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Symplectic capacities of toric manifolds and combinatorial inequalities

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Abstract We shall give concrete estimations for the Gromov symplectic width of toric manifolds in combinatorial data. As by-products some combinatorial inequalities in the polytope theory are obtained. ***To cite this article: G. Lu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 889–892.***

Capacités symplectiques de variétés toriques et inéqualités combinatoires

Résumé On obtient des estimations concrètes pour le largeur symplectique de Gromov pour les variétés toriques par ses données combinatoires. Comme un sous-produit, quelques inéqualités combinatoires dans la théorie de polytope sont obtenus. ***Pour citer cet article : G. Lu, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 889–892.***

The toric manifolds are a very beautiful family of Kähler manifolds. Since they admit a combinatorial description it is very interesting to estimate their (pseudo) symplectic capacities in terms of combinatoral data. Recall that the Gromov symplectic width $\mathcal{W}_G(M,\omega)$ of a $2n$-dimensional symplectic manifold (M,ω) is defined by the supremum of all numbers πr^2 for which there exists a symplectic embedding from a ball $B^{2n}(r)$ in $(\mathbb{R}^{2n},\omega_0)$ of radius r into (M,ω). It is the first symplectic capacity. Recently, the author introduced the notion of *pseudo symplectic capacity* [6]. Let us begin by briefly recalling it. For its properties and applications the reader refer to [6]. Given a connected symplectic manifold (M,ω) of dimension $2n$ and a smooth function H on it let X_H denote the symplectic gradient of H. An isolated critical point p of H is called *admissable* if the spectrum of the linear transformation $DX_H(p): T_pM \to T_pM$ is contained in $\mathbb{C}\setminus\{\lambda \mathrm{i} \mid 2\pi \leqslant \pm\lambda < +\infty\}$. For two given nonzero homology classes $\alpha_0, \alpha_\infty \in H_*(M)$ we denote by $\mathcal{H}_{ad}(M,\omega;\alpha_0,\alpha_\infty)$ (resp. $\widehat{\mathcal{H}}_{\mathrm{ad}}(M,\omega;\alpha_0,\alpha_\infty)$) the set of all smooth functions on M for which there exist two smooth compact submanifolds P and Q of M with connected smooth boundaries and of codimension zero such that the following condition groups (a)–(f) (resp. (a)–(e), (g)) are satisfied:

(a) $P \subset \mathrm{Int}(Q)$ and $Q \subset \mathrm{Int}(M)$;
(b) $H|_P = 0$ and $H|_{M-\mathrm{Int}(Q)} = \max H$;
(c) $0 \leqslant H \leqslant \max H$;
(d) There exist chain representatives of α_0 and α_∞, still denoted by α_0, α_∞, such that $\mathrm{supp}(\alpha_0) \subset \mathrm{Int}(P)$ and $\mathrm{supp}(\alpha_\infty) \subset M \setminus Q$;

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(e) H has only finitely many critical points in $\mathrm{Int}(Q)\setminus P$ and each of them is admissible in the above sense;
(f) The Hamiltonian system $\dot{x}=X_H(x)$ on M has no nonconstant periodic solutions of period less than 1;
(g) The Hamiltonian system $\dot{x}=X_H(x)$ on M has no nonconstant contractible periodic solutions of period less than 1.

If $\alpha_0\in H_0(M)$ can be represented by a point we allow P to be an empty set. If M is a closed manifold and $\alpha_\infty\in H_0(M)$ is represented by a point, we also allow $Q=M$.

The pseudo symplectic capacities of Hofer–Zehnder type are defined by

$$\begin{cases} C^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty):=\sup\{\max H \mid H\in\mathcal{H}_{ad}(M,\omega;\alpha_0,\alpha_\infty)\},\\ \widehat{C}^{(2)}_{HZ}(M,\omega;\alpha_0,\alpha_\infty):=\sup\{\max H \mid H\in\widehat{\mathcal{H}}_{\mathrm{ad}}(M,\omega;\alpha_0,\alpha_\infty)\}.\end{cases}\tag{1}$$

In this Note we denote by pt the generator of $H_0(M)$ represented by a point, and always make the convention that $\sup\emptyset=0$ and $\inf\emptyset=+\infty$.

1. The pseudo symplectic capacity of toric manifolds

For the following related knowedge on the toric manifolds the reader may refer to [1,2,5]. Let Σ be a complete regular fan in $\mathbb{R}^n$ and $G(\Sigma)=\{u_1,\dots,u_d\}$ the set of all generators of 1-dimensional cones in Σ. Denote by P_Σ the toric manifold associated with Σ. It is well known that every Kähler form on P_Σ can be represented by a strictly convex support function φ for Σ and that every strictly convex support function for Σ represents a Kähler form on P_Σ. Therefore, in this Note we shall use the same letter to denote a Kähler form on P_Σ and the corresponding strictly convex support function for Σ when the context makes our meaning clear. In the following we denote by $\mathbb{Z}_{\geqslant 0}$ the set of all nonnegative integers.

THEOREM 1. – *Under the assumptions above let ω be a strictly convex support function for Σ. Then it holds that*

$$\Upsilon(\Sigma,\omega):=\frac{1}{2\pi}\inf\left\{\sum_{k=1}^d\omega(u_k)a_k>0\;\middle|\;\sum_{k=1}^d a_ku_k=0,\ a_k\in\mathbb{Z}_{\geqslant 0},\ k=1,\dots,d\right\}>0,\tag{2}$$

and that for every $n\geqslant 2$,

$$\mathcal{W}_G(\mathrm{P}_\Sigma,\omega)\leqslant C_{HZ}\big(\mathrm{P}_\Sigma,\omega;\,pt,PD\big([\omega]\big)\big)\leqslant 2\pi\cdot\Upsilon(\Sigma,\omega).\tag{3}$$

In particular, let us consider a Delzant polytope in $(\mathbb{R}^n)^*$

$$\triangle=\bigcap_{k=1}^d\big\{x\in\big(\mathbb{R}^n\big)^*\;\big|\;l_k(x):=x(u_k)-\lambda_k\geqslant 0\big\}\tag{4}$$

(*cf.* [1,5]), where d is the number of the $(n-1)$-dimensional faces of $\triangle$, u_k is a uniquely primitive element of the lattice $\mathbb{Z}^n\subset\mathbb{R}^n$ (the inward-pointing normal to the k-th face of $\triangle$), and λ_k is a real number. Denote by $X_\triangle$ the toric manifold associated with the fan generated by $\triangle$, and by $\omega_\triangle$ the canonical symplectic form on it.

THEOREM 2. – *Under the assumptions above, it holds that*

$$\Upsilon(\triangle):=\inf\left\{-\sum_{k=1}^d\lambda_ka_k>0\;\middle|\;\sum_{k=1}^d a_ku_k=0,\ a_k\in\mathbb{Z}_{\geqslant 0},\ k=1,\dots,d\right\}>0,\tag{5}$$

and that for any $n\geqslant 2$,

$$\mathcal{W}_G(X_\triangle,\omega_\triangle)\leqslant C_{HZ}\big(X_\triangle,\omega_\triangle;\,pt,\mathrm{PD}\big([\omega_\triangle]\big)\big)\leqslant 2\pi\cdot\Upsilon(\triangle).\tag{6}$$

Moreover, if $\mathrm{Vert}(\triangle)$ *denotes the set of all vertexes of $\triangle$ and $E_p(\triangle)$ is the shortest distance from the vertex p to the adjacent n vertexes, then for any capacity function c,*

$$2\pi\cdot\max_{p\in\mathrm{Vert}(\triangle)}E_p(\triangle)\leqslant c(X_\triangle,\omega_\triangle).\tag{7}$$

Remark 3. – For the n-simplex $\triangle = \triangle_n$ in $(\mathbb{R}^n)^*$ spanned by the origin and the dual basis $e_1^*, \ldots, e_n^*$ the associated toric manifold $(X_{\triangle_n}, \omega_{\triangle_n})$ is $(\mathbb{CP}^n, 2\omega_{FS})$ with $\int_{\mathbb{CP}^1} \omega_{FS} = \pi$. It is easily seen that $\Upsilon(\triangle_n) = 1$. Thus the estimates in (6) are optimal. In particular, it follows from the proof of Theorem 2 that

$$\mathcal{W}_G\big(\triangle^n(1) \times \square^n(2\pi), \omega_0\big) \leqslant \mathcal{W}_G\big(\triangle^n(1) \times \mathbb{T}^n, \omega_{\mathrm{can}}\big) \leqslant 2\pi,$$

where $\triangle^n(a) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{>0} \mid \sum_{k=1}^n x_k < a\} \subset \mathbb{R}^n$ and $\square^n(a) = \{(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \mid 0 < \theta_k < a, \forall 1 \leqslant k \leqslant n\}$ for any $a > 0$. But from Theorem 5.1 in [10] one can only get $\mathcal{W}_G\big(\triangle^n(1) \times \square^n(2\pi), \omega_0\big) \leqslant 8n\pi$.

Examples. – (i) Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$ and $u_1 = e_1$, $u_2 = -e_1$, $u_3 = e_2$, $u_4 = e_3$, $u_5 = -e_2 - e_3 - 2e_1$. Consider a fan $\Sigma \subset \mathbb{R}^3$ in which $G(\Sigma) = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of all generators of 1-dimensional cones and whose set of primitive collections is $\{\{u_1, u_2\}, \{u_3, u_4, u_5\}\}$. It is easily checked that this fan is complete and regular. Its associated toric manifold P_Σ is the Fano threefold $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2) \oplus 1)$. Note that each strictly convex support function for Σ can be determined by its values at points u_i, $i = 1, \ldots, 5$. Let ω be a Σ-piecewise linear function such that $\omega(u_i) = 1$, $i = 1, \ldots, 5$. It is easy to prove that it is a strictly convex support function for Σ and that $\Upsilon(\Sigma, \omega) = 1/\pi$. Thus by Theorem 1 we get $\mathcal{W}_G(\mathrm{P}_\Sigma, \omega) \leqslant C_{HZ}(\mathrm{P}_\Sigma, \omega; pt, PD([\omega])) \leqslant 2$.

(ii) Consider a Delzant polytope $\triangle \subset (\mathbb{R}^3)^*$ with vertices $v_0 = 0$, $v_1 = e_1^*$, $v_2 = e_2^*$, $v_3 = (1-a)e_2^* + ae_3^*$, $v_4 = ae_3^*$, $v_5 = (1-a)e_1^* + ae_3^*$. Here $0 < a < 1$ and e_1^*, e_2^*, e_3^* are the dual basis of the standard basis e_1, e_2, e_3 in $\mathbb{R}^3$. It is easy to see that the normal vectors to the 2-dimensional faces are $u_1 = e_1^*$, $u_2 = e_2^*$, $u_3 = e_3^*$, $u_4 = -e_3^*$, $u_5 = -e_1^* - e_2^* - e_3^*$. Furthermore, $\triangle$ can be expressed as the intersection of the half spaces $\langle x, u_j\rangle \geqslant 0$, $j = 1, 2, 3$, and $\langle x, u_4\rangle \geqslant -a$, $\langle x, u_5\rangle \geqslant -1$. Thus $\Upsilon(\triangle) = a$ and it follows from Theorem 2 that the associated toric manifold $(X_\triangle, \omega_\triangle)$ has the capacities

$$\mathcal{W}_G(X_\triangle, \omega_\triangle) \leqslant C_{HZ}\big(X_\triangle, \omega_\triangle; pt, PD\big([\omega_\triangle]\big)\big) \leqslant 2\pi a.$$

Notice that the toric manifold $(X_\triangle, \omega_\triangle)$ is exactly the blow-up of $(\mathbb{CP}^3, 2\omega_{FS})$ of weight $2(1-a)$ at a point. That is, it is obtained by removing the interior of a symplectic embedding ball $(B^6(\sqrt{2(1-a)}), \omega_0)$ of radius $\sqrt{2(1-a)}$ in $(\mathbb{CP}^3, 2\omega_{FS})$ and collapsing the bounding sphere to the exceptional divisor by the Hopf map.

2. Seshadri constants

For a compact complex manifold (M, J) of dimension n, and an ample line bundle $L \to M$ Demailly [4] defined the *Seshadri constant* of L at a point $x \in M$ to be the nonnegative real number

$$\varepsilon(L, x) := \inf_{C \ni x} \frac{\int_C c_1(L)}{\mathrm{mult}_x C}, \tag{8}$$

where the infimum is taken over all irreducible curves passing through the point x, and $\mathrm{mult}_x C$ is the multiplicity of C at x. The global Seshadri constant is defined by

$$\varepsilon(L) := \inf_{x \in M} \varepsilon(L, x). \tag{9}$$

For more details the reader should refer to [4,3] and the references therein.

Let the toric manifold P_Σ be as in Theorem 1 and $L_k = L_k(\Sigma) \to \mathrm{P}_\Sigma$ the corresponding line bundles to the standard toric divisors $D_k(\Sigma)$, $k = 1, \ldots, d$. It is well known that the Chern class $c_1(L_k)$ is Poincaré dual to $[D_k] \in H_2(\mathrm{P}_\Sigma, \mathbb{Z})$ for each k. For $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ consider the line bundle $L = L_1^{m_1} \otimes \cdots \otimes L_d^{m_d}$. By the toric manifold theory it is ample if and only if the Σ-piecewise linear function $\varphi_L = \varphi_{(m_1 \cdots m_d)}) \in \mathrm{PL}(\Sigma)$ determined by $\varphi_L(u_k) = m_k$, $k = 1, \ldots, d$, is a strictly convex support function.

THEOREM 4. – *Let* P_Σ *be the toric manifold associated with a complete regular fan* Σ *in* $\mathbb{R}^n$ *and* $L \to \mathrm{P}_\Sigma$ *an ample line bundle on it. If* φ_L *be any strictly convex support function in* $\mathrm{PL}(\Sigma)$ *representing the class* $c_1(L)$ *then*

$$\varepsilon(L) \leqslant 2\pi \cdot \Upsilon(\Sigma, \varphi_L). \tag{10}$$

Furthermore, if $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ *is such that the* Σ*-piecewise linear function* $\varphi_{(m_1\cdots m_d)}$ *in* (10) *is a strictly convex support function, then*

$$\varepsilon\big(L_1^{m_1} \otimes \cdots \otimes L_d^{m_d}\big) \leqslant \inf\left\{\sum_{k=1}^{d} m_k a_k > 0 \,\middle|\, \sum_{k=1}^{d} a_k u_k = 0,\ a_k \in \mathbb{Z}_{\geqslant 0},\ k = 1, \dots, d\right\}.$$

3. The strategies of proof of the main results

We only outline the proof of Theorem 1. For a closed symplectic manifold (M, ω), by Proposition 1.8, Theorem 1.12 and Remark 1.13 in [6] we know that if there exist homology classes $A \in H_2(M, \mathbb{Z})$ and $\alpha_i \in H_*(M, \mathbb{Q})$, $i = 1, \dots, m$, such that the Gromov–Witten invariant $\Psi^{(M,\omega)}_{A,0,m+1}(pt; pt, \alpha_1, \dots, \alpha_m) \neq 0$ then $\mathcal{W}_G(M, \omega) \leqslant C_{HZ}(M, \omega; pt, PD([\omega])) \leqslant \omega(A)$. Since such A has always the representives of rational curves it follows from the Gromov compactness theorem that the infimum $\mathrm{GW}_0(M, \omega; pt, PD([\omega]))$ of all $\omega(A)$ when A taking over such classes is more than zero. If $\mathrm{GW}_0(M, \omega; pt, PD([\omega]))$ is finite the symplectic manifold (M, ω) is called strong 0-symplectic uniruled in Definition 1.16 of [6]. Batyrev's compuation for the quantum cohomology rings of toric manifolds [2] (*cf.* [8] for a rigorous explanation) showed that the toric manifolds are strong 0-symplectic uniruled. Precisely, under the assumptions of Theorem 1 let us denote by $R(\Sigma) = \{\mu = (\mu_1, \dots, \mu_d) \in \mathbb{Z}^d \mid \mu_1 u_1 + \cdots + \mu_d u_d = 0\}$ and $D_k(\Sigma)$ the toric divisors of P_Σ, $k = 1, \dots, d$. For $A \in H_2(\mathrm{P}_\Sigma, \mathbb{Z})$ let $\mu_k(A)$ denote the intersection numbers $A \cdot D_k(\Sigma)$, $k = 1, \dots, d$. Then $(\mu_1(A), \dots, \mu_d(A)) \in R(\Sigma)$ and the map $H_2(\mathrm{P}_\Sigma, \mathbb{Z}) \to R(\Sigma)$, $A \mapsto (\mu_1(A), \dots, \mu_d(A))$ is an isomorphism. Denote by Ξ_Σ the inverse map of the isomorphism. It was proved in [2] that for every $A = \Xi_\Sigma(a) \in \Xi_\Sigma(\mathbb{Z}^d_{\geqslant 0} \cap R(\Sigma)) \subset H_2(\mathrm{P}_\Sigma, \mathbb{Z})$ and any Kähler form ω on P_Σ the Gromov–Witten invariant $\Psi^{(\mathrm{P}_\Sigma,\omega)}_{A,0,m+1}(pt; pt, PD(c_1^{a_1}), \dots, PD(c_d^{a_d})) = 1$, where $m = 1 + \sum_{k=1}^{d} a_k$ and $c_k \in H^2(\mathrm{P}_\Sigma, \mathbb{Z})$ are the Poincare dual of $[D_k(\Sigma)]$, $k = 1, \dots, d$. On the other hand each Kähler form ω on P_Σ may be represented by a strictly convex support function for Σ, also denoted by ω. By the arguments in §3 of [2] we have $\omega(A) = \langle [\omega], A\rangle = \sum_{k=1}^{d} \omega(u_k) a_k$. Now Theorem 1 may be derived from these arguments.

Theorem 2 may be derived from Theorem 1, the main result in [9] and Lemma 3.11 in [7]. The proof of Theorem 4 may be completed by using Proposition 6.3 in [3] and Theorem 1.39 in [6].

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