C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1011–1014

Systèmes dynamiques/*Dynamical Systems*
(Topologie/*Topology*)

Real-analytic, volume-preserving actions of lattices on 4-manifolds

Benson Farb [a], Peter B. Shalen [b]

[a] Dept. of Mathematics, University of Chicago, 5734 University Ave., Chicago, IL 60637, USA
[b] Dept. of Mathematics, University of Illinois at Chicago, Chicago, IL 60680, USA

Received 28 February 2002; accepted 4 March 2002

Note presented by Étienne Ghys.

Abstract We prove that if Γ is a lattice of $\mathbf{Q}$-rank at least 7 in a simple linear Lie group, then any real-analytic, volume-preserving action of Γ on a closed 4-manifold of nonzero Euler characteristic factors through a finite group action. ***To cite this article: B. Farb, P.B. Shalen, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1011–1014.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Actions analytiques réelles, conservant le volume, de réseaux sur les variétés de dimension 4

Résumé Soit Γ un réseau dans un groupe de Lie linéaire simple, dont le rang rationnel est supérieur ou égal à 7, et soit M une variété fermée de dimension 4 dont la caractéristique d'Euler–Poincaré est non nulle. Nous montrons que toute action analytique réelle de Γ sur M, qui conserve le volume, se factorise à travers l'action d'un groupe fini. ***Pour citer cet article : B. Farb, P.B. Shalen, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1011–1014.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

1. Results

Zimmer conjectured in [14] (*see* also [8]) that the standard action of $\mathrm{SL}(n, \mathbf{Z})$ on the n-torus is minimal in the following sense:

CONJECTURE 1.1. – *Any smooth, volume-preserving action of any finite-index subgroup $\Gamma < \mathrm{SL}(n, \mathbf{Z})$ on a closed r-manifold factors through a finite group action if $n > r$.*

While Conjecture 1.1 has been proved for actions which also preserve an extra geometric structure such as a pseudo-Riemannian metric (*see*, e.g., [14]), almost nothing is known in the general case. For $r = 2$ and $n > 4$, the conjecture was proved for real-analytic actions in [5] and [2]. Quite recently, Polterovich [10] has brought ideas from symplectic topology to the problem, using these to give a proof of Conjecture 1.1 for orientable surfaces of genus > 1; his methods actually prove Conjecture 1.1 for the torus as well (*see* [3]). For $r = 3$, Conjecture 1.1 is known only in some special cases where Γ contains some torsion and the action is real-analytic (*see* [2]).

E-mail addresses: farb@math.uchicago.edu (B. Farb); shalen@math.uic.edu (P.B. Shalen).

S1631-073X(02)02347-6/FLA

The main result of this note, Theorem 1.2 below, implies that Conjecture 1.1 is true in the case where $r = 4$, $n \geqslant 8$, M has nonzero Euler characteristic, and the action is real-analytic. To state the general version of the theorem, we follow the conventions used by Witte in [12]. Consider a nonuniform lattice Γ in a simple linear Lie group G with $\mathbf{R}$-rank$(G) \geqslant 2$. Then G may be given the structure of an algebraic group over $\mathbf{Q}$ in such a way that Γ is commensurate with the group of $\mathbf{Z}$-points in G. After passing to a torsion-free subgroup of finite index, one deduces this from Margulis's Arithmeticity Theorem and Remark 6.17 of [12]. We then define the $\mathbf{Q}$-*rank of* to be the $\mathbf{Q}$ rank of G with this $\mathbf{Q}$-structure; it follows from Theorem 2.10 of [12] that this notion of $\mathbf{Q}$-rank is well-defined.

THEOREM 1.2. – *Let Γ be a lattice of $\mathbf{Q}$-rank $\geqslant 7$ in a simple linear Lie group G. Then any real-analytic, volume-preserving action of Γ on a closed 4-manifold of nonzero Euler characteristic factors through a finite group action.*

The main ingredient in the proof of Theorem 1.2 is Theorem 7.1 of [2] on real-analytic actions which preserve a volume form. This theorem, which is the most difficult result in [2], gives an invariant submanifold *of codimension at least* 2 for centralizers of elements with fixed-points. This is precisely where we use the hypothesis in Theorem 1.2 that the action preserves volume. One can then complete the proof by applying results of [11], which show that real-analytic (not necessarily area-preserving) actions of certain lattices on 2-dimensional manifolds must factor through finite groups.

For the case of symplectic actions, some further progress on Conjecture 1.1 can be found in [10].

2. Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we will need two algebraic properties of lattices with large $\mathbf{Q}$-rank.

PROPOSITION 2.1. – *Let Γ be a lattice of $\mathbf{Q}$-rank d in a simple linear Lie group G. Then the following hold*:

(1) *If $d \geqslant 7$ then Γ contains commuting subgroups A and B which are respectively isomorphic to lattices of $\mathbf{Q}$-rank 2 and $d - 3$ in simple linear Lie groups.*
(2) *If $d \geqslant 4$ then Γ contains a torsion-free nilpotent subgroup which is not metabelian.*

Proof. – Without loss of generality, we may assume that G is a $\mathbf{Q}$-algebraic group of $\mathbf{Q}$-rank d and that Γ is the group of $\mathbf{Z}$-points of G.

The proof of the first statement is similar to that of Proposition 2.1 of [2]. Note that, after passing if necessary to a $\mathbf{Q}$-split subgroup of the algebraic $\mathbf{Q}$-group G whose root system is the reduced subsystem of the $\mathbf{Q}$-root system of G, we may assume G is $\mathbf{Q}$-split.

Since G is $\mathbf{Q}$-simple, the $\mathbf{Q}$-root system Φ of G is irreducible, and the Dynkin diagram determined by Φ therefore appears in the list given in Section 11.4 of [7]. By going through this list, one sees that in every case where $d \geqslant 7$, one may "erase a vertex" of the diagram to obtain a graph with 2 components: one with two vertices and another which is a Dynkin diagram with at least $d - 3$ vertices. Let G_1 and G_2 be the root subgroups corresponding to these two components of the Dynkin diagram. Then the group of $\mathbf{Q}$-points of G_1 has $\mathbf{Q}$-rank at least 2, the group of $\mathbf{Q}$-points of G_2 has $\mathbf{Q}$-rank at least $d - 3$, and G_1 commutes with G_2.

Now $\Gamma_i = \Gamma \cap G_i$ is the group of $\mathbf{Z}$-points of the algebraic $\mathbf{Q}$-group G_i. The groups $A = \Gamma_1$ and $B = \Gamma_2$ have the required properties.

To prove the second statement, note that since G has $\mathbf{Q}$-rank $\geqslant 4$, we can find a connected, nilpotent Lie subgroup N of G which is defined over $\mathbf{Q}$ and has derived length $\geqslant 3$, i.e., is not metabelian. As $\Gamma \cap N$ is the group of $\mathbf{Z}$-points of the $\mathbf{Q}$-group N, it is a lattice in N, and in particular is Zariski-dense in N. Hence $\Gamma \cap N$ is nilpotent and has no metabelian subgroup of finite index. As $\Gamma \cap N$ must have a torsion-free subgroup of finite index, the assertion follows. □

We now turn to the proof of Theorem 1.2. We shall say that a group action $\rho:\Gamma\to \mathrm{Diff}(M)$ is *finite* if ρ has finite image, and *infinite* otherwise. We assume that the lattice Γ, of $\mathbf{Q}$-rank $d\geqslant 7$, admits an infinite, volume-preserving, real-analytic action on M, a 4-manifold of nonzero Euler characteristic; this will lead to a contradiction. By part (1) of Proposition 2.1, Γ contains commuting subgroups A and B which are isomorphic to lattices of $\mathbf{Q}$-rank 2 and $d-3\geqslant 4$ respectively.

Let γ_0 be any infinite order element of A. By a theorem of Fuller [4], any homeomorphism of a closed manifold of nonzero Euler characteristic has a periodic point; the proof is an application of the Lefschetz fixed-point theorem and basic number theory. Hence some positive power γ of γ_0 has a fixed point.

We will also need the following two facts. One of the corollaries (*see*, e.g., Corollary II.7 of [12] or Theorem VIII.3.12 of [9]) of the Margulis Superrigidity theorem is that if Λ is commensurable with the group of $\mathbf{Z}$-points a $\mathbf{Q}$-simple algebraic $\mathbf{Q}$-group G with $\mathbf{Q}\text{-rank}(G)\geqslant 1$ and $\mathbf{R}\text{-rank}(G)\geqslant 2$, then any representation of Λ into a compact Lie group must have finite image. Since $\mathbf{R}\text{-rank}(B)\geqslant Q\text{-rank}(B)\geqslant 4$, this fact together with the Superrigidity Theorem itself implies that any representation of B into $\mathrm{GL}(4,\mathbf{R})$ has finite image. Second, since Γ is a lattice in a simple linear Lie group G of $\mathbf{R}$-rank $\geqslant 2$, the Margulis Finiteness Theorem (*see*, e.g., Theorem 8.1 of [15]) gives that Γ is *almost simple* in the sense that any normal subgroup of Γ must be finite or of finite index.

The properties of Γ, A and B that we have stated show that they satisfy the hypotheses of Theorem 7.1 of [2] (with $n=4$). For the reader's convenience we recall the statement here.

THEOREM 7.1 OF [2]. – *Let Γ be an almost simple group. Suppose we are given an infinite, volume-preserving, real-analytic action of Γ on a closed, connected n-manifold M. Suppose further that Γ contains commuting subgroups A and B with the following properties*:

– *There exists an element $\gamma\in A$, noncentral in Γ, having a fixed point in M.*
– *A is isomorphic to a lattice of $\mathbf{Q}$-rank $\geqslant 2$.*
– *B is noncentral in Γ.*
– *Any representation of any finite-index subgroup of B in $\mathrm{GL}(n,\mathbf{R})$ has finite image.*

Then there is a nonempty, connected, real-analytic submanifold $W\subset M$ of codimension at least 2 which is invariant under a finite-index subgroup B' of B. Furthermore, the action of this subgroup on W is infinite.

Remark 2.2. – The action of B' on the surface W produced by this theorem is *not* necessarily area preserving.

We now conclude the proof of Theorem 1.2. Let B' be the subgroup, and W the submanifold, given by Theorem 7.1 of [2]. Then B' is a lattice of $\mathbf{Q}$-rank at least 4, W is a compact, connected manifold of dimension 0, 1 or 2, and the action of B' on W is infinite. If $\dim W=0$ we have an immediate contradiction, since no group admits an infinite action on a point. If $\dim W=1$ then we have a contradiction to Witte's theorem [13] that a lattice of $\mathbf{Q}$-rank $\geqslant 2$ admits no infinite action on S^1. (For a generalization of Witte's result, see Burger–Monod [1] or Ghys [6].) Now suppose that $\dim W=2$, so that W is a compact, connected surface. It follows from part (2) of Proposition 2.1 that B' contains a torsion-free nilpotent subgroup H which is not metabelian. But Rebelo [11] showed that any nilpotent group of real-analytic diffeomorphisms of a compact, connected surface must be metabelian. (Rebelo states his result only in the orientable case. However, an action of a nilpotent group on a non-orientable surface gives rise to an action of a $\mathbf{Z}/2\mathbf{Z}$-extension of that group on the orientable double cover; since a $\mathbf{Z}/2\mathbf{Z}$ extension of a nilpotent group is nilpotent, it follows that Rebelo's result holds in the non-orientable case.) Hence the action of H on W is not effective. Since H is torsion-free, there is an infinite-order element of $H\leqslant B'$ which acts trivially on W, so that the action of B' on W has infinite kernel. Since B' is almost simple by the Margulis finiteness theorem, this kernel must have finite index in B', so that the action of B' on W is finite, and we again have a contradiction. $\square$

Acknowledgements. Nous remercions Étienne Ghys d'avoir lu le manuscrit avec une rapidité exceptionnelle, et de nous avoir transmis des commentaires qui nous ont permis d'améliorer nettement la qualité de l'exposition.

Both authors are supported in part by the NSF.

References

[1] M. Burger, N. Monod, Bounded cohomology of lattices in higher rank Lie groups, J. European Math. Soc. 1 (2) (1999) 199–235.
[2] B. Farb, P. Shalen, Real-analytic actions of lattices, Invent. Math. 135 (2) (1998) 273–296.
[3] B. Farb, P. Shalen, Some remarks on symplectic actions of discrete groups, in preparation.
[4] F.B. Fuller, The existence of periodic points, Ann. of Math. 57 (2) (1953) 229–230.
[5] E. Ghys, Sur les Groupes Engendrés par des Difféomorphismes Proches de l'Identité, Bol. Soc. Bras. Mat. 24 (2) (1993) 137–178.
[6] E. Ghys, Actions de réseaux sur le cercle, Invent. Math. 137 (1) (1999) 199–231.
[7] J. Humphreys, Introduction to Lie Algebras and Representation Theory, GTM 9, 3rd edn., Springer-Verlag, 1972.
[8] F. Labourie, Large groups actions on manifolds, in: Proc. I.C.M., Berlin, 1998, Doc. Math., Extra Vol. II, 1998, pp. 371–380.
[9] G. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag, 1991.
[10] L. Polterovich, Growth of maps, distortion in groups and symplectic geometry, Preprint, November 2001.
[11] J. Rebelo, On nilpotent groups of real analytic diffeomorphisms of the torus, C. R. Acad. Sci. Paris 331 (1) (2000) 317–322.
[12] D. Witte, Introduction to arithmetic groups, Preprint, October 2001.
[13] D. Witte, Arithmetic groups of higher **Q**-rank cannot act on 1-manifolds, Proc. Amer. Math. Soc. 122 (2) (1994) 333–340.
[14] R. Zimmer, Actions of semisimple groups and discrete subgroups, in: Proc. I.C.M., Berkeley, 1986, pp. 1247–1258.
[15] R. Zimmer, Ergodic Theory and Semisimple Groups, Monographs in Math., Vol. 81, Birkhäuser, 1984.