C. R. Acad. Sci. Paris, Ser. I 334 (2002) 903–908

Probabilités/*Probability Theory*

Stochastic integration with respect to Gaussian processes

Laurent Decreusefond

E.N.S.T. 46, rue Barrault, 75634 Paris cedex 13, France

Received 7 October 2001; accepted after revision15 March 2002

Note presented by Paul Malliavin.

Abstract We construct a Stratonovitch–Skorohod-like stochastic integral for general Gaussian processes. We study its sample path regularity and one of its numerical approximating schemes. We also analyze the way it is transformed by an absolutely continuous change of probability and we give an Itô formula. ***To cite this article: L. Decreusefond, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 903–908.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Intégrale stochastique pour les processus gaussiens

Résumé Nous construisons une intégrale stochastique du type Stratonovitch–Skorohod, pour les processus gaussiens généraux. Nous montrons qu'elle peut être approchée par des sommes de type Stratonovitch et nous établissons sa régularité trajectorielle. Nous étudions aussi la façon dont elle se transforme lors d'un changement absolument continu de probabilité. Nous montrons enfin que la formule d'Itô–Stratonovitch est vérifiée. ***Pour citer cet article : L. Decreusefond, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 903–908.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

De nombreux auteurs se sont attachés à construire les fondements d'un calcul stochastique pour le mouvement brownien fractionnaire. Les diverses approches peuvent être séparées en deux principales catégories : celles qui reposent sur les propriétés trajectorielles de ce processus [7,8,16] (notamment la continuité au sens de Hölder de ses trajectoires) et celles qui utilisent sur son caractère gaussien. Ces dernières sont basées sur la représentation $B_H(t) = \int_{-\infty}^{t} K_H(t,s)\,\mathrm{d}B_s$ où K_H est un noyau déterministe. L'expérience des calculs faits pour le mouvement brownien fractionnaire montre qu'il est ardu de travailler directement avec des hypothèses sur le noyau K_H mais qu'il est plus aisé et plus informatif de travailler avec les propriétés de l'opérateur intégral canoniquement associé à ce noyau.

Pour un noyau déterministe qui satisfait l'Hypothèse I (*voir* plus bas), on considère le processus $\widetilde{X}_t = \int_0^t K(t,s)\,\mathrm{d}B_s$. Compte-tenu des résultats de [4], on sait que $\widetilde{X}$ a une version à trajectoires p.s. continues. Par conséquent, l'espace de Wiener sur lequel nous travaillons, est $\Omega = \mathcal{C}_0([0,1];\mathbf{R})$, l'espace de Cameron–Martin est $K(\mathrm{L}^2([0,1])$ et P est la probabilité sur Ω qui fait du processus canonique, X, un

E-mail address: Laurent.Decreusefond@enst.fr (L. Decreusefond).

S1631-073X(02)02360-9/FLA

processus gaussien de même loi que $\widetilde{X}$. Compte-tenu des résultats antérieurs obtenus sur le mouvement brownien fractionnaire, une définition raisonnable de l'intégrale stochastique de u par rapport à X est

$$\int_0^t u_s \, \mathrm{d}X_s = \delta\big(\mathcal{K}_t^* u\big) + \operatorname{trace}(\mathcal{K}\nabla u).$$

$\mathcal{K}_t^*$ est ici l'adjoint dans $\mathrm{L}^2([0,t])$. Ceci exige que $\mathcal{K}^* u$ appartienne au domaine de δ et que $\mathcal{K}\nabla u$ soit p.s. un opérateur à trace. On définit ensuite une intégrale de type Stratonovitch par limite de sommes discrètes qui coïncide avec la précédente lorsque $\mathcal{K}\nabla u$ est nucléaire, mais qui possède un domaine plus facile à caractériser. On peut aussi, grâce aux inégalités de Meyer, établir les hypothèses sur l'intégrand sous lesquelles on a une inégalité maximale pour la partie divergence, voir les Théorèmes 3.2 et 3.1. Les formules de composition des intégrales stochastiques par un shift absolument continu de l'espace de Wiener, montrent que la première de ces intégrales se transforme formellement comme une semi-martingale lors d'un changement de probabilité absolument continue, voir le Théorème 4.1. Pour terminer, nous établissons la formule d'Itô–Stratonovitch.

1. Introduction

In the past few years, several papers have been devoted to the stochastic analysis of fractional Brownian motion (fBm). A natural problem is now to extend the theory of stochastic calculus, developed for fBm, to a larger class of processes. Sample-paths constructions, as those devised in [8,9,16], can be applied to processes with rough paths but this is not the way we chose. As long as fBm is concerned, the other approach to construct a stochastic integral, is based on its representation as a stochastic integral with respect to a standard Brownian B: $B_H(t) = \int_{-\infty}^t K_H(t,s)\,\mathrm{d}B_s$. It is clear that the regularity of the kernel plays a key role in a tentative definition of a stochastic calculus with respect to B_H. The experience of the past works for fractional Brownian motion shows that it is simpler and more insightful to work with the property of the linear map associated to the kernel than with the properties of the kernel itself (*see* [2] for such an approach).

2. Definition

Let K be a Hilbert–Schmidt map from $\mathrm{L}^2([0,1])$ into itself of the form $(Kf)(t) = \int_0^1 K(t,s) f(s)\,\mathrm{d}s$. For any $\alpha > 0$, $1 < p < \infty$, we introduce the Slobodetzki space (*see* [7,16]):

$$\mathcal{S}_{\alpha,p} = \left\{ f : \|f\|^p_{\mathcal{S}_{\alpha,p}} := \iint_{[0,1]^2} \frac{|f(t)-f(s)|^p}{|t-s|^{1+p\alpha}} \,\mathrm{d}s\,\mathrm{d}t < \infty \right\}.$$

For any $\alpha \geqslant 0$, $\mathcal{S}_{\alpha,p} \subset \mathcal{S}_{0,p} = \mathrm{L}^p([0,1])$. Furthermore, $I_{\alpha,p}$ is continuously embedded in $\mathrm{L}^{p(1-\alpha p)^{-1}}([0,1])$ when $\alpha p < 1$ and in $\mathrm{Hol}(\alpha - 1/p)$ when $\alpha p > 1$. We introduce the following assumption.

HYPOTHESIS I. – There exists $\alpha > 0$ such that K is continuous, one-to-one, from $\mathrm{L}^2([0,1])$ into $\mathcal{S}_{\alpha+1/2,2}$. Moreover, K is triangular, i.e., $K(t,s) = 0$ for $s > t$.

Let $\widetilde{X}_t = \int_0^t K(t,s)\,\mathrm{d}B_s$. It is shown in [4] that under Hypothesis I, there exists a measurable version of $\widetilde{X}$ which has a.s. $(\alpha-\varepsilon)$-Hölder continuous sample-paths, for any $\varepsilon > 0$. Hence, the Wiener space is $\Omega = \mathcal{C}_0([0,1];\mathbf{R})$, the Cameron–Martin space is $K(\mathrm{L}^2([0,1]))$ and P, the probability on Ω under which the canonical process X has the law of $\widetilde{X}$. As usual, for $\mathcal{X}$ a separable Hilbert space, $\mathbb{D}_{2,1}(\mathcal{X})$ denote the space of $\mathcal{X}$-valued r.v. which are once Gross–Sobolev differentiable and the derivative of which belongs to $\mathrm{L}^2(\Omega\times[0,1];\mathcal{X})$. When $\alpha \geqslant 1/2$, the map $\mathcal{K}$ defined by $\mathcal{K}f = (Kf)'$, is a continuous map from $\mathrm{L}^2([0,1])$ to $\mathcal{S}_{(\alpha-1/2),2}$. For $\alpha < 1/2$, we will use the following assumption:

HYPOTHESIS II. – For $\alpha < 1/2$, we assume that $\mathcal{K}$ is continuous from $\mathcal{S}_{1/2-\alpha,2}$ into $\mathrm{L}^2([0,1])$ and is a densely defined, closable operator from $\mathrm{L}^2([0,1])$ into itself. We denote by $\mathcal{K}$ its closure and by $\mathcal{K}^*$ its adjoint. We furthermore assume that $\operatorname{Dom}\mathcal{K} \cap \operatorname{Dom}\mathcal{K}^*$ is dense in $\mathrm{L}^2([0,1])$.

Remark 1. – Hypothesis I and II are satisfied for the kernel corresponding to fBm (with stationary increments or not), cf. [6]. For the kernel $K(t,s) = K_{H(t)}(t,s)$, these hypothesis are also satisfied for some $\alpha > 0$, if there exists $\gamma > \alpha$ such that $H(\cdot)$ is a γ-Hölder continuous function and $\inf_t H(t) \geqslant \alpha$. This is the generalization of the fBm introduced in [3]. Note that in all the subsequent theorems, the additional hypotheses are all satisfied by K_H.

For the sake of simplicity, we will speak of the domains of $\mathcal{K}$ and $\mathcal{K}^*$ independently of the position of α with respect to $1/2$. It must be clear that for $\alpha \geqslant 1/2$, $\operatorname{Dom}\mathcal{K} = \operatorname{Dom}\mathcal{K}^* = \mathrm{L}^2([0,1])$. We denote by $\mathcal{K}_t^*$ the adjoint of $\mathcal{K}$ on $\mathrm{L}^2([0,t])$ and by ∇ the L^2-valued Gross–Sobolev derivative of the stochastic calculus of variations (usually denoted $\dot{\nabla}$) – *see* [10,15]. The adjoint of ∇ with respect to P is called divergence and is denoted by δ, its domain is $\operatorname{Dom}\delta$. We denote by $\operatorname{Dom}\delta_{\mathcal{K}^*}$, the set of processes u belonging a.s. to $\operatorname{Dom}\mathcal{K}^*$ and such that $\mathcal{K}^*u$ belongs to $\operatorname{Dom}\delta$. $\operatorname{Dom}\delta_X$, represents the set of processes u in $\operatorname{Dom}\delta_{\mathcal{K}^*}$ such that $\nabla\mathcal{K}^*u$ is P-a.s. a trace class operator. Following the definition of a stochastic integral with respect to fBm first introduced in [6], then in [1] and also in [2] for some Gaussian processes, we set:

DEFINITION 2.1. – Assume that Assumption I and II hold. For $u \in \operatorname{Dom}\delta_X$, we define the stochastic integral on $[0,t]$, of u with respect to X by

$$\int_0^t u_s * \mathrm{d}X_s \stackrel{\text{def}}{=} \delta\big(\mathcal{K}_t^* u\big) + \operatorname{trace}\big(\nabla\big(\mathcal{K}_t^* u\big)\big).$$

It is useful to note that $\operatorname{trace}(\nabla(\mathcal{K}_t^* u)) = \operatorname{trace}((\mathcal{K}\nabla)(u\mathbf{1}_{[0,t]})\mathbf{1}_{[0,t]})$ provided that one of these two terms exists.

Another approach is to follow the Stratonovitch integration principles. We first to approximate B by a linear interpolation to obtain an approximation of X, then obtain an approximation of $\mathrm{d}X$. This leads to consider:

$$\mathrm{BS}_\pi(u) = \sum_{t_i \in \pi} \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \mathcal{K}^* u(s)\,\mathrm{d}s\,\Delta B_i.$$

We say that u is Stratonovitch integrable when the family BS_π converges in probability as $|\pi| \to 0$. The limit is denoted by $\int u(s) \circ \mathrm{d}X_s$. The proofs of the following results if not given here, are established in a paper in preparation [5].

THEOREM 2.2. – *Assume that Hypothesis* I *and* II *hold for some* $\alpha < 1/2$. *Assume furthermore that there exists* $\eta > 0$ *such that* $\mathcal{K}^*$ *is continuous from* $\mathcal{S}_{1+\eta-\alpha,2}$ *into* $\mathcal{S}_{1/2+\eta,2}$. *If* u *belongs to* $\mathbb{D}_{2,1}(\mathcal{S}_{1+\eta-\alpha,2})$, *there exists a measurable and integrable process, denoted by* $\widetilde{D}u$ *such that, for almost any* r,

$$E\big[\big|\mathcal{K}^*\nabla_r u(s) - \widetilde{D}u(r)\big|\big] \leqslant c|s-r|^\eta \|\nabla_r u\|_{\mathrm{L}^2(\Omega;\mathcal{S}_{1+\eta-\alpha,2})}. \tag{1}$$

Moreover,

$$E\left[\left\|\int_0^{\cdot} \widetilde{D}u(r)\,\mathrm{d}r\right\|^2_{\mathrm{Hol}(1/2)}\right] \leqslant c\|u\|^2_{\mathbb{D}_{2,1}(\mathcal{S}_{1+\eta-\alpha,2})}. \tag{2}$$

Furthermore, u *is Stratonovith integrable and the limit of* $\mathrm{BS}_\pi(u)$ *is* $\delta(\mathcal{K}^*u) + \int_0^1 \widetilde{D}u(r)\,\mathrm{d}r := \int u(s) \circ \mathrm{d}X_s$.

THEOREM 2.3. – *Assume that Hypothesis* I *holds for* $\alpha > 1/2$. *Assume furthermore that* $\mathcal{K}^*$ *is continuous from* $\mathrm{L}^p([0,1])$ *into* $\mathcal{S}_{\alpha-1/2,p}$ *for some* $p > (\alpha - 1/2)^{-1}$. *If* u *belongs to* $\mathbb{D}_{p,1}(\mathrm{L}^p([0,1]))$, *then there exists a measurable and integrable process, denoted by* $\widetilde{D}u$ *such that, for almost any* r,

$$E\left[\left|\nabla_r \mathcal{K}^* u(s) - \widetilde{D}u(r)\right|\right] \leqslant c|s-r|^{\alpha-1/2-1/p}\|\nabla_r u\|_{\mathrm{L}^p(\Omega\times[0,1])}. \tag{3}$$

Moreover,

$$E\left[\left\|\int_0^{\cdot} \widetilde{D}u(r)\,\mathrm{d}r\right\|^p_{\mathrm{Hol}(1-1/p)}\right] \leqslant c\|u\|^p_{\mathbb{D}_{p,1}(\mathrm{L}^p([0,1]))}. \tag{4}$$

Furthermore, u is Stratonovith integrable and the limit of $\mathrm{BS}_\pi(u)$ *is* $\delta(\mathcal{K}^* u) + \int_0^1 \widetilde{D}u(r)\,\mathrm{d}r$.

Remark 2. – As in [11], it follows that if u and $\mathcal{K}^*$ satisfies the hypothesis of the previous theorems and if $\nabla\mathcal{K}^* u$ is a trace class operator then, the two integrals $\int u(s) * \mathrm{d}X_s$ and $\int u(s)\circ \mathrm{d}X_s$ coincide.

3. Regularity of the trajectories

We now turn to the regularity of the stochastic integral. As will be seen, $\mathcal{K}^*$ is always continuous from $(\mathcal{S}_{\alpha-1/2,2})'$ into $\mathrm{L}^2([0,1])$. We denote by $\|\mathcal{K}_1^*\|_{\alpha,2}$ its norm.

THEOREM 3.1. – *For any* $\alpha\in(0,1/2)$, *assume that assumptions* I *and* II *hold. Let* u *belong to* $\mathbb{D}_{2,1}(\mathcal{S}_{\eta+(1/2-\alpha),2})\cap \mathrm{Dom}\,\delta_{\mathcal{K}^*}$ *with* $\eta>0$. *The process* $\{\delta(\mathcal{K}_t^* u),\ t\in[0,1]\}$ *admits a modification with* γ*-Hölder continuous paths for any* $0<\gamma<\eta$ *and we have the maximal inequality*:

$$\left\|\delta\left(\mathcal{K}_{\cdot}^* u\right)\right\|_{\mathrm{L}^2(\Omega;\mathrm{Hol}(\gamma))} \leqslant c\left\|\mathcal{K}_1^*\right\|^2_{\alpha,2}\|u\|_{\mathbb{D}_{2,1}(\mathcal{S}_{\eta+(1/2-\alpha),2})}.$$

Proof. – Note that since K is triangular, we have for any $f\in\mathrm{Dom}\,\mathcal{K}^*$, $\mathcal{K}_t^* f \equiv \mathcal{K}_1^*(f\mathbf{1}_{[0,t]})\mathbf{1}_{[0,t]}$. Thus, $\delta(\mathcal{K}_t^* u)-\delta(\mathcal{K}_s^* u)=\delta(\mathcal{K}_1^* u\mathbf{1}_{[s,t]})$.

Since the divergence operator is continuous from $\mathbb{D}_{2,1}(\mathrm{L}^2([0,1]))$ in $\mathrm{L}^2(\Omega)$, we have

$$E\left[\left|\delta\left(\mathcal{K}_1^*(u\mathbf{1}_{[s,t]})\right)\right|^2\right] \leqslant c\left(E\left[\int_0^1\left|\mathcal{K}_1^*(u\mathbf{1}_{[s,t]})\right|^2(\tau)\,\mathrm{d}\tau\right] + E\left[\iint_{[0,1]^2}\left|\mathcal{K}_1^*(\nabla_r u\mathbf{1}_{[s,t]})\right|^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\right]\right). \tag{5}$$

Under assumption II, $\mathcal{K}$ is continuous from $\mathrm{L}^2([0,1])$ into $\mathcal{S}_{\alpha-1/2,2}$ thus $\mathcal{K}^*$ is continuous from $(\mathcal{S}_{\alpha-1/2,2})'\simeq\mathcal{S}_{1/2-\alpha,2}$ into $\mathrm{L}^2([0,1])$. We have:

$$E\left[\left|\delta\left(\mathcal{K}_1^*(u\mathbf{1}_{[s,t]})\right)\right|^2\right] \leqslant c\|\mathcal{K}_1^*\|_{\alpha,2}\left(E\left[\|u\mathbf{1}_{[s,t]}\|^2_{\mathcal{S}_{1/2-\alpha,2}}\right] + E\left[\int_0^1\|\nabla_r u\mathbf{1}_{[s,t]}\|^2_{\mathcal{S}_{1/2-\alpha,2}}\,\mathrm{d}r\right]\right).$$

According to Theorem 2.1 of [12] with $t=1/2-\alpha$, $r=1/2-\gamma$ and $s=1/2-\alpha+\eta$, we have

$$\left\|\delta\left(\mathcal{K}_1^*(u\mathbf{1}_{[s,t]})\right)\right\|_{\mathrm{L}^2} \leqslant c\left\|\mathcal{K}_1^*\right\|_{\alpha,2}\|\mathbf{1}_{[s,t]}\|_{\mathcal{S}_{1/2-\gamma}}\|u\|_{\mathbb{D}_{2,1}(\mathcal{S}_{\eta+1/2-\alpha,2})}.$$

Simple computations give that $\|\mathbf{1}_{[s,t]}\|_{\mathcal{S}_{1/2-\gamma}}\leqslant c|t-s|^\gamma$ and the maximal inequality follows by a functional analytic version of the Kolmogorov criterion [7]. □

THEOREM 3.2. – *For any* $\alpha\in[1/2,1)$, *assume that assumption* I *holds. Let* u *belong to* $\mathbb{D}_{p,1}(\mathrm{L}^p)\cap\mathrm{Dom}\,\delta_{\mathcal{K}^*}$ *with* $\alpha p>1$. *The process* $\{\delta(\mathcal{K}_t^* u),\ t\in[0,1]\}$ *admits a modification with* $(\gamma-1/p)$*-Hölder continuous paths for any* $0\leqslant\gamma<\alpha$ *and we have the maximal inequality:*

$$\left\|\delta\left(\mathcal{K}_{\cdot}^* u\right)\right\|_{\mathrm{L}^p(\Omega;\mathrm{Hol}(\alpha-1/p))} \leqslant c\left\|\mathcal{K}_1^*\right\|_{\gamma,2}\|u\|_{\mathbb{D}_{p,1}(\mathrm{L}^p)}.$$

Proof. – The proof of Theorem 3.2 is identical to that of Theorem 3.1 until Eq. (5). Since $\alpha>1/2$, it is clear that $\mathcal{K}$ is continuous from $\mathrm{L}^2([0,1])$ into $\mathcal{S}_{\alpha-1/2,2}$ thus that $\mathcal{K}^*$ is continuous from $\mathcal{S}'_{\alpha-1/2,2}$ in $\mathrm{L}^2([0,1])$. Since $\mathcal{S}_{\alpha-1/2,2}$ is continuously embedded in $\mathrm{L}^{(1-\alpha)^{-1}}$, it follows that $\mathrm{L}^{1/\alpha}=(\mathrm{L}^{1/(1-\alpha)})'$ is continuously embedded in $\mathcal{S}_{1/2-\alpha,2}$. Since u belongs to $\mathbb{D}_{p,1}(\mathrm{L}^p)$, Hölder inequality implies that $\|u\mathbf{1}_{[s,t]}\|_{\mathrm{L}^{1/\alpha}}\leqslant\|u\|_{\mathrm{L}^p}|t-s|^{\alpha-1/p}$. A similar inequality is valid for the other term of (5) and the proof is then complete. □

Pour citer cet article : L. Decreusefond, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 903–908

4. Change of variable

Another feature of this stochastic integral is that its composition with absolutely continuous shifts of the Wiener space is simple to compute.

THEOREM 4.1. – *Let $T(\omega)=\omega+Kv(\omega)$ be such that v belongs to $\mathbb{D}_{p,1}(\mathrm{L}^2)$ for some $p>1$ and $T^*P \ll P$. Let u be such that u and $u\circ T$ belong to* $\operatorname{Dom}\delta_{\mathcal{K}^*}$ *and $\nabla\mathcal{K}^*u$ and $\nabla(\mathcal{K}^*u\circ T)$ are a.s. trace class operators. Then,*

$$\left(\int u(s)\circ \mathrm{d}X_s\right)\circ T=\int (u\circ T)(s)\circ \mathrm{d}X_s+\int \mathcal{K}^*(u\circ T)(s)v(s)\,\mathrm{d}s.$$

Proof. – According to Proposition B.6.12 and B.6.8 of [14], we have

$$\delta\big(\mathcal{K}^*u\big)\circ T=\delta\big(\mathcal{K}^*(u\circ T)\big)+\int \mathcal{K}^*(u\circ T)(s)v(s)\,\mathrm{d}s+\operatorname{trace}\big(\big(\nabla\mathcal{K}^*u\big)\circ T\cdot\nabla v\big)\quad\text{and}$$

$$\operatorname{trace}\big(\big(\nabla\mathcal{K}^*u\big)\circ T\cdot\nabla v\big)=\operatorname{trace}\big(\nabla\big(\mathcal{K}^*u\circ T\big)\big)-\operatorname{trace}\big(\nabla\mathcal{K}^*u\big)\circ T.$$

The proof is completed by substituting the latter equation into the former. □

For u deterministic and v adapted, this means that the law of the process $\{\int_0^t u_s\,\mathrm{d}X_s-\int_0^t\mathcal{K}^*u(s)v(s)\,\mathrm{d}s,\ t\geqslant 0\}$, under T^*P, is identical to the P-law of the process $\{\int_0^t u_s\,\mathrm{d}X_s,\ t\geqslant 0\}$.

5. Itô's formula

We are now interested in non-linear transformations of Itô-like processes: $Z(t)=z+\int_0^t u(s)\circ\mathrm{d}X_s$, for a sufficiently regular u. The key remark is that the triangularity of the kernel K implies that $\mathcal{K}_1^*(u\mathbf{1}_{[0,t]})=\mathcal{K}_t(u)$ on $[0,t]$.

THEOREM 5.1. – *Let F be a $\mathcal{C}_b^2$-function and suppose that u is a smooth-cylindric process, i.e., $u(t)=f(\delta h_1,\dots,\delta h_n)v(t)$, where f is a rapidly decreasing function from $\mathbf{R}^n$ into $\mathbf{R}$, v belongs to* $\operatorname{Dom}\mathcal{K}$ *and for any j, h_j belongs to* $\operatorname{Dom}\mathcal{K}$. *Let $Z_t=z+\int_0^t u(s)\,\mathrm{d}X_s$. Then, $u\cdot F'\circ Z$ belongs to* $\operatorname{Dom}\delta_X$ *and*

$$F(Z_t)=F(z)+\int_0^t u(s)F'(Z_s)\circ\mathrm{d}X_s. \tag{6}$$

Sketch of the proof. – The proof follows the lines of the method initiated in [13]. Let ψ be an $\mathbf{R}$-valued smooth-cylindric functional,

$$\begin{aligned}&E\big[\big(F(Z_{t+\varepsilon})-F(Z_t)\big)\psi\big]\\&\quad=E\big[F'(Z_t)(Z_{t+\varepsilon}-Z_t)\psi\big]+E\left[(Z_{t+\varepsilon}-Z_t)^2\int_0^1F''\big(uZ_t+(1-u)Z_{t+\varepsilon}\big)(1-u)\,\mathrm{d}u\cdot\psi\right]:=A_1+A_2.\end{aligned}$$

In view of the definition of the stochastic integral with respect to X and of the triangularity of $\mathcal{K}$,

$$Z_{t+\varepsilon}-Z_t=\delta\big(\mathcal{K}_1^*u\mathbf{1}_{[t,t+\varepsilon]}\big)+\int_t^{t+\varepsilon}D_su(s)\,\mathrm{d}s,\quad\text{where } D=\mathcal{K}\nabla,$$

thus

$$\begin{aligned}A_1&=E\big[\delta\big(\mathcal{K}_1^*u\mathbf{1}_{[t,t+\varepsilon]}\big)F'(Z_t)\psi\big]+E\left[\psi F'(Z_t)\int_t^{t+\varepsilon}D_su(s)\,\mathrm{d}s\right]\\&=E\left[F'(Z_t)\int_t^{t+\varepsilon}u(s)D_s\psi\,\mathrm{d}s\right]+E\left[\int_t^{t+\varepsilon}u(s)D_sF'(Z_t)\,\mathrm{d}s\,\psi\right]+E\left[\psi F'(Z_t)\int_t^{t+\varepsilon}D_su(s)\,\mathrm{d}s\right].\end{aligned}$$

It is then clear that $\varepsilon^{-1}A_1$ converges to $E[D_t(u(t)F'(Z_t))\psi]+E[F'(Z_t)u(t)D_t\psi]$. Since F'' is bounded, $A_2\leqslant[(Z_{t+\varepsilon}-Z_t)^2\psi]$, and similar computations show that $\varepsilon^{-1}A_2$ converges to 0 as ε goes to 0. One can then use the fundamental lemma of calculus and we obtain:

$$
\begin{aligned}
E\big[\big(F(Z_t)-F(z)\big)\psi\big] &= E\left[\int_0^t D_t\big(u(t)F'(Z_t)\big)\,\mathrm{d}t\,\psi\right] + E\left[\int_0^t F'(Z_t)u(t)D_t\psi\,\mathrm{d}t\right]\\
&= E\left[\int_0^t D_t\big(u(t)F'(Z_t)\big)\,\mathrm{d}t\,\psi\right] + E\big[\delta\big(\mathcal{K}_t^*(uF'\circ X)\big)\big]\\
&= E\left[\int_0^t u(s)F'(Z_s)\circ\mathrm{d}X_s\,\psi\right].
\end{aligned}
$$

Eq. (6) follows by identification. □

By density of the smooth-cylindric functionals, one has the more general result:

THEOREM 5.2. – *Assume that we have, either*

– *Hypothesis* I *and* II *hold for some* $\alpha < 1/2$ *and there exists* $\eta > 0$ *such that* $\mathcal{K}^*$ *is continuous from* $\mathcal{S}_{1+\eta-\alpha,2}$ *into* $\mathcal{S}_{1/2+\eta,2}$, *and u belongs to* $\mathbb{D}_{8,2}(\mathcal{S}_{1+\eta-\alpha,2})$, *or*

– *Hypothesis* I *holds for* $\alpha \geqslant 1/2$ *and* $\mathcal{K}^*$ *is continuous from* $\mathrm{L}^p([0,1])$ *into* $\mathcal{S}_{\alpha-1/2,p}$ *for some* $p > (\alpha-1/2)^{-1}$, *and u belongs to* $\mathbb{D}_{4p,2}(\mathrm{L}^p([0,1]))$.

Then, for $F \in \mathcal{C}_b^2$, $u\cdot F'\circ Z$ *belongs to* $\operatorname{Dom}\delta_X$ *and* (6) *is true.*

The proof boils down to prove that

$$
\|u\cdot F'\circ Z\|_{\mathbb{D}_{p,1}(V)} \leqslant c\|u\|_{\mathbb{D}_{4p,2}(V)},
$$

where $V = \mathcal{S}_{1+\eta-\alpha,2}$, $p = 2$ when $\alpha < 1/2$, and $V = \mathrm{L}^p([0,1])$, when $\alpha > 1/2$. This is done by repetitive use of the previous Theorems 2.2, 2.3, 3.1 and 3.2.

References

[1] E. Alòs, J.A. Olivier León, D. Nualart, Stochastic Stratonovitch calculus for fractional Brownian motion with Hurst parameter less than 1/2, Taiwanese J. Math. 5 (3) (2001) 609–632.

[2] E. Alòs, O. Mazet, D. Nualart, Stochastic calculus with respect to Gaussian processes, Ann. Probab. 29 (2001) 766–801.

[3] A. Benassi, P. Bertrand, S. Cohen, J. Istas, Identification d’un processus gaussien multifractionnaire avec des ruptures sur la fonction d’échelle, C. R. Acad. Sci. Paris, Série I 329 (5) (1999) 435–440.

[4] L. Decreusefond, Regularity properties of some stochastic Volterra integrals with singular kernel, Potential Anal. 16 (2002) 139–149.

[5] L. Decreusefond, Stochastic calculus for Volterra processes (2002), in preparation.

[6] L. Decreusefond, A.S. Üstünel, Stochastic analysis of the fractional Brownian motion, Potential Anal. 10 (2) (1999) 177–214.

[7] D. Feyel, A. de La Pradelle, On fractional Brownian processes, Potential Anal. 10 (3) (1999) 273–288.

[8] M. Gradinaru, F. Russo, P. Vallois, Generalized covariations, local time and Stratonovitch Itô’s formula for fractional Brownian motion with Hurst index $\geqslant 1/4$, Tech. Report 16, Université Paris XIII, 2001.

[9] T.J. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoamericana 14 (2) (1998) 215–310.

[10] D. Nualart, The Malliavin Calculus and Related Topics, Springer-Verlag, 1995.

[11] D. Nualart, M. Zakai, On the relation between the Stratonovich and Ogawa integrals, Ann. Probab. 17 (4) (1989) 1536–1540.

[12] J. Tambača, Estimates of the Sobolev norm of a product of two functions, J. Math. Anal. Appl. 255 (1) (2001) 137–146.

[13] A.S. Üstünel, The Itô formula for anticipative processes with nonmonotonous time scale via the Malliavin calculus, Probab. Theory Related Fields 79 (1988) 249–269.

[14] A.S. Üstünel, M. Zakai, Transformation of Measure on Wiener Space, Springer-Verlag, Berlin, 2000.

[15] A.S. Üstünel, An Introduction to Analysis on Wiener Space, Lectures Notes in Math., Vol. 1610, Springer-Verlag, 1995.

[16] M. Zähle, Integration with respect to fractal functions and stochastic calculus, I, Probab. Theory Related Fields 111 (3) (1998) 333–374.