Seifert fibrations of P (when such fibrations exist). For each representative, we consider a set consisting of a regular fibre and all the exceptional fibres. If C is a fibre of this set, we take a closed regular neighborhood P_C of C in $\mathring{P}$. We call these solid tori the *circular pieces* of M. We say that P_C is *contained* in the main piece P. If C is a regular fibre, we call P_C a *regular circular piece*, otherwise we call it an *exceptional circular piece*.

We are interested in the following question: *a knot type $\mathcal{K}$ being given, which pieces of M contain representatives of $\mathcal{K}$*? This is the placement question alluded to in the title of the note.

If $\mathcal{K}$ is local, one sees immediately that all the pieces contain representatives of $\mathcal{K}$. For this reason, we restrain our study to the global types of knots. Moreover, we suppose that there exist pieces containing representatives of $\mathcal{K}$, which is equivalent to $\mathcal{K}$ being isolable.

In order to answer to our placement question, we need one more definition:

DEFINITION 1.4. – Let $\mathcal{P}(M)$ denote the set of pieces of M. We define a partial ordering $\prec$ on $\mathcal{P}(M)$, saying that $P_1 \prec P_2$ if and only if we are in one of the following situations:
(1) P_1 is a regular circular piece and P_2 is an exceptional circular piece corresponding to the same fibration of a main piece.
(2) P_1 is a regular circular piece and P_2 is a toral piece adjacent to the main piece which contains P_1.
(3) P_1 is a regular circular piece and P_2 is a main piece adjacent to the main piece which contains P_1.
(4) P_1 is a circular piece and P_2 is the main piece which contains P_1.
(5) P_1 is a toral piece, P_2 is a main piece adjacent to P_1.

Our main results are the following two theorems:

THEOREM 1.5. – *Let M be a compact connected orientable irreducible* 3*-manifold. We suppose M is not a solid torus, a thickened torus or a lens-space. Let $\mathcal{K}$ be a global and isolable knot type in M. Then* *there exists a piece $P(\mathcal{K})$ with the property that a piece P contains a representative of $\mathcal{K}$ if and only if $P \succeq P(\mathcal{K})$.*

THEOREM 1.6. – *With the same hypotheses as before, if P is a piece that contains two representatives K_1 and K_2 of $\mathcal{K}$, then K_1 is isotopic to K_2 inside P.*

We call the piece $P(\mathcal{K})$ the *address* of the knot type $\mathcal{K}$ and of any representative of $\mathcal{K}$. It is a piece canonically associated to $\mathcal{K}$, by Theorem 1.5. Moreover, Theorem 1.6 shows that the knot type of a representative of $\mathcal{K}$ inside its address does not depend on the chosen representative. This is the canonical placement alluded to in the title.

As a corollary of the theorems, we deduce the following characterization of the partial ordering on $\mathcal{P}(M)$:

COROLLARY 1.7. – *If P_1, P_2 are two pieces of M, then $P_1 \preceq P_2$ if and only if P_1 can be isotoped inside P_2.*

The "only if" part is easy, we need Theorem 1.5 for the converse.

The techniques of proof are of geometric topology. One cannot work directly in the fundamental group, as the knots we consider may be contractible (for example take a Whitehead knot inside a circular piece). We present the structures of the proofs in Sections 3 and 4, after having explained in Section 2 the way we arrived at our question from the study of singularities of complex plane curves.

In [9], we called the global knots *sedentary* and the local ones *nomad*. This was motivated by the fact that a ball containing a knot can be imagined as a vehicle in which to visit any part of M and that Theorem 1.5 explains simply that a sedentary knot has an address.

2. Relation with the theory of singularities of plane curves

Theorem 1.5 was proved in less generality (with the additional hypothesis that the boundary of M consists only of tori) in [9]. We conjectured it starting from the study of germs of complex analytic plane curves. Let us explain this briefly. Details and references can be found in [9].

A reduced germ of plane curve Γ being given, there is an algebraic way to associate to it a combinatorial tree $\mathcal{T}(\Gamma)$ from the knowledge of the characteristic Newton–Puiseux sequences and of the intersection numbers of its components. In [9] we call it the reduced Eggers tree. If Δ is an irreducible curve, one can associate to it algebraically a vertex or an edge of $\mathcal{T}(\Gamma)$. We want to give a topological interpretation of this construction. In order to do this we cut $\Gamma \cup \Delta$ with a Milnor sphere $\mathbf{S}^3$ and we consider $K(\Delta) := \mathbf{S}^3 \cap \Delta$ as a knot inside $M(\Gamma) := \mathbf{S}^3 - \overset{\circ}{N}(K(\Gamma))$. We see first that $\mathcal{T}(\Gamma)$ is isomorphic to the adjacency graph of a msJSJ for $M(\Gamma)$, whose vertices correspond bijectively to the main pieces and whose edges correspond to adjacency tori between those pieces. Moreover, $K(\Delta)$ can be isotoped inside the main or toral piece corresponding to its associated vertex or edge in $\mathcal{T}(\Gamma)$. Then a natural question is to know if this placement is in some sense canonical topologically. Using Theorem 1.5, we see that this is indeed the case.

More generally, our theorems can be useful for the topological study of curves contained in germs of normal surfaces. Indeed, if Γ is now a (possibly empty) reduced germ of curve drawn on the germ Σ of normal surface, and Δ is an irreducible curve on Σ, we can consider again the intersections with a Milnor sphere, and with the evident notations, $K(\Delta)$ as a knot inside $M(\Gamma) := K(\Sigma) - \overset{\circ}{N}(K(\Gamma))$. Following Neumann [7], we see that $M(\Gamma)$ is irreducible. Moreover, $K(\Delta)$ is global and isolable, so we can apply our results whenever the hypothesis of Theorem 1.5 are verified by $M(\Gamma)$. In this case, the address of $K(\Delta)$ is necessarily circular or toral.

3. The ingredients of the proof of Theorem 1.5

In the following sections we suppose that the hypotheses of Theorem 1.5 are verified. Let us describe first the principle of the proof. If P is a piece and K is a knot in P, then it can be easily seen that it is global in P if and only if it is global in M, which in turn is equivalent to the irreducibility of $M - \overset{\circ}{N}(K)$. So this last manifold admits also a msJSJ. We consider a special one, called $T_{M,K}$, obtained in the following way:

LEMMA 3.1. – *Let P be a piece of M and $K \subset P$ be a global knot. Take $T_{P,K}$, a msJSJ for $P - \overset{\circ}{N}(K)$, and define T' to be ∂P if P is a circular piece, $\partial P \cap \overset{\circ}{M}$ if P is a toral piece and $\emptyset$ otherwise. Then one can obtain a msJSJ called $T_{M,K}$ for $M - \overset{\circ}{N}(K)$, starting from $T_{P,K} \cup (T \cap (M - P)) \cup T'$, and possibly eliminating components of $(T \cap (M - P)) \cup T'$ which are adjacent to P.*

Then we consider pairs of global knots contained in distinct pieces $K_1 \subset \overset{\circ}{P_1}$, $K_2 \subset \overset{\circ}{P_2}$ and we suppose that they are isotopic in M. By the unicity up to isotopy of the msJSJ of $M - \overset{\circ}{N}(K_2)$, there exists an isomorphism $\phi : (M, K_1, T_{M,K_1}) \to (M, K_2, T_{M,K_2})$ isotopic to id_M. We study then the possible images of well chosen components of T_{M,K_1}. We do this for different choices of the pieces P_1, P_2.

As a simple example of this method, let us prove a first step towards Theorem 1.5:

LEMMA 3.2. – *Let P_1 and P_2 be two distinct main pieces of M such that there is no main or toral piece adjacent to both of them. Consider global knots K_1 and K_2 inside $\overset{\circ}{P_1}$, respectively $\overset{\circ}{P_2}$. Then K_1 and K_2 are not isotopic in M.*

Proof. – By contradiction, we suppose that there exists an isomorphism ϕ like before, isotopic to the identity. Let $T_{\max} \subset T$ be the subsystem of tori which are adjacent neither to P_1 nor to P_2. By hypothesis, $T_{\max}$ is not empty and P_1 and P_2 are in distinct components of $M - T_{\max}$. Let $T_{\min} \subset T_{\max}$ be a minimal subsystem with this property. Then $M - T_{\min}$ has exactly two components and so, for every torus $\tau \subset T_{\min}$, we can speak of the side oriented towards K_1 and of the side oriented towards K_2. We look then at the image $\phi(\tau)$ of such a torus.

If $\phi(\tau) \subset P_2$, then τ and $\phi(\tau)$ are disjoint, and because τ is incompressible, by a lemma of Waldhausen [11], they cobound a thickened torus. This thickened torus contains necessarily a component τ' of T parallel to a component of ∂P_2, and by the characterisation of incompressible surfaces in thickened tori proven also in [11], we deduce that τ is parallel to $\tau' \neq \tau$, which contradicts the minimality in M of the system T.

So $\phi(\tau) \subset T$ and again by the minimality of T, we deduce that $\phi(\tau) = \tau$. This is true for each component of $T_{\min}$. As $\phi(K_1) = K_2$, we deduce that ϕ changes the orientations of the components of $T_{\min}$, which contradicts another lemma of Waldhausen [11]. □

We use the same method to prove the following lemmas, from which Theorem 1.5 can be easily deduced. In all of them, we suppose that P_1 and P_2 are two distinct pieces and $K_1 \subset \overset{\circ}{P_1}$, $K_2 \subset \overset{\circ}{P_2}$ are global knots which are isotopic in M. As an important ingredient in their proofs, we use the knowledge of the isotopy classes of Seifert fibrations in compact manifolds with boundary (*see* [3,4]).

LEMMA 3.3. – *If P_1 and P_2 are both circular, then P_1 and P_2 are contained in the same main piece P, they correspond to the same Seifert fibration of P and one of them is a regular circular piece.*

LEMMA 3.4. – *If P_1 and P_2 are both toral, then there exists a main piece P simultaneously adjacent to P_1 and P_2, a Seifert fibration of $P \cup P_1 \cup P_2$, regular fibres C_1, C_2 of it which possess regular neighborhoods $N(C_1) \subset P_1$, $N(C_2) \subset P_2$ containing K_1 and K_2 respectively.*

LEMMA 3.5. – *If P_1 and P_2 are non-adjacent main pieces, then there exists a main piece P which is simultaneously adjacent to P_1 and P_2, a regular neighborhood $N(P)$ containing K_1 and K_2, a Seifert fibration of $N(P)$ and two regular fibres C_1 and C_2 of it which possess regular neighborhoods containing K_1 and K_2 respectively.*

LEMMA 3.6. – *If P_1 and P_2 are main pieces and the knots K_1, K_2 are not isotopic to a knot contained in a regular circular piece, then P_1 and P_2 are adjacent through a component τ of T and there exists a regular neighborhood $N(\tau)$ of τ which contains both K_1 and K_2.*

4. The ingredients of the proof of Theorem 1.6

First we prove Theorem 1.6 when P is the address of K_1 and K_2. This situation can be characterized in a refinement of Lemma 3.1:

LEMMA 4.1. – *Let P be a piece of M and $K \subset P$ be a global knot. With the same notations as in Lemma* 3.1, *the system of tori $T_{P,K} \cup (T \cap (M - P)) \cup T'$ is a msJSJ for $M - \overset{\circ}{N}(K)$ if and only if P is the address of K.*

Using this lemma, we prove the following result:

LEMMA 4.2. – *If P is a piece of M and K_1, K_2 are global knots inside P whose address is P, then there exists a diffeomorphism $\phi : (M, K_1, \partial P) \to (M, K_2, \partial P)$ isotopic to id_M.*

We distinguish then two cases, according to the nature of the piece P.

If P is not circular, the components of $\partial P \cap \overset{\circ}{M}$ are incompressible in M. We use then the following result of Laudenbach [6], as reformulated in [2]: if $S \subset M$ is a closed connected orientable incompressible surface which is not a fiber of a fibration over the circle, and $\psi : (M, S) \to (M, S)$ is an isomorphism isotopic to id_M, then there exists an isotopy to id_M which preserves S at each level. In our case, this allows to show that there exists an isotopy of ϕ to id_M which preserves ∂P at each level. During this isotopy, K_2 remains inside P.

If P is a circular piece, let P' be the main piece which contains P. Then it can be shown that ϕ sends the Seifert fibration of $P' - \overset{\circ}{P}$ onto an isotopic one. Taking the restriction to ∂P and the fact that a meridian for P is sent to an isotopic curve, we see that $\phi : \partial P \to \partial P$ is the identity on the rational homology, and so is isotopic to the identity. We conclude using the fact that an automorphism of a solid torus whose restriction to the boundary is isotopic to the identity, is itself isotopic to the identity.

To prove Theorem 1.6 in general, we use the previously analyzed case and the lemmas of Section 3.

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References

[1] A. Hatcher, Notes on basic 3-manifold topology, Accessible on http://math.cornell.edu/~hatcher.
[2] A. Hatcher, D. McCullough, Finiteness of classifying spaces of relative diffeomorphism groups of 3-manifolds, Geometry and Topology 1 (1997) 91–109.
[3] W.H. Jaco, Lectures on Three-Manifold Topology, Regional Conference Series in Mathematics, Vol. 43, American Mathematical Society, 1980.
[4] W.H. Jaco, P.B. Shalen, Seifert fibred spaces in three-manifolds, Mem. Amer. Math. Soc. 21 (1979) 220.
[5] K. Johannson, Homotopy Equivalences of 3-Manifolds with Boundaries, Lecture Notes in Math., Vol. 761, Springer-Verlag, 1979.
[6] F. Laudenbach, Topologie de dimension trois, homotopie et isotopie, Astérisque 12 (1974).
[7] W. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Amer. Math. Soc. 268 (2) (1981) 299–344.
[8] W.D. Neumann, G.A. Swarup, Canonical decompositions of 3-manifolds, Geometry and Topology 1 (1997) 21–40.
[9] P. Popescu-Pampu, Arbres de contact des singularités quasi-ordinaires et graphes d'adjacence pour les 3-variétés réelles, Thesis, Univ. Paris 7, Novembre 2001.
[10] F. Waldhausen, Eine Klasse von 3-dimensionalen Mannigfaltigkeiten, Invent. Math. 3 (1967) 308–333 and Invent. Math. 4 (1967) 87–117.
[11] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. Math. 87 (1968) 56–88.