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Continued fractions and solutions of the Feigenbaum–Cvitanović equation

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Abstract In this paper, we develop a new approach to the construction of solutions of the Feigenbaum–Cvitanović equation whose existence was shown by H. Epstein. Our main tool is the analytic theory of continued fractions. ***To cite this article: A.V. Tsygvintsev et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 683–688.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Fractions continues et des solutions de l'équation de Feigenbaum–Cvitanović

Résumé Dans ce travail, nous énonçons une nouvelle méthode de construction des solutions de l'équation de Feigenbaum–Cvitanović dont l'existence a été montrée par H. Epstein. On utilise la théorie analytique des fractions continues. ***Pour citer cet article : A.V. Tsygvintsev et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 683–688.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

(Les numéros d'équations renvoient au texte anglais.) Dans cette Note nous étudions des solutions de l'équation de Feigenbaum–Cvitanović (1.1) où g est une application de l'intervalle $[-1, 1]$ dans lui-même. Nous considérons seulement des solutions de (1.1) de la forme $g(x) = F(x^d)$, $d > 1$, où F est une fonction analytique et décroissante dans l'intervalle $[0, 1)$ et sans points critiques. Soit $U = F^{-1}$ une fonction réciproque de F, alors U satisfait (1.2). Une fonction f analytique dans $\mathbb{C}_+ \cup \mathbb{C}_- \cup (a, b)$, $a, b \in \mathbb{R}$ avec la propriété $f(\mathbb{C}_+) \subset \mathbb{C}_-$, $f(\mathbb{C}_-) \subset \mathbb{C}_+$ est appelée une fonction anti-Herglotzienne.

THÉORÈME 0.1 (Epstein, 1986). – *La solution U de l'équation* (1.2) *existe pour tout $d > 1$ et peut être étendue à une fonction anti-Herglotzienne dans le domaine* $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, \lambda^{-2})$.

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L'équation (1.2) écrite pour la fonction $u = U^{1/d}$ est de la forme (3.1) où u satisfait $u(1) = 0$, $u(-\lambda) = 1$ et $u'(-\lambda) = -\lambda^{d-1}$. Soit $d \geqslant 2$, $f_\lambda = (1-\lambda)^{-1}(\lambda + z)(1-z)^{-1}$ et $\mathcal{D} = \mathbb{C}_+ \cup \mathbb{C}_- \cup (-1, +\infty)$ alors la fonction $w = u \circ f_\lambda^{-1}$ a les propriétés suivantes :

(i) w et w^2 sont des fonctions anti-Herglotziennes dans $\mathcal{D}$;
(ii) w est réelle pour $z \in (-1, +\infty)$, $w(0) = 1$;
(iii) La partie réelle de w dans $\mathcal{D}$ est positive.

En appliquant le théorème de Wall [5, p. 279] il est facile de montrer que la fonction w avec des propriétés (i)–(iii) peut être représentée à l'aide d'une fraction continue (1.4) $u(z) = \{g_1, g_2, \ldots \mid z\}$ qui converge uniformément sur tous les sous-ensembles compacts de $\mathbb{C}$. L'avantage principal de cette forme est que les coefficients g_i qui figurent dans (1.4) satisfont $0 \leqslant g_i \leqslant 1$. Si l'on revient à l'équation (3.1) nous pouvons donc établir le résultat suivant

THÉORÈME 0.2. – *Dans le cas $d \geqslant 2$ la solution $u(z)$ de l'équation* (3.1) *s'ecrit sous la forme $u(z) = \{g_1, g_2, \ldots \mid f_\lambda(z)\}$.*

Par conséquent on en déduit que la fonction u peut être bornée par des fonctions rationnelles anti-Herglotziennes. À titre d'exemple, dans le cas plus simple ces bornes sont données par (2.3) où l'on remplace z par $f_\lambda(z)$. Avec l'équation (3.1) elles nous permet d'obtenir l'approximation uniforme suivante $\lambda^d \leqslant c$, $c = 0.26308\ldots$, $d \geqslant 2$.

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1. Introduction

In the present paper we consider the Feigenbaum–Cvitanović functional equation

$$g(x) = -\lambda^{-1} g\big(g(\lambda x)\big), \quad g(0) = 1, \quad -g(1) = \lambda \in (0,1), \tag{1.1}$$

where g is a map of the interval $[-1, 1]$ into itself. We only consider solutions g such that, on $[0, 1)$, $g(x) = F(x^d)$, $d > 1$, with F analytic, decreasing, and without critical points. Let $U = F^{-1}$ be the inverse function of F. Then U will satisfy

$$U\big(U(-\lambda x)^{1/d}\big) = \lambda U(x), \quad U(-\lambda) = 1, \quad U'(-\lambda) = -d\lambda^{d-1}. \tag{1.2}$$

We denote $\mathbb{C}_+ = -\mathbb{C}_- = \{\zeta \in \mathbb{C},\ \operatorname{Im}\zeta > 0\}$.

THEOREM 1.1 (Epstein, 1986). – *The solution U of Eq.* (1.2) *exists for all $d > 1$ and extends to a function holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_- \cup (-\lambda^{-1}, \lambda^{-2})$ which is injective there and $U(\mathbb{C}_+) \subset \mathbb{C}_-$, $U(\mathbb{C}_-) \subset \mathbb{C}_+$.*

An analytic function f which maps $\mathbb{C}_+$ into $\mathbb{C}_+$ and $\mathbb{C}_-$ into $\mathbb{C}_-$ is called a *Herglotz* function ($-f$ is an *anti-Herglotz* function). See [1] for details.

In the proof of Theorem 1.1 Epstein used in an essential way the integral representation for anti-Herglotz functions, which provides us with *a priori* bounds of the form $r(x) \leqslant U(x) \leqslant R(x)$ where $r(x)$, $R(x)$ are known rational functions (*see* [2–4]). These bounds contain some initial basic information about the behaviour of U.

The purpose of the present paper is to make more precise the analytic nature of the solution of the functional equation (1.2) and give some new rational bounds on it. Our main tool here is the analytic theory of continued fractions, a detailed account of which may be found in the book by Wall [5]. The foundation of this theory is the natural correspondence between continued fractions, analytic functions and their integral representation given by a moment problem. For concreteness, let us consider the Stieltjes integral of the form

$$f(z)=\int_0^1 \frac{\mathrm{d}\theta(u)}{1+uz}, \quad \int_0^1 \mathrm{d}\theta(u)=1, \tag{1.3}$$

where $\theta(u)$ is a bounded nondecreasing function in $(0,1)$. Then f has the continued fraction representation [5]

$$f(z)=\cfrac{1}{1+\cfrac{g_1 z}{1+\cfrac{(1-g_1)g_2 z}{1+\cdots}}}, \quad \text{denoted by } f(z)=\{g_1,g_2,\ldots \mid z\}, \tag{1.4}$$

where the real coefficients $g_p\in[0,1]$, $p=1,2,\ldots$, are certain rational functions of the *moments* μ_p of $\theta(u)$ defined by $\mu_p=\int_0^1 u^p\,\mathrm{d}\theta(u)$, $p\geqslant 1$. In Section 2 we show that the function $u=U^{1/d}$, where U is a solution of (1.2), always can be reduced, by a certain conformal change, to the form (1.4). In Section 3 we derive a new uniform bound on λ^d for $d\geqslant 2$ which constitutes the main result of the present paper. Section 4 contains some numerical results.

2. Wall functions and their continued fraction representation

Denote by $\mathcal{D}$ the portion of the complex plane $\mathbb{C}$ exterior to the cut along the real axis from $-\infty$ to -1. Let $f(z)$ satisfying $f(0)=1$, be an anti-Herglotz function analytic in $\mathcal{D}$ and positive for $z>-1$. We call f a *Wall* function if its square f^2 is again an anti-Herglotz function. Let $\mathcal{W}$ be the set of all Wall functions.

THEOREM 2.1. – *For an arbitrary $f\in\mathcal{W}$ we have $f(z)=\{g_1,g_2,\ldots \mid z\}$ for some uniquely defined sequence of coefficients $g_i\in[0,1]$, $i\geqslant 1$. The corresponding continued fraction converges uniformly on every compact domain in $\mathcal{D}$.*

Proof. – Let $f\in\mathcal{W}$ then from the definition of a Wall function it follows that f is an anti-Herglotz function which is analytic in the domain $\mathcal{D}$, real for real values of z and having a positive real part. Consider the function $\phi=\sqrt{1+z}$ where the square root is positive on the axis $(-1,+\infty)$. The function so obtained is a Herglotz function since the argument of ϕ is one half the argument of $1+z$. We will check that the product $\Phi=\phi f$ is a function which is analytic in the domain $\mathcal{D}$, real for real values of z and having a positive real part. Let $f=f_1+\mathrm{i}f_2$, $\phi=\phi_1+\mathrm{i}\phi_2$ where f_1, ϕ_1 and f_2, ϕ_2 are the corresponding real and imaginary parts of the functions f and ϕ. Then $\mathrm{Re}(\Phi)=\phi_1 f_1-\phi_2 f_2>0$ in $\mathcal{D}$. Indeed, $\phi_1 f_1>0$ since both f and ϕ have positive real parts and $-\phi_2 f_2\leqslant 0$ since the imaginary parts of f, ϕ always have a different sign in $\mathcal{D}$. By the theorem of Wall [5, p. 279] the function $\Phi=\phi f$ can be represented in the form $\Phi=\sqrt{1+z}\{g_1,g_2,\ldots \mid z\}$ for some uniquely defined sequence $g_i\in[0,1]$, $i\geqslant 1$, where the corresponding continued fraction converges uniformly on every compact domain in $\mathcal{D}$. Cancelling $\sqrt{1+z}$, it follows that f is of the form (1.4). This concludes the proof. □

Remark 1. – It is easy to see from the integral representation (1.3) that $f(z)=\{g_1,g_2,\ldots \mid z\}$ is an anti-Herglotz function.

We will use the following notation for partial approximants of (1.4)

$$\{g_1,\ldots,g_k \mid z\}=\cfrac{1}{1+\cfrac{g_1 z}{1+\cfrac{(1-g_1)g_2 z}{1+\cfrac{\cdots}{1+(1-g_{k-1})g_k z}}}}, \quad k\geqslant 1,$$

which are all rational anti-Herglotz functions.

The continued fraction $f(z)=\{g_1,g_2,\dots\mid z\}$ can be approximated by rational functions as given by the following theorem

THEOREM 2.2. – (a) *Let $k=2n+1$, $n=0,1,2,\dots$, then*

$$A_k(z)\leqslant f(z)\leqslant B_k(z),\quad -1<z<+\infty, \tag{2.1}$$

where $A_k(z)=\{g_1,\dots,g_k\mid z\}$, $B_k(z)=\{g_1,\dots,g_k,1\mid z\}$

(b) *Let $k=2n$, $n=1,2,\dots$, then*

$$A_k^+(z)\leqslant f(z)\leqslant B_k^+(z),\quad 0\leqslant z<+\infty,\quad \textit{and}\quad A_k^-(z)\leqslant f(z)\leqslant B_k^-(z),\quad -1<z<0, \tag{2.2}$$

where $A_k^+(z)=\{g_1,\dots,g_k,1\mid z\}$, $B_k^+(z)=\{g_1,\dots,g_k\mid z\}$ and $A_k^-(z)=B_k^+(z)$, $B_k^-(z)=A_k^+(z)$.

The proof is based on Theorems 1.11 and 14.2, 14.3 of [5] and is straightforward.

Applying Theorem 2.2 in the case $k=1$ we obtain $A_1(z)\leqslant f(z)\leqslant B_1(z)$, $z\in(-1,\infty)$, where

$$A_1(z)=(1+g_1z)^{-1},\qquad B_1(z)=\big(1+(1-g_1)z\big)(1+z)^{-1}. \tag{2.3}$$

3. Applications to the Feigenbaum–Cvitanović equation

In this section we fix U to be a solution of the functional equation (1.2) given by Theorem 1.1 for a certain $d>1$. Then Eq. (1.2) written in terms of the function $u=U^{1/d}$ takes the form

$$\lambda^{-1}u\big(u(-\lambda z)\big)=u(z),\quad \lambda\in(0,1), \tag{3.1}$$

where $u(z)=U(z)^{1/d}$ satisfies the conditions $u(1)=0$, $u(-\lambda)=1$, $u'(-\lambda)=-\lambda^{d-1}$ from (1.2) and is an anti-Herglotz function analytic in the domain $\mathbb{C}_+\cup\mathbb{C}_-\cup(-\lambda^{-1},1)$.

Let us consider the conformal mapping $f_\lambda(z)=(1-\lambda)^{-1}(\lambda+z)(1-z)^{-1}$ which sends the real interval $(-\lambda^{-1},1)$ into $(-1,+\infty)$ bijectively and satisfies $f_\lambda(-\lambda)=0$, $f_\lambda(-\lambda^{-1})=-1$. Note that both f_λ and f_λ^{-1} are Herglotz functions. Then $u\circ f_\lambda^{-1}\in\mathcal{W}$ for an arbitrary $d\geqslant 2$ which proves the following proposition

PROPOSITION 3.1. – *Let $u(z)=U(z)^{1/d}$ be the solution of the functional equation* (3.1) *with corresponding parameters $d\geqslant 2$, $\lambda\in(0,1)$. Then $u(z)=\{g_1,g_2,\dots\mid f_\lambda(z)\}$ with certain uniquely defined coefficients $g_k\in[0,1]$, $k\geqslant 1$.*

It turns out that g_1 depends only on the numbers λ and d. In fact, using the condition $u'(-\lambda)=-\lambda^{d-1}$, by a simple differentiation of u we obtain

$$g_1=\lambda^{d-1}\big(1-\lambda^2\big). \tag{3.2}$$

Using the rational bounds given by Theorem 2.2 it is easy to write the corresponding bounds on the function $u(z)$: $A(f_\lambda(z))\leqslant u(z)\leqslant B(f_\lambda(z))$, $z\in(-\lambda^{-1},1)$, where $A(z)$, $B(z)$ are the lower and upper bounds given by formulas (2.1), (2.2) for a certain $k\geqslant 1$. Let us denote $\mathcal{A}(z)=A(f_\lambda(z))$, $\mathcal{B}(z)=B(f_\lambda(z))$ then we have $u(-\lambda z)\in[\mathcal{A}(-\lambda z),\mathcal{B}(-\lambda z)]$ and $u(u(-\lambda z))\in[\mathcal{A}(\mathcal{B}(-\lambda z)),\mathcal{B}(\mathcal{A}(-\lambda z))]$ for $z\in(-\lambda^{-1},1)$. Thus, if u satisfies Eq. (3.1) then the following inequalities must be satisfied

$$\mathcal{B}\big(\mathcal{A}(-\lambda z)\big)\geqslant\lambda\mathcal{A}(z),\quad \mathcal{A}\big(\mathcal{B}(-\lambda z)\big)\leqslant\lambda\mathcal{B}(z),\quad \forall z\in\big(-\lambda^{-1},1\big). \tag{3.3}$$

The rational functions $\mathcal{A}(z)$, $\mathcal{B}(z)$ contain $g_1,\dots,g_k$, λ as parameters so that (3.3) imposes certain restrictions on these numbers and can be used together with (3.2) to obtain relations between g_i, λ and d.

The degree of accuracy will depend essentially on the number k but even for small k this method gives satisfactory results. We consider the simplest case $k = 1$. Taking $\mathcal{A}(z)$, $\mathcal{B}(z)$ corresponding to bounds given by (2.3) and using the first inequality from (3.3) for $z \to -\lambda^{-1}$, we obtain

$$\mathcal{B}(0) \geqslant \lambda \mathcal{A}\left(-\lambda^{-1}\right) \quad \text{or} \quad 1 - \lambda + (1 - g_1)\lambda \geqslant \lambda(1 - g_1)^{-1},$$

which implies

$$0 \leqslant g_1 \leqslant \theta(\lambda), \quad \text{where } \theta(\lambda) = \lambda^{-1}\left(1 + \lambda - \sqrt{1 - 2\lambda + 5\lambda^2}\right)/2. \tag{3.4}$$

Using (3.2), this gives the following lower bound on d

$$d \geqslant \log\left(\frac{\lambda\theta(\lambda)}{1-\lambda^2}\right) \Big/ \log(\lambda), \quad d \geqslant 2.$$

The inequality (3.4) can be written in the form $\lambda^d \leqslant \zeta(\lambda)$, $\zeta(\lambda) = \lambda\theta(\lambda)/(1-\lambda^2)$. It is easy to show that $\zeta'(\lambda)$ vanishes at a unique point λ_c in $(0, 1)$ and $\zeta''(\lambda_c) < 0$. Hence in this interval ζ has just one maximum. From this we derive the following uniform bound

$$\lambda^d \leqslant c, \quad c = 0.26308\ldots, \quad \text{for } d \geqslant 2,$$

where we omit the explicit expression for c (which is an algebraic number) due to its complexity. We believe that a more careful analysis of inequalities (3.3) for $k > 1$ can considerably improve this estimate.

4. Numerical applications

In this section we will discuss briefly a numerical technique based on the application of Proposition 3.1. We fix $d \geqslant 2$ and look for a rational approximation of the solution $u(z)$ of Eq. (3.1) in the form $r(z) = \{g_1, \ldots, g_n | f_{\lambda_a}(z)\}$, $n > 1$, with $g_1 = \lambda_a^{d-1}(1-\lambda_a^2)$ for a certain sequence of real numbers $g_k \in [0, 1]$ and $\lambda_a \in \mathbb{R}$. One can show that the substituting of $r(z)$ into the left hand side of (3.1) with $\lambda = \lambda_a$ gives us the new function $R(z) = \{\tilde{g}_1, \tilde{g}_2, \ldots | f_{\lambda_a}(z)\}$. The numbers $\tilde{g}_k$ are not necessary in the interval $[0, 1]$ and are certain functions of $\lambda_a, g_1, \ldots, g_n$. Let us suppose that the first n coefficients of the continued fraction representations of $r(z)$ and $R(z)$ are the same, i.e., $g_i = \tilde{g}_i$, $1 \leqslant i \leqslant n$. This system of equations can be solved using, for example, Newton's method. Numerical experiments show that it seems to have a solution for all $d \geqslant 2$ and $n > 1$ with $\lambda_a, g_1, \ldots, g_n$ belonging to the interval $[0, 1]$ which is as expected in view of our Proposition 3.1. Since $r(z)$ is a rational anti-Herglotz function, it can be always be represented in the simple form $r(z) = c + \sum_{i=1}^{K} \frac{r_i}{z - z_i}$ with some $c, z_i \in \mathbb{R}$ and $r_i \in \mathbb{R}_+$. In the classical Feigenbaum case, $d = 2$, this method gives, for $n = 7$, the following rational approximation for $u(z)$

$$u(z) = 2.822306 + \frac{1.144796}{z + 4.06181} + \frac{0.003406}{z - 1.021068} + \frac{0.139511}{z - 1.373872} + \frac{15.718028}{z - 7.253131}, \tag{4.1}$$

with the corresponding value $\lambda_a = 0.399778\ldots$ and g_i given by

$$g_1 = 0.33588,\ g_2 = 0.27793,\ g_3 = 0.29515,\ g_4 = 0.39803,\ g_5 = 0.50228,\ g_6 = 0.23676,\ g_7 = 0.02773.$$

The value of λ_a is in a good correspondence with the value of $\lambda = 0.399535\ldots$ for the families of quadratic maps, known from the classical theory of Feigenbaum. For $r(z)$ given by (4.1) we have $\max|r(z) - R(z)| < 6 \times 10^{-7}$ for $-\lambda_a \leqslant z \leqslant 0$ and $\max|r(z) - R(z)| < 5 \times 10^{-2}$ for $0 \leqslant z \leqslant 1$.

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