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A sufficient condition for the existence of approximate inertial manifolds containing the global attractor *

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Abstract Using the recently introduced concept of inertial manifold with delay we present a new method of construction of approximate inertial manifolds (AIMs). In the case when the global attractor can be embedded into a finite-dimensional C^2-manifold we construct AIMs of the same dimension which contain the attractor. ***To cite this article: A. Rezounenko, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1015–1020.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une condition suffisante pour l'existence de variétés inertielles approchées contenant l'attracteur global

Résumé Utilisant le récent concept de variété inertielle avec retard, nous présentons une nouvelle méthode de construction de variétés inertielles approchées. Dans le cas où l'attracteur global peut être plongé dans une variété C^2 de dimension finie, nous construisons une variété inertielle approchée de la même dimension contenant l'attracteur. ***Pour citer cet article : A. Rezounenko, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1015–1020.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Nous étudions le système dynamique engendré par l'équation (2.1) avec A opérateur positif, à spectre discret et une application lipschitzienne B de domaine $D(A^\alpha)(0 \leqslant \alpha < 1)$ à valeurs dans un espace de Hilbert séparable H.

Pour N entier, on appelle P_N la projection orthogonale sur l'espace engendré par les N premiers vecteurs propres de A ; on pose $Q_N = I - P_N$; Nous notons $\Phi : P_N \mathrm{H} \times Q_N D(A^\alpha) \to Q_N D(A^\alpha)$ l'application qui définit la variété inertielle avec retard (*voir* [9,19,18]).

Le but essentiel de cette Note est d'étudier l'application ψ définie par (3.1) et son graphe M^φ (*voir* (3.2)) pour une application lipschitzienne $\varphi : P_N \mathrm{H} \to Q_N D(A^\alpha)$. Nous démontrons le

THÉORÈME 0.1. – *Soit* $\{\varphi\}_{N=N_0}^{\infty}$ *une suite d'applications* $\varphi_N \in \mathrm{Lip}(P_N \mathrm{H}, Q_N D(A^\alpha))$ *telle que pour tout* $l_N < C$ *pour tout* $N \geqslant N_0$, *où* C *est une constante et* l_N *est la constante de Lipschitz de* φ_N. *Alors la*

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* For my parents Vyacheslav and Larisa.

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suite M^{φ_N} *définie dans* (3.2) *forme la famille des variétés inertielles approchées d'ordre exponentiel donné par* (3.4) *où* ψ *est remplacé par* ψ_N *et* C^i_R, σ_0, ρ *sont indépendants de* N.

Si on suppose $B \in \mathrm{C}^2(D(A^\alpha), \mathrm{H})$ et borné sur les boules de $D(A^\alpha)$ on utilise la définition suivante :

DÉFINITION 0.2 (*Voir* [20]). – Soit $\mathcal{K}$ un compact de $D(A^\alpha)$ invariant pour l'opérateur d'évolution $\{S_t\}$ engendré par (2.1). Nous disons que *la dynamique sur* $\mathcal{K}$ *est de dimension finie*, si pour une valeur de $N \geqslant 1$, il existe une équation différentielle ordinaire $\dot{x} = h(x)$ avec $h \in \mathrm{Lip}(R^N, R^N)$ et l'opérateur d'évolution associé $\{S_t^N\}$, et également un plongement lipschitzien $g : \mathcal{K} \to R^N$ tel que $g(S_t u) = S_t^N g(u)$ pour tout $u \in \mathcal{K}$ et $t \geqslant 0$.

Nous remarquons qu'une condition suffisante d'existence de dynamique de dimension finie sur $\mathcal{K}$ est $\mathcal{K} \subset M_2$, où M_2 est une variété C^2 de dimension finie dans $D(A^\alpha)$ [20, Théorème 1.5].

Le théorème suivant est l'application essentielle du Théorème 0.1.

THÉORÈME 0.3. – *Soit la dynamique sur* $\mathcal{K}$ *de dimension finie. Alors il existe une famille* $\{M^i_{\mathcal{K}}\}_{i=m}^{\infty}$ *de variétés inertielles approchées d'ordre exponentiel tel que chaque* $M^i_{\mathcal{K}}$ *contienne le compact* $\mathcal{K}$.

Si on choisit $\mathcal{K} = \mathcal{A}$ ou $\mathcal{A}$ est l'attracteur global, nous obtenons le résultat énoncé dans le titre de cette Note. Dans ce cas nous pouvons dire plus : c'est non seulement un voisinage de l'AIM qui est attiré, mais l'AIM elle-même attire toutes les orbites.

1. Introduction

The concept of global attractor has received much attention and is extremely useful in the study of the asymptotic behaviour of solutions of dissipative partial differential equations (PDEs) (*see*, e.g., [1,22,15, 3,4] and references therein). Many PDEs have recently been shown to possess global attractors which are finite-dimensional. It opens the way for the reduction of the dynamics of infinite-dimensional dissipative equations to a finite-dimensional system. More precisely, one looks for a finite-dimensional system that will adequately capture all the asymptotic properties of the original system.

One of the direct approaches to this problem is the theory of inertial manifolds (IM) (*see*, e.g., [13, 14,22,2,7,3]). IM is a finite-dimensional invariant Lipschitz manifold which attracts exponentially all orbits and contains the global attractor. Unfortunately, the existence of IMs usually holds under the very restrictive spectral gap condition. It seems that this condition is not just a technical obstacle. To consider the case when the spectral gap condition does not hold the concepts of approximate inertial manifolds (*see*, e.g., [11,13,8, 5]) and of exponential attractors [10] have been introduced. An approximate inertial manifold can be defined as a finite-dimensional Lipschitz manifold and a thin surrounding neighborhood into which any orbit enters in a finite time. It is clear that the global attractor is included in this neighborhood. Unfortunately, AIM does not possess the property of invariance, so AIM itself does not reflect the asymptotic behaviour of the system, but its attracting neighborhood does.

The aim of this work is to construct AIMs which contain the global attractor and hence provide a finite-dimensional system which adequately presents the asymptotic nature of the original flow. We do this in the case when the attractor can be embedded into a finite-dimensional C^2-manifold. We notice that due to Foias and Olson [12] any compact set (e.g., global attractor) of finite fractal dimension can be embedded into a finite-dimensional Hölder manifold.

Our construction is based on the recently introduced concept of inertial manifold with delay [9] and its connection [17] with AIMs.

2. Preliminaries

We study the dynamical system generated by the evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = B(u) \quad \text{for } t > \sigma. \tag{2.1}$$

Here A is a positive operator with a discrete spectrum in a separable Hilbert space H, i.e., there exists an orthonormal basis $\{e_k\}$ of H such that

$$Ae_k = \lambda_k e_k \quad \text{with } 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \ \lim_{k\to\infty} \lambda_k = \infty.$$

We also assume in (2.1) that B is a (nonlinear) mapping from $D(A^\alpha)$ to H ($0 \leqslant \alpha < 1$) such that

$$\|B(w)\| \leqslant M \cdot \bigl(1 + \|A^\alpha w\|\bigr), \quad w \in D\bigl(A^\alpha\bigr),$$
$$\|B(w_1) - B(w_2)\| \leqslant M \cdot \|A^\alpha (w_1 - w_2)\| \quad \text{for } w_1, w_2 \in D\bigl(A^\alpha\bigr).$$

Here and below $\|\cdot\|$ is the norm of H, and M is a positive constant. We will also use $|\cdot|_\alpha \equiv \|A^\alpha \cdot\|$ and $(\cdot,\cdot)$ the Hermitian product in H.

As in the finite-dimensional case we will use the following definition:

DEFINITION 2.1. – A function $u = u(t) \in \mathrm{C}(\sigma, T; D(A^\alpha))$ is a *solution* of (2.1) in the interval $[\sigma, T]$ if

$$u(t) = \mathrm{e}^{-(t-\sigma)A} u(\sigma) + \int_\sigma^t \mathrm{e}^{-(t-\tau)A} B\bigl(u(\tau)\bigr)\,\mathrm{d}\tau.$$

Using the standard fixed point method one easily proves existence and uniqueness of the solution of (2.1). Therefore, we can define an evolution semigroup S_t in $D(A^\alpha)$ by $S_t[v] \equiv u(t)$, where $u(t)$ is the solution of (2.1) with $u(0) = v,\ t \geqslant 0$.

Let us fix an integer N and denote by $P = P_N$ the orthogonal projection onto the space spanned by the first N eigenvectors of A. Let $Q_N = I - P_N$.

We essentially rely on the concept of Inertial Manifold with Delay (IMD) introduced by Debussche and Temam [9]. We use a version of the Lyapunov–Perron method presented in [2] and rely on the following version [18] of the main theorem from [9].

THEOREM 2.2. – *There exists T_0 such that for any $T \in (0, T_0]$, any $p \in P_N\mathrm{H}$ and $q \in Q_N D(A^\alpha)$, there exists a unique solution $u = u(t)$ of* (2.1) *defined on $[-T, \infty)$ such that $P_N u(0) = p,\ Q_N u(-T) = q$. Moreover, if we set $\Phi(p,q) \equiv Q_N u(0)$, this defines a Lipschitz mapping from $P_N\mathrm{H} \times Q_N D(A^\alpha)$ to $Q_N D(A^\alpha)$, i.e., for any $(p^i, q^i) \in P_N\mathrm{H} \times Q_N D(A^\alpha),\ i = 1, 2$, we have*:

$$\bigl|\Phi\bigl(p^1, q^1\bigr) - \Phi\bigl(p^2, q^2\bigr)\bigr|_\alpha \leqslant L_1(T)\bigl|p^1 - p^2\bigr|_\alpha + L_2(T)\bigl|q^1 - q^2\bigr|_\alpha. \tag{2.2}$$

Additionally, for any given constant $c > \ln 2$ the following two conditions

$$\lambda_N^{1-\alpha} > 4M\,\mathrm{e}^c\left(\mathrm{e}^c - \mathrm{e}^{-c} + \frac{\alpha^\alpha}{1-\alpha} c^{1-\alpha}\right) \quad \textit{and} \quad \ln 2 < \lambda_{N+1} T \leqslant c$$

imply that the Lipschitz constants satisfy $L_i < 1,\ i = 1, 2$, in (2.2).

The graph of Φ is called Inertial Manifold with Delay (IMD). For all details and properties the reader is referred to [9,19,18].

3. Approximate inertial manifolds

For any pair of spaces V_1 and V_2 we denote by $\mathrm{Lip}(V_1, V_2)$ the class of Lipschitz mappings from V_1 to V_2. A continuous group $S_t : V_1 \to V_2$ is called a *Lipschitz flow* if $S_t \in \mathrm{Lip}(V_1, V_2)$ for any $t \in R$.

Consider any $\varphi \in \mathrm{Lip}(P_N\mathrm{H}, Q_N D(A^\alpha))$. The main goal of the article is to study the following mapping, induced by φ:

$$\psi(p) \equiv \Phi\bigl(p; \varphi(p)\bigr). \tag{3.1}$$

Here Φ is defined as in Theorem 2.2. We also define the finite-dimensional manifold

$$M^\varphi \equiv \bigl\{p + \psi(p) : p \in P_N\mathrm{H}\bigr\} \equiv \bigl\{p + \Phi\bigl(p; \varphi(p)\bigr) : p \in P_N\mathrm{H}\bigr\}. \tag{3.2}$$

We will suppose that (2.1) possesses an absorbing ball in $D(A^\alpha)$ (*see* [22] for numerous examples). Using the existence of an absorbing ball for the equation we can classically truncate the nonlinear term B outside the ball so that it will be replaced by a function which is equal to B inside the absorbing ball but which has bounded support. We denote by R the radius of a ball containing this support.

Using the same line of arguments as in [5] and [17, Theorems 3.1 and 3.2] we can prove that M^φ is an approximate inertial manifold.

THEOREM 3.1. – *Let us fix* $\varphi \in \mathrm{Lip}(P_N\mathrm{H}, Q_N D(A^\alpha))$. *There exist constants* ρ_1 *and* Λ (*both depending on* M, α *and* λ_1 *only*) *such that if*

$$\lambda_{N+1}^{1-\alpha} \geqslant \Lambda\rho^{-1}, \quad T = \rho\lambda_{N+1}^{-\alpha}, \quad 0 < \rho \leqslant \rho_1, \tag{3.3}$$

then the mapping ψ *defined in* (3.1) *possesses the following property*

$$\bigl\|A^\alpha\bigl(Q_N u(t) - \psi\bigl(P_N u(t)\bigr)\bigr)\bigr\| \leqslant C_R^1 \cdot \exp\left\{-\frac{\sigma_0}{\rho}\lambda_{N+1}^\alpha(t - t_*)\right\} + C_R^2 \cdot \exp\left\{-\frac{\rho}{2}\lambda_{N+1}^{1-\alpha}\right\}, \tag{3.4}$$

for all $t \geqslant t_* + T/2$. *Here* σ_0 *is an absolute constant and* $u = u(t)$ *is any solution of* (2.1) *such that* $\|A^\alpha u(t)\| \leqslant R$ *for* $t_* \leqslant t < \infty$.

The main result of this section is the following

THEOREM 3.2. – *Let* $\{\varphi_N\}_{N=N_0}^\infty$ *be a sequence of mappings* $\varphi_N \in \mathrm{Lip}(P_N\mathrm{H}, Q_N D(A^\alpha))$ *such that* $\ell_N < C$ *for all* $N \geqslant N_0$. *Here* C *is a constant,* ℓ_N *is the Lipschitz constant of* φ_N. *Then the sequence* M^{φ_N} *defined in* (3.2) *forms the family of AIMs of an exponential order such that* (3.4) *holds with* ψ_N *instead of* ψ *and with* C_R^i, σ_0, ρ *independent of* N.

Remark 1. – The conditions of Theorem 3.1 guarantee that $L_i < 1$, $i = 1, 2$ (*see* (2.2)), so for any $p \in P_N\mathrm{H}$ we can use the mapping Φ, defined in Theorem 2.2, as a strict contraction in $Q_N D(A^\alpha)$. In this case we get the existence of a unique mapping $\overline{\varphi} : P_N\mathrm{H} \to Q_N D(A^\alpha)$ satisfying $\overline{\varphi}(p) \equiv \Phi(p; \overline{\varphi}(p))$. The graph of $\overline{\varphi}$ is called the Steady approximate inertial manifold and it is investigated in [17].

Remark 2. – The family of AIMs constructed in [5] can be obtained if we choose in (3.2) $\varphi(p) \equiv 0$.

4. AIMs containing compact invariant sets

In this section we additionally assume that $B \in \mathrm{C}^2(D(A^\alpha), \mathrm{H})$ and B is bounded on balls in $D(A^\alpha)$. This, together with the existence of an absorbing ball, after the truncation of B, guarantees the existence of the compact global attractor $\mathcal{A}$ [6].

DEFINITION 4.1 ([20]). – Let $\mathcal{K}$ be a compact in $D(A^\alpha)$ which is invariant for S_t. We say that the dynamics on $\mathcal{K}$ is finite-dimensional if for some $N \geqslant 1$ there exist an ordinary differential equation $\dot{x} = h(x)$ with $h \in \mathrm{Lip}(R^N, R^N)$ and the corresponding evolution operator $\{S_t^N\}$, and also a Lipschitz embedding $g : \mathcal{K} \to R^N$ such that $g(S_t u) = S_t^N g(u)$ for any $u \in \mathcal{K}$ and $t \geqslant 0$.

DEFINITION 4.2 ([20]). – We say that the limiting dynamics of (2.1) is finite-dimensional if the dynamics is finite-dimensional on the attractor $\mathcal{A}$.

We will use the equivalence of the following statements (*see* [20, Theorem 1.6]):

(FD) The dynamics on $\mathcal{K}$ is finite-dimensional.

(Fl) The semi-flow $\{S_t\}$ on $\mathcal{K}$ is injective and can be extended to a Lipschitz one in the $D(A^\alpha)$-metrics of the flow.

(GrF) For some $n \geqslant 1$, for the spectral projection P_n, one has $|u - v|_\alpha \leqslant C \cdot |P_n(u - v)|_\alpha$ on $\mathcal{K}$ with $C = C(\mathcal{K}, n)$.

Remark 3. – The property (GrF) means that $P_n : \mathcal{K} \to D(A^\alpha)$ is a Lipschitz embedding, so $\mathcal{K}$ is a part of some Lipschitz graph over $P_n\mathrm{H}$.

We notice that a sufficient condition for the dynamics on $\mathcal{K}$ to be finite-dimensional is $\mathcal{K} \subset \mathcal{M}_2$, where $\mathcal{M}_2$ is a finite-dimensional C^2-manifold in $D(A^\alpha)$ [20, Theorem 1.5].

The main result of this section is the following

THEOREM 4.3. – *Assume that the dynamics is finite-dimensional on $\mathcal{K}$. Then there exists an approximate inertial manifold which contains the compact $\mathcal{K}$, i.e., $\mathcal{K} \subset M_{\mathcal{K}} \equiv \{p + \psi(p) : p \in P_m\mathrm{H}\}$. Here $m \geqslant$* $\max\{n, N\}$, *where n is defined in* (GrF) *and N satisfies the first inequality in* (3.3).

Remark 4. – If (GrF) is satisfied for some n, then it is evidently satisfied for any $m > n$.

Hence, using Theorems 3.2 and 4.3 we can prove the following:

THEOREM 4.4. – *Let the dynamics on $\mathcal{K}$ be finite-dimensional. Then there exists a family $\{M^i_{\mathcal{K}}\}_{i=m}^\infty$ of approximate inertial manifolds of an exponential order such that each $M^i_{\mathcal{K}}$ contains the compact $\mathcal{K}$.*

Remark 5. – If we choose $\mathcal{K} = \mathcal{A}$, where $\mathcal{A}$ is the global attractor, we get the result stated in the title of the article (*see* also Definition 4.2). In this case we can say even more: not just a neighborhood of the AIM is attracting but the AIM itself attracts all the orbits.

Proof of Theorem 4.3. – For an invariant compact $\mathcal{K}$ we construct our AIM $\{M_{\mathcal{K}}\}$ as follows:

Step 1. Let $V \equiv P_m\mathcal{K}$, where $m \geqslant \max\{n, N\}$. Define the mapping $f_0(p) \equiv P_m^{-1}p - p$ for any $p \in V$. We have $f_0 \in \mathrm{Lip}(V, Q_m D(A^\alpha))$ (*see* Remark 3).

Step 2. Define $g_0 \equiv S_{-T} : \mathcal{K} \to \mathcal{K}$. The property (Fl) gives that g_0 is a Lipschitz mapping in $D(A^\alpha)$.

Step 3. Define $g(p) \equiv Q_m g_0(p + f_0(p)) : V \to Q_m D(A^\alpha)$. It is also a Lipschitz mapping due to the Lipschitz properties of f_0 and g_0.

Step 4. Now we extend g to the mapping $\varphi \in \mathrm{Lip}(P_m H, Q_m D(A^\alpha))$. This is possible due to the idea of the proof [21, Theorem 3, Chapter VI] (*see* also [20, Lemma 2.4]).

Step 5. Define $\psi(p) \equiv \Phi(p; \varphi(p)) \in \mathrm{Lip}(P_m\mathrm{H}, Q_m D(A^\alpha))$. We note that to construct the mapping Φ we choose $T = \rho\lambda_{m+1}^{-\alpha}$ (*see* (3.3)). Theorem 3.1 gives that $M_{\mathcal{K}} \equiv \{p + \psi(p) : p \in P_m\mathrm{H}\}$ is an approximate inertial manifold.

Let us prove the inclusion $\mathcal{K} \subset M$. Consider any $u^0 \in \mathcal{K}$ and the solution $u(t) \subset \mathcal{K}$ such that $u(0) = u^0$. We can write (*see* Remark 3) $u^0 = p + f_0(p)$, where f_0 is defined in step 1. To construct $\Phi(p; \varphi(p))$ one uses the unique solution v satisfying $P_m v(0) = p$ and $Q_m v(-T) = \varphi(p)$. This is the solution $u(t)$ (*see* steps 2–4). Hence, by the definition $\Phi(p; \varphi(p)) = Q_m u(0) = f_0(p)$. Now the inclusion $\mathcal{K} \subset \{p + f_0(p) :$ $p \in P_m\mathrm{H}\}$ and $\psi(p) = f_0(p)$ for any $p \in P_m\mathcal{K}$ give $\mathcal{K} \subset M_{\mathcal{K}}$. The proof is complete.

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